

# ROBUST DETECTION AND SOCIAL LEARNING IN TANDEM NETWORKS

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## ABSTRACT

We consider a binary hypothesis testing problem in a tandem network where the distribution of the agent observations under each hypothesis comes from an uncertainty class. When agents know their positions in the tandem, and the contamination of the uncertainty classes are non-zero, we show that asymptotic learning of the true hypothesis under social learning is not possible even when the log likelihood ratio of the nominal distributions of the uncertainty classes is unbounded. Furthermore, asymptotic learning in social learning is achievable if and only if the uncertainty classes contamination converge to zero. When agents do not know their positions, the minimax error probability is bounded from zero, and we provide tight bounds for it.

*Index Terms*— social learning, decentralized detection, tandem networks

## 1. INTRODUCTION

Social networks have grown immensely in popularity over the last ten years, and have become an easily accessible source of information for many people, some of whom rely on such networks to inform them of global affairs and news updates [1]. A potential application of social networks is in social learning and sensing, where inference about a phenomenon of interest is made through the help of agents in the network [2,3]. For example, when users send a picture of litter in a park to a social sensing platform [4], the park management can immediately deploy cleaners to the right locations to remove the litter in order to create a more pleasant environment for park goers. Another use might be for agents to report congested road conditions, so that other agents know to avoid those roads and prevent aggravating the traffic congestion [5].

A social network propagates information via information exchange between agents in the network, with the information potentially modified by each agent before passing on to another agent. A social network can be modeled as a graph of agents, where some agents make observations about a phenomenon of interest, and shares part of their observed information with other agents connected to it. Each agent then fuses its own information (if any) with information obtained from its neighbors in the network to form its own opinion. The goal of each agent is to make an inference about the phenomenon, with the help of the opinions of other agents in the network. However, information provided by an agent may not be reliable or an agent may not have sufficient prior knowledge of the agent from whom it receives a piece of information from to accurately make inferences using that information [4].

In this paper, we formulate and study the robust social learning problem in the very simplistic tandem network. A tandem network consists of agents connected in a serial fashion, where each agent

receives information from a previous agent, makes its own observation about a phenomenon of interest modeled by a binary hypothesis, and makes a decision of the hypothesis based on both its observation and the information from the previous agent. The agent's decision is based on a local error criterion, which it selfishly tries to optimize. This behavior is present in social networks, where users are mainly concerned with spreading only locally accurate information. In this paper, we call this *social learning* [6–9], in contrast to the case where agents' decision rules are designed to minimize the error criterion of the last agent in the network, which is known as *decentralized detection* [10, 11]. The tandem network approximates a single information flow in a social network, and has been widely studied in [12–16]. In [16], the tandem network is studied under social learning rules, and conditions for the error approaching zero as the number of agents grows large are derived. The reference [11] shows that the rate of error decay is at most subexponential. Feedforward networks, in which an agent obtains information from a subset of previous agents not necessarily just the immediate predecessor, have been studied in [6, 9]. In the above papers, it is assumed that each agent knows the distribution of its private observation, and that of its predecessor, as well as its location in the network. In this paper, we investigate what happens when one or both of these assumptions do not hold.

The robust detection framework was first proposed by [17], which studies the case of a single agent. The underlying probability distributions governing the agent observations are assumed to belong to different uncertainty classes under different hypotheses, and it is shown that under a minimax error criterion, the optimal decision rule for the agent is a likelihood ratio test based on the pair of least favorable distributions (LFDs). Subsequently, the reference [18] investigates robust detection in a finite parallel configuration, with and without a fusion center. In this paper, we consider robust detection and social learning in a tandem network. Our main contributions are the following.

1. We obtain the LFDs for tandem networks, and show that when the uncertainty classes for all agent observations are the same, and agents know their positions in the tandem, asymptotic learning under decentralized detection and social learning is not possible even when the log likelihood ratio of the nominal distributions is unbounded if the contamination of both uncertainty classes are non-zero. This is in contrast to the case where the contamination of the uncertainty classes are zero [11, 16], in which case asymptotic learning happens if the log likelihood ratio is unbounded.
2. When agents know their positions in the tandem, we show that asymptotic learning under social learning is achievable if and only if the log likelihood ratio of the nominal distributions is unbounded, and the contamination of the uncertainty classes converge to zero.

3. When the agents do not know their positions in the tandem, asymptotic learning is not possible. We provide lower bounds for the false alarm and missed detection probabilities, and show that if the contamination of the uncertainty classes converge to zero, the error probabilities converge to these bounds.

The rest of this paper is organized as follows. In 2, we describe our model and notation, as well as prove the result in [18] for tandem networks. In 3, we study various asymptotic properties of the tandem network. We then relax the assumptions that every agent knows its position in the network and derive bounds for the asymptotic minimax error, as well as the conditions necessary to achieve those bounds. In 4, we provide some numerical results. In 5, we conclude with some brief comments.

## 2. PROBLEM FORMULATION

We consider a tandem network consisting of  $N$  agents, and a binary hypothesis testing problem in which the true hypothesis  $H$  is  $H_j$  with prior probability  $\pi_j \in (0, 1)$ , for  $j = 0, 1$ . Conditioned on  $H = H_j$ , each agent  $i$  in the tandem network makes an observation  $Y_i$ , defined on a common measurable space  $(\mathcal{Y}, \mathcal{A})$ , and with distribution  $P_j^i$  belonging to an uncertainty class

$$\mathcal{P}_j^i = \left\{ Q \mid Q = (1 - \epsilon_j^i)P_j + \epsilon_j^i R, R \in \mathcal{R} \right\},$$

where  $\mathcal{R}$  is the set of all probability measures on  $(\mathcal{Y}, \mathcal{A})$ ,  $P_j \in \mathcal{R}$  is the nominal probability distribution, and  $\epsilon_j^i \in [0, 1]$  is a positive constant that is sufficiently small so that  $\mathcal{P}_0^i$  and  $\mathcal{P}_1^i$  are disjoint. We assume that all distributions in  $\mathcal{P}_0^i$  and  $\mathcal{P}_1^i$  are absolutely continuous with one another, and the distribution  $P_j^i$  from which the observation  $Y_i$  is drawn from is unknown. The parameter  $\epsilon_j^i$  is also known as the contamination for the uncertainty class  $\mathcal{P}_j^i$ . When  $\epsilon_j^i = 0$ , we recover the classical Bayesian hypothesis testing problem.

For  $i = 1, \dots, N$ , each agent  $i$  makes a decision  $U_i = \phi_i(Y_i, U_{i-1})$  about the hypothesis  $H$ , where  $U_0 \equiv 0$ . For  $j = 0, 1$ , let  $P_j^{(N)} = P_j^1 \times P_j^2 \times \dots \times P_j^N$ . In the decentralized detection problem, our aim is to find a set of decision rules  $\phi^{(N)} = (\phi_1, \phi_2, \dots, \phi_N)$  such that the maximum probability of error

$$\begin{aligned} P_e^{(N)}(\phi^{(N)}) = & \pi_0 \sup_{P_0^{(N)} \in \mathcal{P}_0^{(N)}} P_F(\phi^{(N)}, P_0^{(N)}) \\ & + \pi_1 \sup_{P_1^{(N)} \in \mathcal{P}_1^{(N)}} P_M(\phi^{(N)}, P_1^{(N)}) \end{aligned} \quad (1)$$

is minimized. In (1),  $P_F$  and  $P_M$  are the false alarm and missed detection probabilities respectively, and can be determined recursively as

$$\begin{aligned} P_F(\phi^{(i)}, P_0^{(i)}) = & P_F(\phi^{(i-1)}, P_0^{(i-1)})P_0^i(\phi_i(Y_i, 1) = 1) \\ & + (1 - P_F(\phi^{(i-1)}, P_0^{(i-1)}))P_0^i(\phi_i(Y_i, 0) = 1) \end{aligned}$$

and

$$\begin{aligned} P_M(\phi^{(i)}, P_1^{(i)}) = & P_M(\phi^{(i-1)}, P_1^{(i-1)})P_1^i(\phi_i(Y_i, 0) = 0) \\ & + (1 - P_M(\phi^{(i-1)}, P_1^{(i-1)}))P_1^i(\phi_i(Y_i, 1) = 0), \end{aligned}$$

with  $P_F(\phi_1, P_0^1) = P_0^1(\phi_1(Y_1) = 1)$  and  $P_M(\phi_1, P_1^1) = P_1^1(\phi_1(Y_1) = 0)$ .

In the social learning problem, given the decision rules of the previous agents  $1, \dots, i-1$ , each agent  $i$  seeks to find  $\phi_i$  to minimize

$$\begin{aligned} P_e^i(\phi_i \mid \phi^{(i-1)}) = & \pi_0 \sup_{P_0^{(i)} \in \mathcal{P}_0^{(i)}} P_F(\phi^{(i)}, P_0^{(i)}) \\ & + \pi_1 \sup_{P_1^{(i)} \in \mathcal{P}_1^{(i)}} P_M(\phi^{(i)}, P_1^{(i)}). \end{aligned} \quad (2)$$

Let  $p_j$  be the density (with respect to some measure) of  $P_j$ , for  $j = 0, 1$ . The LFDs for two given uncertainty classes  $\mathcal{P}_0$  and  $\mathcal{P}_1$  is defined by [17] to be the pair of distributions  $(Q_0, Q_1)$  with densities  $(q_0, q_1)$  such that

$$q_0(y) = \begin{cases} (1 - \epsilon_0)p_0(y) & \text{for } p_1(y)/p_0(y) < c'' \\ (1/c'')(1 - \epsilon_0)p_1(y) & \text{for } p_1(y)/p_0(y) \geq c'' \end{cases}$$

$$q_1(y) = \begin{cases} (1 - \epsilon_1)p_1(y) & \text{for } p_1(y)/p_0(y) > c' \\ c'(1 - \epsilon_1)p_0(y) & \text{for } p_1(y)/p_0(y) \leq c' \end{cases}$$

where  $0 \leq c' < c'' \leq \infty$  are determined such that  $q_0$  and  $q_1$  are probability densities. Let  $b = (1 - \epsilon_1)/(1 - \epsilon_0)$ . Then we have

$$\frac{q_1(y)}{q_0(x)} = \begin{cases} bc' & \text{for } p_1(y)/p_0(y) \leq c' \\ bp_1(y)/p_0(y) & \text{for } c' < p_1(y)/p_0(y) < c'' \\ bc'' & \text{for } p_1(y)/p_0(y) \geq c'' \end{cases} \quad (3)$$

When  $N = 1$ , the minimax error  $\inf_{\phi} P_e^{(1)}(\phi)$  is achieved by letting  $\phi$  to be the likelihood ratio test using  $(Q_0, Q_1)$ . A similar result is proven in [18] for a parallel network configuration. In the following, we show the same result for a tandem network (in fact, the result is easily generalized to include all tree configurations). In the rest of this paper, for any random variable  $Y$  with distribution drawn from a given pair of uncertainty classes, we let  $l^*(Y)$  be the likelihood ratio  $q_1(Y)/q_0(Y)$ , where  $q_0$  and  $q_1$  are the respective densities of the LFDs of the aforementioned uncertainty classes. We first state a lemma given in [17].

**Lemma 1.** *Suppose that the LFDs for  $(\mathcal{P}_0, \mathcal{P}_1)$  are  $(Q_0, Q_1)$ . Then, for any  $Q_j' \in \mathcal{P}_j$ , where  $j = 0, 1$ , we have  $Q_0'(l^*(Y) > t) \leq Q_0(l^*(Y) > t) \leq Q_1(l^*(Y) > t) \leq Q_1'(l^*(Y) > t)$ .*

For all  $i \geq 1$ , let  $(Q_0^i, Q_1^i)$  be the LFDs for  $(\mathcal{P}_0^i, \mathcal{P}_1^i)$ , and  $Q_j^{(N)} = Q_j^1 \times Q_j^2 \times \dots \times Q_j^N$  for  $j = 0, 1$ .

**Theorem 1.** *Let  $\phi^{(N)}$  be any set of monotone likelihood ratio tests based on  $Q_0^{(N)}$  and  $Q_1^{(N)}$  for the tandem topology. Then for all  $(P_0^{(N)}, P_1^{(N)}) \in \mathcal{P}_0^{(N)} \times \mathcal{P}_1^{(N)}$ , we have*

$$P_F(\phi^{(N)}, Q_0^{(N)}) \geq P_F(\phi^{(N)}, P_0^{(N)})$$

and

$$P_M(\phi^{(N)}, Q_1^{(N)}) \geq P_M(\phi^{(N)}, P_1^{(N)}).$$

*Proof. (Outline)* We will only show the first inequality as the proof for the second is similar. We proceed by mathematical induction on  $N$ . From Lemma 1, the inequality holds for  $N = 1$ . We now assume that the theorem holds for  $N < i$ . The likelihood ratio test for agent  $i$  is of the form

$$U_i = \begin{cases} 1 & \text{if } l^*(U_{i-1}, Y_i) > t_i \\ 0 & \text{otherwise} \end{cases}$$

where  $T_i$  is some threshold. Since the decision of agent  $i - 1$  and the observation of agent  $i$  are independent, we have

$$l^*(U_{i-1}, Y_i) = l^*(Y_i)l^*(U_{i-1}).$$

As  $l^*(U_{i-1})$  and  $l^*(Y_i)$  are both stochastically larger under  $Q_0^{(i)}$  than under any other distribution  $P_0^{(i)}$ , from Lemma 2 of [18], their product is as well. Therefore, we have

$$\begin{aligned} Q_0^{(i)}(U_i = 1) &= Q_0^{(i)}(l^*(U_{i-1}, Y_i) > t_i) \\ &\geq P_0^{(i)}(l^*(U_{i-1}, Y_i) > t_i) \\ &= P_0(U_i = 1) \end{aligned}$$

and the theorem holds for every  $N$ .  $\square$

Let  $\phi_*^{(N)}$  be the set of decision rules based on the LFDs  $(Q_0^{(N)}, Q_1^{(N)})$ . From Theorem 1, we have

$$\begin{aligned} &\inf_{\phi^{(N)}} \left\{ \sup_{P_0^{(N)} \in \mathcal{P}_0^{(N)}} P_F(\phi^{(N)}, P_0^{(N)}) + \sup_{P_1^{(N)} \in \mathcal{P}_1^{(N)}} P_M(\phi^{(N)}, P_1^{(N)}) \right\} \\ &\leq \sup_{P_0^{(N)} \in \mathcal{P}_0^{(N)}} P_F(\phi_*^{(N)}, P_0^{(N)}) + \sup_{P_1^{(N)} \in \mathcal{P}_1^{(N)}} P_M(\phi_*^{(N)}, P_1^{(N)}) \\ &= P_F(\phi_*^{(N)}, Q_0^{(N)}) + P_M(\phi_*^{(N)}, Q_1^{(N)}) \\ &= \inf_{\phi^{(N)}} [P_F(\phi^{(N)}, Q_0^{(N)}) + P_M(\phi^{(N)}, Q_1^{(N)})] \\ &\leq \inf_{\phi^{(N)}} \left[ \sup_{P_0^{(N)} \in \mathcal{P}_0^{(N)}} P_F(\phi^{(N)}, P_0^{(N)}) \right. \\ &\quad \left. + \sup_{P_1^{(N)} \in \mathcal{P}_1^{(N)}} P_M(\phi^{(N)}, P_1^{(N)}) \right], \end{aligned}$$

where the last equality holds because  $\phi_*^{(N)}$  are the optimal decision rules given that agent observations have distributions  $(Q_0^{(N)}, Q_1^{(N)})$  [10]. Therefore, the inequality signs are all equalities, and the minimax error in the decentralized detection problem is equal to the minimum error when all the distributions of the observations are exactly equal to the LFDs. The same conclusion holds for the social learning problem.

### 3. ASYMPTOTIC PROPERTIES

In this section, we study the asymptotic minimax error probability of a tandem network for the decentralized detection and social learning problems. We consider the cases where agents have knowledge of their positions in the tandem or not separately.

#### 3.1. Known Agent Positions

Since the minimax error is equal to the minimum error when all the distributions of the observations are exactly equal to the LFDs, we just have to study the network assuming this is the case. Asymptotic learning is said to occur if the minimax error probability in (1) or (2) converges to zero as  $N$  or  $i$  increases, in the decentralized detection and social learning problem respectively. For simplicity, we let  $\epsilon_j^1 = \dots = \epsilon_j^N = \epsilon_j$  for  $j = 0, 1$ .

**Proposition 1.** *Suppose that  $\epsilon_j^1 = \dots = \epsilon_j^N = \epsilon_j$  for  $j = 0, 1$ . Then, asymptotic learning for decentralized detection occurs if and only if either  $\log p_1(y)/p_0(y)$  is not upper bounded and  $\epsilon_0 = 0$ , or  $\log p_1(y)/p_0(y)$  is not lower bounded and  $\epsilon_1 = 0$ .*

*Proof. (Outline)* We consider three different cases, depending on whether  $\epsilon_0$  or  $\epsilon_1$  is zero.

Case 1:  $\epsilon_0 = \epsilon_1 = 0$ . This reduces to the classical Bayesian hypothesis testing problem. From [16], the minimax error probability is bounded above zero if and only if the log-likelihood ratio of  $P_1$  and  $P_0$  is bounded.

Case 2: Either  $\epsilon_0 = 0$  or  $\epsilon_1 = 0$ , but not both. In this case, we know exactly one of the  $\mathcal{P}_i$ . Without loss of generality, let this be  $\mathcal{P}_0$ . Then  $(Q_0, Q_1) = (P_0, Q_1)$  are the LFDs of  $\mathcal{P}_0$  and  $\mathcal{P}_1$ . Since  $\epsilon_0 = 0$  and  $\epsilon_1 \neq 0$ , we have  $c'' = \infty$  and  $c' > 0$  in (3). Note that for  $p_1(y)/p_0(y) > c'$ , we have  $l^*(y) = bp_1(y)/p_0(y)$ . Hence, if  $p_1(y)/p_0(y)$  is bounded from above, then for  $p_1(y)/p_0(y) > c'$ , we have  $0 < bc' \leq l^*(y) = bp_1(y)/p_0(y) < \infty$ . This implies that the minimax error is bounded away from zero since  $\log(l^*(y))$  is bounded (the proof is similar to case 1). On the other hand, suppose that  $\log p_1(y)/p_0(y)$  is not bounded from above. Thus,  $\log q_1(y)/q_0(y)$  is bounded from below but not from above, and using the decision rules proposed in [16], we can make the maximum error arbitrarily small as the number of agents tends to infinity.

Case 3:  $\epsilon_0 > 0$  and  $\epsilon_1 > 0$ . Here,  $c' > 0$  and  $c'' < \infty$  and so the log-likelihood of  $Q_1$  and  $Q_0$  is bounded. Similar to case 2, the minimax error is bounded above zero as  $N \rightarrow \infty$  as the log-likelihood of  $Q_1$  and  $Q_0$  is bounded.  $\square$

Similarly, we have the following result for social learning.

**Proposition 2.** *Suppose that  $\epsilon_j^1 = \dots = \epsilon_j^N = \epsilon_j$  for  $j = 0, 1$ . Asymptotic learning in social learning occurs if and only if the log-likelihood ratio of  $P_0$  and  $P_1$  is unbounded and both of  $\epsilon_0$  and  $\epsilon_1$  are equal to zero.*

Proposition 2 assumes that the contamination for the uncertainty classes of the agent observations are fixed. In the following result, we allow the contamination to vary amongst the agents. The proof is omitted due to space constraints.

**Proposition 3.** *The minimax error probability converges to zero in social learning if and only if the log-likelihood ratio of  $P_1$  and  $P_0$  is unbounded, and there exists an infinite subsequence of agents with both  $\epsilon_0^i$  and  $\epsilon_1^i$  converging to zero.*

#### 3.2. Unknown Agent Positions

In a social network, users have to make their decisions not knowing how many times a decision has been propagated from the source node. We first start with the case where agent observations have the nominal distributions. If each agent has no knowledge of its position, we assume that the optimal decision rules are likelihood ratio tests of the following form:

$$U_i = \begin{cases} 0 & \text{if } p_1(Y_i)/p_0(Y_i) < t_1 \\ U_{i-1} & \text{if } t_1 \leq p_1(Y_i)/p_0(Y_i) < t_0 \\ 1 & \text{if } p_1(Y_i)/p_0(Y_i) \geq t_0 \end{cases} \quad (4)$$

where  $t_0$  and  $t_1$  are fixed constants. The proof of this statement is omitted due to space reasons.

**Proposition 4.** *Suppose that the contamination of the uncertainty classes for all agents are zero. Then, the false alarm error and*

missed detection probabilities are bounded above zero and converge linearly to

$$\frac{P_0(p_1(Y_1)/p_0(Y_1) \geq t_0)}{P_0(p_1(Y_1)/p_0(Y_1) \geq t_0) + 1 - P_0(p_1(Y_1)/p_0(Y_1) \geq t_1)} \quad (5)$$

and

$$\frac{P_1(p_1(Y_1)/p_0(Y_1) \geq t_0)}{P_1(p_1(Y_1)/p_0(Y_1) \geq t_0) + 1 - P_1(p_1(Y_1)/p_0(Y_1) \geq t_1)} \quad (6)$$

respectively, where  $t_0$  and  $t_1$  are the thresholds in (4).

*Proof.* Let false alarm probability  $P_0(U_{i-1} = 1)$  and missed detection probability  $P_1(U_{i-1} = 0)$  of agent  $i-1$  be denoted as  $P_F^{i-1}$  and  $P_M^{i-1}$  respectively. The total error probability of agent  $i-1$  is equal to  $\pi_0 P_F^{i-1} + \pi_1 P_M^{i-1}$ , and the false alarm error probability of agent  $i$  is

$$\begin{aligned} P_F^i &= P_0(u_{i-1} = 1)P_0(p_1(Y_i)/p_0(Y_i) \geq t_1) \\ &\quad + P_0(u_{i-1} = 0)P_0(p_1(Y_i)/p_0(Y_i) \geq t_0) \\ &= P_F^{i-1}P_0(p_1(Y_i)/p_0(Y_i) \geq t_1) \\ &\quad + (1 - P_F^{i-1})P_0(p_1(Y_i)/p_0(Y_i) \geq t_0) \\ &= [P_0(p_1(Y_i)/p_0(Y_i) \geq t_1) - P_0(p_1(Y_i)/p_0(Y_i) \geq t_0)]P_F^{i-1} \\ &\quad + P_0(p_1(Y_i)/p_0(Y_i) \geq t_0). \end{aligned}$$

This is a recurrence relation, and it converges linearly to (5). A similar derivation holds for the missed detection probability, and the proposition is proved.  $\square$

We now turn our attention to the robust social learning problem. We assume that  $\epsilon_0^i$  is the same for any  $i$ , and a similar condition holds for  $\epsilon_1^i$ . We also assume that each agent knows the value of  $\epsilon_0^i$  and  $\epsilon_1^i$ . Then, using the above decision rules with  $P_0$  and  $P_1$  replaced by the LFDs  $Q_0$  and  $Q_1$  respectively, we can apply Theorem 1 and conclude that the minimax error converges linearly to

$$\begin{aligned} &\frac{\pi_0 Q_0(l^*(Y_i) \geq t_0)}{Q_0(l^*(Y_i) \geq t_0) + 1 - Q_0(l^*(Y_i) \geq t_1)} \\ &+ \frac{\pi_1 Q_1(l^*(Y_i) \geq t_0)}{Q_1(l^*(Y_i) \geq t_0) + 1 - Q_1(l^*(Y_i) \geq t_1)}, \quad (7) \end{aligned}$$

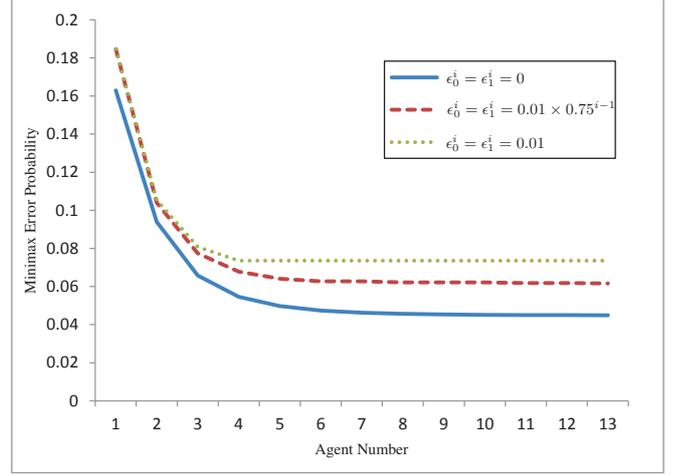
where the thresholds  $t_0$  and  $t_1$  are chosen to minimize (7).

We now consider the case where the contamination values can vary. We limit our analysis to the case where  $\epsilon_0^i = \epsilon_1^i = \epsilon^i$  for every agent  $i$ . In the following result, we allow the contamination values to converge to zero. The proof is omitted due to space constraints.

**Proposition 5.** *Suppose that for all  $i \geq 1$ ,  $\epsilon_0^i = \epsilon_1^i = \epsilon^i$ , and each agent uses the decision rule (4). If  $\epsilon^i \rightarrow 0$  as  $i \rightarrow \infty$ , the false alarm and missed detection probabilities in social learning converges to (5) and (6) respectively.*

#### 4. NUMERICAL RESULTS

In this section, we provide numerical results for the case where the agent observations' nominal distributions are zero mean Gaussian distributions with variance 25 and 1 under hypothesis  $H_0$  and  $H_1$  respectively. We consider three cases under the social learning framework: for every  $i$ , we have either (i)  $\epsilon_0^i = \epsilon_1^i = 0$ , (ii)  $\epsilon_0^i = \epsilon_1^i = 0.01 \times 0.75^{i-1}$ , or (iii)  $\epsilon_0^i = \epsilon_1^i = 0.01$ . Figure 1 shows the minimax error probability of these three cases. It can be seen that at all



**Fig. 1.** Social learning minimax error probability under different levels of contamination for the uncertainty classes.

points on the curve, the error probability of the case with no contamination in the uncertainty classes is no worse than that of the case with a decaying level of contamination, which in turn is no worse than the case with a constant level of contamination. This is quite an intuitive result, as less uncertainty implies that an agent is able to better optimize its local error probability, which will in turn aid the next agent in better optimizing its local error probability under the social learning rules.

Because the log-likelihood ratio of the conditional probability distributions are bounded from above but not below, each agent can receive arbitrarily strong observations in favor of  $H_0$ , but not  $H_1$ . Hence, the false alarm error probabilities for all the three cases converge to zero, and the asymptotic minimax error probability is equal to half of the asymptotic missed detection probability. It should be noted that if the probability distributions are such that the log-likelihood ratios are unbounded from above as well, the asymptotic minimax error probability of the case with no contamination and the case with decaying contamination is expected to converge to zero, while that of the case with a constant amount of contamination is expected to be still bounded above zero.

#### 5. CONCLUSION

We have shown that with some uncertainty in the observation distributions of a tandem network, the minimax error probability is obtained by assuming that each observation is distributed according to the LFDs of the uncertainty class. For a tandem topology with constant contamination in the uncertainty classes under both hypothesis, the asymptotic minimax error probability is bounded above zero. However, the social learning minimax error probability converges to zero if the amount of contamination decays to zero, and the log-likelihood ratio of the nominal distributions are unbounded. The above conclusions hold only if agents know their positions in the tandem. In many social learning scenarios, agents have no knowledge of their positions. We have provided error bounds for this case, and shown that if the uncertainty classes' contamination converges to zero, the minimax error probability also converges to this bound.

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