# **REGULARIZING INVERSE PROBLEMS IN IMAGE PROCESSING** WITH A MANIFOLD-BASED MODEL OF OVERLAPPING PATCHES

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#### ABSTRACT

Local patch-based models have been shown to be effective in numerous image processing applications and have become the core of the state-of-the-art denoising, inpainting and structural editing algorithms. Most such modeling approaches mainly rely on searching for similar patches in the set of available patches. However, the apparent similarity between sufficiently small (e.g.,  $5 \times 5$  pixels) image regions motivates modeling them with a low-dimensional manifold instead and suggests the existence of a simple parametrization for it. Although there exist manifold models for a single patch, it has remained an open problem how to efficiently represent an entire image in terms of its overlapping patches drawn from the underlying non-linear manifold. We propose to consider an image to lie on the intersection of separate manifolds corresponding to different overlapping patches, which we approximate with affine subspaces in a kernel-induced feature space. In contrast to our previous work on this topic, here we solve the intersection and preimage problems simultaneously, ensuring the existence of a suitable solution in the input space. This significantly improves the performance and robustness of our method. Our method incorporates any desired equality constraints on the image, and thus can be used to regularize any linear inverse problem with the manifold intersection model. Our experimental results show nearly perfect compressive sensing reconstruction of images whose patches are well described by a manifold model, as well as exceptional performance in denoising and inpainting

*Index Terms*— Patch-based image processing, inverse problems, kernel methods, manifold models.

#### 1. INTRODUCTION

A linear inverse problem is typically formulated as a problem of reconstruction of a signal  $y^*$  from its observations **b**:

$$\mathbf{b} = \mathbf{W}\mathbf{y}^* + \mathbf{n}.$$
 (1)

Typically, the low rank of the matrix  $\mathbf{W}$  makes it underdetermined with an infinite number of solutions (as in compressive sensing or inpainting). In denoising, even though  $\mathbf{W} = \mathbf{I}$ , the identity matrix, additive noise n still makes the problem (1) ill-posed.

In order to restrict the set of possible solutions additional assumptions on the sought signal  $y^*$  that will promote its desired qualitative characteristics should be made along with establishing a quantitative criterion for choosing the optimal one. For example, one may minimize total variation (TV) to sharpen edge transitions of the image. The appropriateness of the chosen model for representing a particular class of signals is a major factor that determines the overall performance of a signal processing algorithm.

While naturally suitable for texture synthesis [1, 2], inpainting [3, 4, 5], and structural image editing [6, 7], modeling images with

small overlapping patches was found recently to be surprisingly effective in regularizing other inverse problems. For example, the idea of adaptively averaging similar image patches for denoising [8, 9] was further generalized in the state-of-the-art algorithm, BM3D [10]. This model was also adapted for superior performance in deblurring, compressive sensing reconstruction, superresolution and other problems in image processing [11, 12].

While these algorithms proceed by searching for and building a filter based on similar patch exemplars, the implicit assumption being made is one of smoothness of the set of patches. Indeed, it has been shown that patches, even though sampled from a relatively high-dimensional space ( $\mathbb{R}^{25}$  for  $5 \times 5$  pixel patches), can be parametrized by far fewer continuous coordinates [13]. In other words, they belong to an underlying manifold that imposes mutual constraints on the pixels of each patch. This manifold model was studied in [14], and the manifold of high-contrast patches was further shown in [15] to have the topology of a Klein bottle. Thus, when algorithms like BM3D find and adapt similar exemplars, we would argue that they implicitly construct an estimate of the nearest point on the image patch manifold. Hence, we might hope to do better by making this implicit goal explicit in our algorithm.

However, one of the obstacles to using manifold models for patches is the problem of describing the whole image in terms of the manifold-modeled patches. For example, complex nonparametric Bayesian models [16] and Gaussian mixture models [17, 18] have been applied to describe the manifold corresponding to a single patch, but neither extends readily to the case of overlapping patches. Peyré in [13] regularizes inverse problems by requiring the overlapping patches to trace a two-dimensional trajectory along the manifold. However, the main drawback of this method is the computational expense of optimizing over all such trajectories on the denselysampled non-linear patch manifold.

The idea of representing images as lying on the intersection of several manifolds, each corresponding to different overlapping patches, was proposed in our previous work [19] along with an efficient kernel-based method of finding such intersections. Although the solution can be expressed in closed form in the kernel-induced feature space, solving the preimage problem to bring it to the input space can significantly degrade the quality of the final result and imposes additional computational burden. Moreover, the requirement that training sets of samples have the dimension of the whole image rather then individual patches makes the method impractical for processing large images.

In this paper, we will show a way to overcome these drawbacks and present a robust and memory-efficient method for regularization of any linear inverse problem in image processing with the overlapping patch manifold model. While learning the manifold geometry with kernel PCA we combine both stages of the algorithm (namely, finding the intersection and its preimage) in a single iterative procedure, which ensures the existence of a suitable solution in the original space. Furthermore, we learn the patch manifold only once and

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distribute its description to all patch positions to enable processing of images of significantly larger size. Finally, the iterative nature of the algorithm allows us to incorporate additional constraints and apply the same framework not only for denoising but also for regularization of compressive sensing reconstruction, inpainting, and other inverse problems. The obtained results exhibit exceptional robustness to noise, often show nearly perfect reconstruction, and are comparable with state-of-the-art methods.

The organization of this paper is as follows. In Section 2, we will introduce the proposed model of images with an explanatory example and then will formalize it in Section 3. Our kernel-based approach for finding the intersection of patch-manifolds is explained in Section 4. Section 5 shows the results of applying our algorithm for regularization of several problems in image processing.

#### 2. AN INTERSECTING MANIFOLDS MODEL OF IMAGES

Consider the entire space of D-pixel images. For any  $p \times q$ -area of the image pixel grid, there is a corresponding d = pq-dimensional subspace of  $\mathbb{R}^D$ . The manifold model for image patches allows us to assume that such a  $p \times q$  patch lies on or close to a  $\delta$ -dimensional non-linear manifold  $\mathcal{M}$  (with  $\delta < d$ ) within this subspace. At the same time, the other D - d pixels of the image are unconstrained by this patch, so the whole image is allowed to lie on a  $(D - d) + \delta$ dimensional manifold  $\mathcal{M}_m \subset \mathbb{R}^D$ . Because there is one such manifold constraint corresponding to each of M overlapping patches, the image itself lies at or close to the intersection of all these manifolds  $\mathcal{M}_m, m = 1, \ldots, M$ .

To illustrate this, consider the toy example of an image with only three pixels. It can be regarded as a combination of two overlapping 2-pixel patches (Fig. 1). Suppose that each of them is restricted to lie on some 1-dimensional manifold, e.g., a unit circle in  $\mathbb{R}^2$ . This is equivalent to constraining the whole image to lie on the side of a cylinder in  $\mathbb{R}^3$ . Therefore, any images that conform to this model lie on the intersection of both cylinders and solve the system of non-

linear equations  $\begin{cases} x^2 + y^2 = 1\\ y^2 + z^2 = 1 \end{cases}$ 



**Fig. 1.** Left: Covering of a three-pixel image with two overlapping patches. Middle: Two cylinders in  $\mathbb{R}^3$  created by constraining each of the image patches to lie on the unit circle. Right: The result of using our algorithm to map randomly-generated points to the nearest points on the manifolds' intersection (see Section 5.1).

# 3. INTERSECTION OF MANIFOLDS AS AN OPTIMIZATION PROBLEM

In this section, we will examine how to translate the manifold intersection criterion into a regularization term for inverse problems. First, to formalize the model, consider M (possibly overlapping) patches of an image  $\mathbf{y}$ . Let  $\mathbf{E}_m$ ,  $m = 1, \ldots, M$ , be  $d \times D$  patch extraction matrices with entries  $(\mathbf{E}_m)_{i,j} = 1$  if the  $j^{\text{th}}$  pixel of an image corresponds to the  $i^{\text{th}}$  pixel of the  $m^{\text{th}}$  patch and 0 otherwise. Now the  $m^{\text{th}}$  patch of the image  $\mathbf{y}$  can be written (in vector form) as  $\mathbf{E}_m \mathbf{y}$ . Although the final solution admits any combination of patches, in our examples in Section 5, we will cover an image with L < pq randomly-offset grids, each segmenting the image into a layer of non-overlapping  $p \times q$  patches (disregarding all partial patches). We then combine the patches from all L layers to obtain the M overlapping patches.

The assumption that the desired image  $\mathbf{y} \in \mathbb{R}^D$  lies on or close to the intersection of several manifolds suggests a regularization for inverse problems that minimizes the Euclidean distances to each manifold  $d(\mathbf{y}, \mathcal{M}_m) = \inf_{\mathbf{x} \in \mathcal{M}_m} d(\mathbf{y}, \mathbf{x})$ . In contrast to our previous work [19], here we recognize that since each manifold  $\mathcal{M}_m$ is parallel to D - d axes, these coordinates do not affect the distance. Therefore,  $d(\mathbf{y}, \mathcal{M}_m) = d(\mathbf{E}_m \mathbf{y}, \mathcal{M})$ , where  $\mathcal{M} \subset \mathbb{R}^d$  is the patch manifold. For example, we could measure the distances from  $\mathbf{E}_1 \mathbf{y}$  and  $\mathbf{E}_2 \mathbf{y}$  to the unit circle rather then the distances from  $\mathbf{y}$  to each cylinder above.

Our proposed patch-based regularization term thus becomes:

$$\min_{\mathbf{y}} \sum_{m=1}^{M} w_m d^2 \left( \mathbf{E}_m \mathbf{y}, \mathcal{M} \right), \tag{2}$$

where the weights  $w_m \ge 0$  can be chosen to control the distances from a solution to each manifold if their intersection set is empty. We will set  $w_m = 1$  for all m in our experiments in Section 5.

This regularization term will encourage all overlapping patches to conform to the manifold model simultaneously, finding an intersection if it exists, as we desired. We examine how to efficiently minimize it in the next section.

# 4. FINDING THE INTERSECTION OF THE MANIFOLDS 4.1. Brief review of kernel methods

Our approach to minimizing Eq. 2 uses the kernel trick [20] as an efficient and elegant way to learn the non-linear structure of the manifold  $\mathcal{M}$ . The main idea of kernel methods is to map data points by some non-linear transformation  $\Phi : \mathbb{R}^d \to \mathcal{H}$  to a  $D_{\mathcal{H}}$ -dimensional feature space  $\mathcal{H}$  (with  $D_{\mathcal{H}} > d$ ), in which they can be analyzed with linear algorithms. Efficiency is gained by the fact that the images  $\Phi(\mathbf{x})$  do not need to be computed explicitly. Instead, the kernel function  $\kappa(\mathbf{x}_i, \mathbf{x}_j) = \langle \Phi(\mathbf{x}_i), \Phi(\mathbf{x}_j) \rangle_{\mathcal{H}}$  is defined to represent the inner products in the feature space. Eventually, this yields a nonlinear solution when mapped back to the original space. Please see [20] for more details of kernel methods.

In particular, Kernel Principal Component Analysis (KPCA) [21], a kernel trick extension of the PCA algorithm, is one of the most powerful known methods for learning a manifold from its samples. Indeed, other popular manifold learning algorithms, such as Laplacian Eigenmaps [22], Locally Linear Embedding [23], and ISOMAP [24], were shown in [25, 26] to be special cases of it. Its effectiveness has been proven in many signal processing settings [27, 28, 29].

#### 4.2. Optimization problem in feature space

The key idea of KPCA is that, for an appropriate kernel, in the induced feature space, the manifold becomes an affine  $\delta_{\mathcal{H}}$ -dimensional subspace  $\mathcal{U}$  that can be parametrized by PCA. Then the initial criterion (2) becomes equivalent to unconstrained minimization of:

$$J(\mathbf{y}) = \frac{1}{2} \sum_{m=1}^{M} w_m d^2 \left( \Phi(\mathbf{E}_m \mathbf{y}), \mathcal{U} \right), \qquad (3)$$

where  $d^2 (\Phi (\mathbf{E}_m \mathbf{y}), \mathcal{U}) = \|\Phi (\mathbf{E}_m \mathbf{y}) - P_{\mathcal{U}} (\mathbf{E}_m \mathbf{y})\|_{\mathcal{H}}^2$  is the squared distance from the image  $\Phi (\mathbf{E}_m \mathbf{y})$  to its projection  $P_{\mathcal{U}} (\mathbf{E}_m \mathbf{y})$  onto the subspace  $\mathcal{U}$  in feature space. We note that this functional is similar to the preimage regularization term in the Robust KPCA algorithm of Nguyen and De la Torre [30], to which it reduces for M = 1. Effectively, we will solve both intersection and preimage problems simultaneously, which will ensure the existence of a suitable solution in the input space.

Let  $\Phi(\mathbf{X})$  be a  $D_{\mathcal{H}} \times n$  matrix, whose columns  $\Phi(\mathbf{x}_i) \in \mathcal{H}$ are the images of the training samples  $\mathbf{x}_i \in \mathbb{R}^d$ , i = 1, ..., n, we will use to learn the manifold  $\mathcal{M}$ . The corresponding subspace  $\mathcal{U}$  in the feature space is then estimated via its principal components  $\mathbf{U} = \Phi(\mathbf{X}) \boldsymbol{\alpha}$  and the sample mean  $\mathbf{m} = \frac{1}{n} \Phi(\mathbf{X}) \mathbf{1}$ . Here 1 denotes the vector of ones and  $\boldsymbol{\alpha}$  is an  $n \times \delta_{\mathcal{H}}$  matrix of scaled eigenvectors of the centered Gram matrix  $\mathbf{K} = (\mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}^{\mathrm{T}}) \mathbf{K} (\mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}^{\mathrm{T}})$ , where  $\mathbf{K}_{i,j} = \kappa(\mathbf{x}_i, \mathbf{x}_j)$ . See [20, 21, 25] for mode details of KPCA.

Now, for a patch  $\mathbf{E}_m \mathbf{y}$ , the projection of  $\Phi(\mathbf{E}_m \mathbf{y})$  onto  $\mathcal{U}$  is:

$$P_{\mathcal{U}}(\mathbf{E}_{m}\mathbf{y}) = \mathbf{U}\mathbf{U}^{\mathrm{T}}\Phi(\mathbf{E}_{m}\mathbf{y}) + (\mathbf{I} - \mathbf{U}\mathbf{U}^{\mathrm{T}})\mathbf{m} \qquad (4)$$
$$= \Phi(\mathbf{X})\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{k}_{m} + \Phi(\mathbf{X})\boldsymbol{\mu},$$

where  $\mathbf{k}_m$  is a vector with entries  $[\mathbf{k}_m]_i = \kappa (\mathbf{x}_i, \mathbf{E}_m \mathbf{y})$  for  $i = 1, \ldots, n$ , and  $\boldsymbol{\mu} = \frac{1}{n} (\mathbf{I} - \boldsymbol{\alpha} \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{K}) \mathbf{1}$ . The distance from  $\Phi (\mathbf{E}_m \mathbf{y})$  to the subspace  $\mathcal{U}$  in Eq. 3 is then:

$$d^{2} \left( \Phi \left( \mathbf{E}_{m} \mathbf{y} \right), \mathcal{U} \right) = \left\| \Phi \left( \mathbf{E}_{m} \mathbf{y} \right) - \mathcal{P}_{\mathcal{U}} \left( \mathbf{E}_{m} \mathbf{y} \right) \right\|^{2}$$
$$= \kappa \left( \mathbf{E}_{m} \mathbf{y}, \mathbf{E}_{m} \mathbf{y} \right) - \mathbf{k}_{m}^{\mathrm{T}} \boldsymbol{\alpha} \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{k}_{m} - 2 \mathbf{k}_{m}^{\mathrm{T}} \boldsymbol{\mu} + \boldsymbol{\mu}^{\mathrm{T}} \mathbf{K} \boldsymbol{\mu}.$$
(5)

# 4.3. Iterative optimization of the regularization term

Provided that the kernel function is differentiable,  $J(\mathbf{y})$  can be minimized, for example, with the gradient descent algorithm. However, an iterative fixed-point method can also be used for rbf kernels [27, 30]. In this paper, we restrict our attention to the Gaussian kernel  $\kappa(\mathbf{x}, \mathbf{y}) = \exp\left(-\frac{1}{\sigma} \|\mathbf{x} - \mathbf{y}\|^2\right)$  and derive both types of solution for it.

Using Eq. 5 and noting that  $\nabla_{\mathbf{y}} f(\mathbf{E}_m \mathbf{y}) = \mathbf{E}_m^{\mathrm{T}} \nabla_{\mathbf{E}_m \mathbf{y}} f(\mathbf{E}_m \mathbf{y})$ the gradient  $\nabla_{\mathbf{y}} J(\mathbf{y})$  becomes:

$$\nabla_{\mathbf{y}} J(\mathbf{y}) = \frac{1}{2} \sum_{m=1}^{M} w_m \mathbf{E}_m^{\mathrm{T}} \nabla_{\mathbf{E}_m \mathbf{y}} d^2 \left( \Phi(\mathbf{E}_m \mathbf{y}), \mathcal{M} \right)$$
$$= -\sum_{m=1}^{M} w_m \mathbf{E}_m^{\mathrm{T}} \mathbf{k}_m' \boldsymbol{\nu}_m, \tag{6}$$

where  $\boldsymbol{\nu}_{m} = \boldsymbol{\alpha} \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{k}_{m} + \boldsymbol{\mu}$  and  $\mathbf{k}_{m}^{'}$  denotes a  $D \times n$  matrix with columns  $\kappa^{'}(\mathbf{x}_{i}, \mathbf{E}_{m}\mathbf{y}) = \frac{2}{\pi} [\mathbf{k}_{m}]_{i} (\mathbf{x}_{i} - \mathbf{E}_{m}\mathbf{y}), i = 1, \dots, n.$ 

Now, defining h to be some (possibly variable) step size, the result of the  $k^{\text{th}}$  iteration is found as:

$$\mathbf{y}^{(k)} = \mathbf{y}^{(k-1)} - h \cdot \nabla_{\mathbf{y}} J\left(\mathbf{y}^{(k-1)}\right) \,. \tag{7}$$

Alternatively, setting Eq. 6 to **0** and solving it for **y** results in the following recursive relation:

$$\mathbf{y}^{(k)} = \frac{\sum_{m=1}^{M} w_m \left[ \mathbf{E}_m^{\mathrm{T}} \mathbf{X} \mathbf{D}_m \boldsymbol{\nu}_m^{(k-1)} + \mathbf{v}_m^{(k-1)} \right]}{\sum_{m=1}^{M} w_m \mathbf{k}_m^{\mathrm{T}} \boldsymbol{\nu}_m^{(k-1)}}, \qquad (8)$$

where  $\mathbf{v}_m^{(k-1)} = \mathbf{k}_m^{\mathrm{T}} \boldsymbol{\nu}_m^{(k-1)} \left( \mathbf{I} - \mathbf{E}_m^{\mathrm{T}} \mathbf{E}_m \right) \mathbf{y}^{(k-1)}$  and  $\mathbf{D}_m$  is an  $n \times n$  diagonal matrix with entries  $[\mathbf{D}_m]_i = [\mathbf{k}_m]_i$ . Since the terms of Eq. 6 are computed for each manifold separately, this gives the flexibility of using patches of different sizes and shapes and of choosing different kernels to achieve the best approximation of every manifold, if desired.

#### 4.4. Regularizing inverse problems with our criterion

Finally, we look at how to use  $J(\mathbf{y})$  (Eq. 3) as a regularization term for inverse problems, i.e., to solve  $\min_{\mathbf{y}} J(\mathbf{y}) \ s.t. \mathbf{W}\mathbf{y} = \mathbf{b}$ . For this, we propose to use a modified Landweber iteration [31, 32]:

$$\mathbf{y}^{(k)} = \lambda \left[ \left( \mathbf{I} - \mathbf{W}^{\dagger} \mathbf{W} \right) \widetilde{\mathbf{y}}^{(k-1)} + \mathbf{W}^{\dagger} \mathbf{b} \right] + (1 - \lambda) \widetilde{\mathbf{y}}^{(k-1)}, \quad (9)$$

where  $\tilde{\mathbf{y}}^{(k-1)}$  denotes the result of computing either Eq. 7 or 8 based on the value  $\mathbf{y}^{(k-1)}$ , and  $\mathbf{W}^{\dagger}$  is the Moore-Penrose pseudoinverse of  $\mathbf{W}$ . The regularization parameter  $\lambda$ ,  $0 \le \lambda \le 1$  is set to 1 in the noiseless case. When noise in  $\mathbf{b}$  would prevent the solution from lying on the intersection of all manifolds,  $\lambda < 1$  can be used to relax adherence to the constraint  $\mathbf{Wy} = \mathbf{b}$ .

Although the uniqueness of the solution is not guaranteed, starting the iterations with the least squares solution to (1), i.e.,  $\mathbf{y}^{(0)} = \mathbf{W}^{\dagger}\mathbf{b}$ , and using  $\lambda = 1$ , results in nearly perfect compressive sensing reconstruction of the examples in Section 5.3. For inpainting, in Section 5.4, we initialize the missing pixels by linearly interpolating the boundaries of the gap in both vertical and horizontal directions with successive averaging, and we set  $\lambda = 1$  to keep the values of known pixels unchanged on each iteration. Finally, for denoising,  $\mathbf{y}_{noisy} = \mathbf{I}\mathbf{y}^* + \mathbf{n}$ , in Section 5.2, Eq. 9 simply reduces to  $\mathbf{y}^{(k)} = \lambda \mathbf{y}_{noisy} + (1 - \lambda) \mathbf{\tilde{y}}^{(k-1)}$ . We initialize the iterations with  $\mathbf{y}^{(0)} = \mathbf{y}_{noisy}$  and set  $\lambda = 0$ , allowing the algorithm to converge to the nearest local minimum of  $J(\mathbf{y})$  to  $\mathbf{y}_{noisy}$ , i.e., the nearest intersection point of the manifolds.

#### 5. EXPERIMENTS AND DISCUSSION

First, we consider an illustrative toy example of finding the intersection of two cylinders in  $\mathbb{R}^3$ , described in Section 2. Then we apply our algorithm for denoising, compressive sensing, and inpainting of natural images whose patches well conform to a manifold model (Fig. 2). For all examples in this section, we use the gradient descent version of the algorithm (Eq. 7) with the fixed step size h = 1.



Fig. 2. Natural images with enlarged regions used as examples.

#### 5.1. Intersection of manifolds in 3D

As an initial proof of concept that our algorithm accurately finds the true manifolds' intersection, we use it to map a cloud of randomly generated points centered at the origin to their closest points on the intersection of two cylinders. The geometry of the cylinders is learned from samples of the unit circle in  $\mathbb{R}^2$  as the training set of patches. The results accurately trace the sought intersection, as shown on the right panel of Fig. 1. Notice that learning the resulting non-differentiable curve directly in  $\mathbb{R}^3$  would be a significantly more difficult problem requiring a larger set of higher-dimensional training samples.

#### 5.2. Image denoising

In this section, we look at using our overlapping-patch regularization in an image denoising problem. In this and future examples, we will learn the manifolds from training sets of 1500 5 × 5 patches, extracted from other similar images. For the zebra image, we use the Gaussian kernel parameter  $\sigma = 55$  and choose  $\delta_{\mathcal{H}} = 75$ . Similarly, for the roof tiles image,  $\sigma = 16$  and  $\delta_{\mathcal{H}} = 105$ . All training and testing images are scaled to the range [-1, 1].

The results of denoising the images corrupted with zero-mean Gaussian noise are presented in Fig. 3. To quantify the performance of our algorithm, we use peak signal-to-noise ratio (PSNR), defined as  $PSNR = 10 \log \frac{\max_{i,j} \mathbf{I}_{i,j}^2}{\frac{1}{N} \sum_{i,j} (\mathbf{I}_{i,j} - \mathbf{J}_{i,j})^2}$ , where **I** and **J** are the original and reconstructed *N*-pixel images, respectively. Modeling the manifold structure from the training set of patches allowed us to



**Fig. 3**. Denoising of natural images. Our algorithm accurately reconstructs high contrast edges, as well as details of texture, and performs similarly to state-of-the-art BM3D in terms of PSNR with enhanced visual quality. Numbers represent corresponding PSNR.

achieve results comparable or slightly better in terms of PSNR than the state-of-the-art BM3D algorithm [10] with better visual quality.

Moreover, the best potential quality of reconstruction, all other conditions being equal, can be achieved by considering much fewer overlapping patches than maximally available (see Fig. 4). In both examples, images are covered by L = 8 overlapping layers of patches instead of the maximum 25, which significantly decreases computation time. On the other hand, using only L = 1 layer in the case of non-overlapping patches produces much worse results, proving the overlapping patch model advantageous.



Fig. 4. Reconstruction PSNR of the zebra image as a function of the number of layers of patches. A significant improvement is gained by using overlapping patches (L > 1), but no major improvement could be achieved by considering more than eight layers of patches.

### 5.3. Compressive sensing reconstruction

In this section, we look at the application of our regularization to a compressive sensing reconstruction problem. We apply the method of iterative projections onto the constraint subspace as described in Section 4.4 with parameter  $\lambda = 1$  to reconstruct  $64 \times 64 = 4096$  pixel images from 400 Bernoulli random measurements (a measurement ratio of < 10%). Since the underlying manifold model can be chosen to provide an accurate description of the considered class of images, our algorithm results in nearly perfect reconstruction (Fig. 5). It greatly outperforms basis pursuit run on 64 8 × 8 non-overlapping patches, assuming sparsity in a dictionary learned with the K-SVD algorithm, clearly demonstrating the advantage of using overlapping vs. non-overlapping patches. Our results have better visual quality and higher PSNR than those obtained with recursive spatially adaptive filtering based on the BM3D algorithm [12] from the same number of low-frequency Fourier measurements.

#### 5.4. Image inpainting

Finally, in this section we assess the performance of our regularization term on an image inpainting task. We see that the results (Fig. 6) are similar to or better then those of the exemplar-based method of Criminisi et. al. [3].



**Fig. 5.** Compressive sensing reconstruction. Results of our algorithm are consistent with the learned model and nearly perfectly match the original images. Spatially adaptive filtering is based on the state-of-the-art BM3D algorithm [12]. Numbers represent PSNR.



Fig. 6. Results of image inpainting. Our algorithm performs similarly to or better than the patch-based method of Criminisi et. al. [3].

#### 6. CONCLUSION

In this paper, we proposed a unified method for regularization of any linear inverse problem in image processing based on an intersecting manifolds model of overlapping patches. We applied the kernel trick to efficiently approximate the patch-manifold in the induced feature space and combined the iterative preimage method with an intersection finding algorithm, which ensured the existence of a solution in the input space and increased overall robustness. We then proposed a Landweber iteration to allow this regularization term to be used with a variety of inverse problems in image processing. Provided that the set of image patches is well-approximated by a manifold that can be linearized in the kernel-induced feature space, our algorithm often achieves almost perfect reconstruction. The experimental results show that our approach vastly outperforms methods based on a non-overlapping patch model. Moreover, given an appropriate patch manifold model, it can even slightly outperform in PSNR (with enhanced visual quality) other algorithms that take a non-manifoldbased approach to modeling overlapping image patches. This shows that a manifold model of overlapping patches is an excellent choice for regularizing inverse problems in image processing. Finding the optimal kernels for particular classes of images as well as establishing the theoretical convergence properties of our method are directions for future work.

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