FRAMES FROM GENERALIZED GROUP FOURIER TRANSFORMS AND $SL_2(\mathbb{F}_Q)$

Matthew Thill   Vidya Muthukumar   Babak Hassibi

Department of Electrical Engineering
California Institute of Technology
Pasadena, CA 91125

ABSTRACT

We explore the problem of deterministically constructing frames and matrices with low coherence, which arises in areas such as compressive sensing, spherical codes, and MIMO communications. In particular, we present a generalization of the familiar harmonic frame by selecting a subset of rows of the generalized discrete Fourier transform matrix over finite groups. We apply our methods to the group $SL_2(\mathbb{F}_q)$ and show how to produce frames with remarkably low coherence, for which we provide upper bounds.

Index Terms— Coherence, frame, unit norm tight frame, group representation, special linear group, spherical codes, compressive sensing.

1. INTRODUCTION AND PREVIOUS WORK

Let $\mathbf{M} \in \mathbb{C}^{m\times n}$ be a complex matrix with columns $\{f_i\}_{i=1}^n$ which form a frame. The frame is called *tight* if $\mathbf{M}\mathbf{M}^*$ is a scalar multiple of the identity $\mathbf{I_m}$, and *equiangular* if $|\langle f_i, f_j\rangle| = \alpha$ for some constant α and all $i \neq j$. We will typically take our frames to be *unit norm*: $||f_k||_2 = 1, \forall k$. The *coherence* μ of $\mathbf{M}$ to be the maximum correlation between any two distinct columns:

$$\mu = \max_{i\neq j} \frac{|\langle f_i, f_j\rangle|}{||f_i||_2 \cdot ||f_j||_2}. \tag{1}$$

If a frame is both tight and equiangular, then it achieves the following lower bound on coherence, known as the *Welch bound* [13]:

Theorem 1 *Let $\mathbb{E}$ be a field, and $\{f_k\}_{k=1}^n$ be a frame for $\mathbb{E}^m$. Then*

$$\max_{i\neq j} \frac{|\langle f_i, f_j\rangle|}{||f_i||_2 \cdot ||f_j||_2} \geq \sqrt{\frac{n-m}{m(n-1)}}, \tag{2}$$

with equality if and only if $\{f_k\}_{k=1}^n$ is both tight and equiangular.

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Designing matrices and frames with low coherence is a problem that has applications arising in a wide range of fields, including compressive sensing [3–8], spherical codes [10,13], MIMO communications [11,12], quantum measurements [14, 15], etc.

The study of frames is also interesting in its own right and has received substantial attention in both engineering and applied math communities (see [17–19]). Much prior work has been done in studying structured frames, including some which are tight and/or equiangular [13, 20, 21] and several of these have employed group theoretic methods [1, 16, 22].

In general, tight equiangular frames do not exist for all values of m and n, but it can be shown that if there is a small number of inner product magnitudes between the elements of a tight frame, then it will tend to have low coherence. Thus, it is of interest to construct tight frames with few mutual inner products between the elements.

In our previous work [23, 24], we developed methods of constructing a harmonic frame by choosing a subset of m rows of the $n \times n$ discrete Fourier transform matrix, so that our frame takes the form

$$\mathbf{M} = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & \omega^{k_1} & \omega^{k_1\cdot 2} & \dots & \omega^{k_1\cdot(n-1)} \\ 1 & \omega^{k_2} & \omega^{k_2\cdot 2} & \dots & \omega^{k_2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{k_m} & \omega^{k_m\cdot 2} & \dots & \omega^{k_m\cdot(n-1)} \end{bmatrix} \tag{3}$$

where $\omega = e^{\frac{2\pi i}{n}}$, $k_i \in \{0, 1, ..., n-1\}$, and we have normalized the columns. If we index the columns by $\ell \in \{0, 1, ..., n-1\}$, then the inner product between the ℓ_1^{th} and ℓ_2^{th} columns takes the form

$$\frac{1}{m}\sum_{i=1}^{m} (\omega^{k_i\ell_1})^* \omega^{k_i\ell_2} = \frac{1}{m}\sum_{i=1}^{m} \omega^{(\ell_2-\ell_1)\cdot k_i}.$$

Thus, there is one inner product value for every choice of $\ell \equiv \ell_2 - \ell_1 \mod n$ (here, $\ell \equiv 0$ corresponds to taking the inner product of a column with itself). Thus we can write our inner products of distinct columns as

$$\frac{1}{m}\sum_{i=1}^{m} \omega^{\ell\cdot k_i},\ \ell = 1, 2, ..., n-1.$$

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We proposed the following scheme: Take n to be a prime so that the nonzero integers $\{1, ..., n-1\}$ form a cyclic group under modulo n multiplication, denoted $(\mathbb{Z}/n\mathbb{Z})^{\times}$. We then take m to be a divisor of $n-1$. Since it is cyclic, $(\mathbb{Z}/n\mathbb{Z})^{\times}$ contains a unique cyclic subgroup of size m, and we choose the integers $\{k_i\}$ to be the elements of this subgroup. We proved the following bounds on coherence:

Theorem 2 *If n is prime, m a divisor of $n-1$, and $\{k_i\}$ the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ of size m, then setting $\omega = e^{\frac{2\pi i}{n}}$, $r := \frac{n-1}{m}$ and $\mu := \max_{\ell \in \{1,...,n-1\}} \frac{1}{m} \left|\sum_{i=1}^{m} \omega^{\ell k_i}\right|$,*

$$\mu \leq \frac{1}{r}\left((r-1)\sqrt{\frac{1}{m}\left(r - \frac{1}{m}\right)} + \frac{1}{m}\right). \tag{4}$$

If m is odd,

$$\mu \leq \frac{1}{r}\sqrt{\left(\frac{1}{m} + \left(\frac{r}{2} - 1\right)\beta\right)^2 + \left(\frac{r}{2}\right)^2 \beta^2}, \tag{5}$$

where $\beta = \sqrt{\frac{1}{m}\left(r + \frac{1}{m}\right)}$.

In practice, these bounds come very close to the Welch bound.

2. GENERALIZED GROUP FOURIER TRANSFORM

We can now hope to generalize these results in the following manner: If we consider the (scaled) i^{th} row of the matrix $\mathbf{M}$ in (3) above,

$$\begin{bmatrix} 1 & \omega^{k_i} & \omega^{k_i \cdot 2} & \dots & \omega^{k_i \cdot (n-1)} \end{bmatrix},$$

this corresponds to an *irreducible unitary representation* of the cyclic group of n elements (denoted $\mathbb{Z}/n\mathbb{Z}$, the integers $\{0, 1, ..., n-1\}$ under addition modulo n). In this case, the representation (call it ρ) simply maps

$$\ell \mapsto \rho(\ell) := \omega^{\ell k_i}$$

for every $\ell \in \{0, 1, ..., n-1\}$. One easily checks that this is a well-defined representation:

$$\rho(\ell_1 + \ell_2) = \omega^{(\ell_1+\ell_2)k_i} = \omega^{\ell_1 k_i} \cdot \omega^{\ell_2 k_i} = \rho(\ell_1)\rho(\ell_2).$$

As long as the integers k_i are unique modulo n, the rows of $\mathbf{M}$ correspond to inequivalent irreducible representations of $\mathbb{Z}/n\mathbb{Z}$.

It turns out that this interpretation of a harmonic frame can be extended by using representations of groups other than $\mathbb{Z}/n\mathbb{Z}$. In particular, let G be a finite group with n_r inequivalent, irreducible representations, $\rho_1, ..., \rho_{n_r}$, with corresponding degrees $d_1, ..., d_{n_r}$. Recall that a *representation* ρ of degree d is a function $\rho : G \to GL(V)$, where V is a d-dimensional complex vector space, such that

$$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2),\ \forall g_1, g_2 \in G.$$

In other words, the $\rho(g_i)$ are invertible $d \times d$ matrices, and the group operation is replaced by matrix multiplication. ρ is *irreducible* if the $\rho(g_i)$ do not simultaneously fix any nontrivial subspace of V (ie, the matrices $\rho(g_i)$ cannot be simultaneously block-diagonalized). Two representations ρ_1 and ρ_2 are *equivalent* if there is an invertible linear transformation T such that $\rho_1(g) = T\rho_2(g)T^{-1}$, $\forall g \in G$ (ie, they are simultaneously similar as matrices), and they are *inequivalent* otherwise. Furthermore, it can be shown that every representation is equivalent to a *unitary representation*, in which $\rho(g)^* = \rho(g)^{-1}$ $(= \rho(g^{-1}))$ for all g (ie, every $\rho(g)$ can be represented as a unitary matrix). Thus, we will assume that all of our representations are unitary without loss of generality. More thorough descriptions of these results can be found in [25, 26].

Representation theory teaches us that for any finite group G, there are finitely many inequivalent irreducible representations whose degrees satisfy

$$d_1^2 + ... + d_{n_r}^2 = |G|. \tag{6}$$

It should be clear from the definitions that any representation ρ of G can be uniquely decomposed into a direct sum of irreducible representations by a similarity transformation (up to isomorphism), though in general these irreducible components need not be inequivalent.

Now, just as we selected the rows of $\mathbf{M}$ in (3) to be scaled rows of the discrete Fourier matrix, we can create a more general class of frames by selecting a subset of rows of the generalized *group Fourier transform matrix*:

$$\mathcal{F} = \begin{bmatrix} \sqrt{d_1}\text{vec}(\rho_1(g_1)) & \dots & \sqrt{d_1}\text{vec}(\rho_1(g_n)) \\ \sqrt{d_2}\text{vec}(\rho_2(g_1)) & \dots & \sqrt{d_2}\text{vec}(\rho_2(g_n)) \\ \vdots & \ddots & \vdots \\ \sqrt{d_m}\text{vec}(\rho_m(g_1)) & \dots & \sqrt{d_m}\text{vec}(\rho_m(g_n)) \end{bmatrix} \tag{7}$$

where $|G| = n$, $g_1, ..., g_n$ are the distinct elements of G, and $\rho_1, ..., \rho_m$ are inequivalent, irreducible representations of G. Note that the columns of $\mathcal{F}$ have not been normalized, but we shall see that each one has norm $\sqrt{\sum_{i=1}^{m} d_i^2}$.

Note that the matrix $\mathcal{F}$ has dimensions $\left(\sum_{i=1}^{m} d_i^2\right) \times n$. When all of the inequivalent irreducible representations are used ($m = n_r$), giving us the full group Fourier transform matrix, then (6) clearly tells us that $\mathcal{F}$ will be a square matrix. And indeed, if $G = \mathbb{Z}/n\mathbb{Z}$ so that g_j is simply the integer j, ρ_i the function which maps $j \mapsto \omega^{ij}$, and $d_i = 1$, $\forall i$, then we recover our original DFT matrix whose rows form $\mathbf{M}$ in (3). Furthermore, it can be shown using elementary representation theory that the columns of $\mathcal{F}$ form a tight frame. (In other words, the rows in (7) are orthonormal).

The inner product between the i^{th} and j^{th} columns of $\mathcal{F}$

Table 1. Character Table of $SL_2(\mathbb{F}_q)$, q even

Class Representative:	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}a & 0\\ 0 & a^{-1}\end{bmatrix}$ B $\begin{bmatrix}s & 0\\ 0 & s^{-1}\end{bmatrix}$ B^{-1}	
No. of such classes:	1	1	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$
Size of class:	1	q^2-1	$q(q+1)$	$q(q-1)$
1_G	1	1	1	1
St_G	q	0	1	-1
ρ_χ	$q+1$	1	$\chi(a)+\chi(a^{-1})$	0
π_η	$q-1$	-1	0	$-\eta(s)-\eta(s^{-1})$

can be written as follows:

$$\sum_{t=1}^{k} d_t \text{vec}(\rho_t(g_i))^* \text{vec}(\rho_t(g_j)) = \sum_{t=1}^{k} d_t \text{Tr}(\rho_t(g_i)^* \rho_t(g_j)) \quad (8)$$

$$= \sum_{t=1}^{k} d_t \text{Tr}(\rho_t(g_i^{-1} g_j)) = \sum_{t=1}^{k} d_t \chi_t(g_i^{-1} g_j). \quad (9)$$

Thus, as in the standard Fourier case, the inner product between the i^{th} and j^{th} columns does not depend on g_i and g_j individually but rather on the product $g_i^{-1} g_j = g_k$. So as before, we only have at most $n-1$ inner products corresponding to each nonidentity group element g_k. We remark that (9) follows from the fact that each ρ_t is assumed to be a unitary representation. Here, $\chi_t(g) = \text{Tr}(\rho_t(g))$ is the *character function* of G associated to ρ_t. Character theory is a powerful tool in the study of representations, and in fact the character of a representation completely determines how it will decompose into irreducible components. For many groups it is easy to find cataloged tables of the characters of all the inequivalent irreducible representations. What we have shown above is that we can compute the inner products between the columns of $\mathcal{F}$ using only knowledge of the character table of G. As we will see, this can be very useful in determining which representations $\rho_1, ..., \rho_m$ to use in constructing $\mathcal{F}$.

There is a strong connection between the characters of G and its *conjugacy classes*.

Definition 1 *We say that g_1 and g_2 are conjugate in G, or in the same conjugacy class, if there is some $g_3 \in G$ such that $g_1 = g_3 g_2 g_3^{-1}$. We call g_1 a representative of the conjugacy class.*

Lemma 1 *If χ is the character of a representation ρ of G, and g_1 and g_2 are conjugate in G, then $\chi(g_1) = \chi(g_2)$.*

Proof: This follows immediately from the definition $\chi(g) = \text{Tr}(\rho(g))$, and the properties of the trace. □

Corollary 1 *The number of distinct values of the inner products between columns of $\mathcal{F}$ is equal to the number of conjugacy classes of G.*

Proof: From Lemma 1 and (9), we see that the inner product between the i^{th} and j^{th} columns of $\mathcal{F}$ depends only on the conjugacy class of $g_i^{-1} g_j$. □

Lemma 2 *The the norm of each column of $\mathcal{F}$ is $\sqrt{\sum_{i=1}^{m} d_i^2}$.*

Proof: The inner product of a column with itself corresponds to taking $i = j$ in (9) above. In this case, $g_i^{-1} g_j = 1$, the identity element of G. Since $\rho_t(1)$ is simply the $d_t \times d_t$ identity matrix, we have $\chi_t(1) = \text{Tr}(\mathbf{I}_{d_t}) = d_t$, and the squared norm of the i^{th} column is $\sum_{t=1}^{m} d_t^2$. □

3. REPRESENTATIONS OF $SL_2(\mathbb{F}_Q)$

We will apply the above framework to the case where $G = SL_2(\mathbb{F}_q)$, and show how to obtain frames with low coherence. Let $\mathbb{F}_q$ be the finite field containing q elements. Such a field exists and is unique whenever q is a power of some prime p. Then $SL_2(\mathbb{F}_q)$ is the set of 2×2 determinant-1 matrices with entries in $\mathbb{F}_q$,

$$SL_2(\mathbb{F}_q) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{F}_q,\ ad - bc = 1 \right\}.$$

It is not difficult to check that this is a group containing $|SL_2(\mathbb{F}_q)| = q(q+1)(q-1)$ elements.

As with many groups, the character table of $SL_2(\mathbb{F}_q)$ is well-described. For example, when q is even (q is a power of 2), the characters are shown in Table 1. There are four types of conjugacy classes in $SL_2(\mathbb{F}_q)$ for q even, corresponding to how the matrices diagonalize. The first is simply the identity matrix, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. The second consists of the matrices that are not diagonalizable, and have the Jordan canonical form $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, which is the conjugacy class representative. (Note how these first two conjugacy classes contain all the matrices with repeat eigenvalues of 1). Each conjugacy class of the third type has a representative which is a diagonal matrix: $\begin{bmatrix}a & 0\\ 0 & a^{-1}\end{bmatrix}$, where $a \in \mathbb{F}_q \setminus \{0, 1\}$. No two of these representatives are conjugate to each other. The elements of the fourth type of conjugacy class are those that take the form

$B\begin{bmatrix} s & 0 \\ 0 & s^{-1}\end{bmatrix}B^{-1}$, where now $B \in SL_2(\mathbb{F}_{q^2})$ and $s \in \mathbb{F}_{q^2}$ is a *norm-1* element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$, that is, $s^{q+1} = 1$. This happens when the characteristic polynomial of the matrix has no solution in $\mathbb{F}_q$. Note that here $\mathbb{F}_{q^2}$ is the finite field of q^2 elements, which contains $\mathbb{F}_q$ as a subfield. As in the previous case, there is a distinct conjugacy class for each such s.

Likewise, there are four types of characters of $SL_2(\mathbb{F}_q)$ for q even. We will omit the descriptions of the representations to which they correspond, but they can be found in [27–29] The first is the character of the identity representation, 1_G, which maps every element to 1. Its character is the first shown in Table 1. The second character, shown in the subsequent row, is that of the Steinberg representation, denoted St_G. Both of these are degree-1 representations.

The third and fourth types of characters in the final two rows of the table are more interesting. The third corresponds to what is called an *induced representation*, denoted here as ρ_χ. This representation has degree $q+1$, but is built from a nontrivial degree-1 representation χ of the group $\mathbb{F}_q^\times$. Here, $\mathbb{F}_q^\times$ denotes the set of nonzero elements of $\mathbb{F}_q$, which form a group under multiplication. This group happens to be cyclic of size $q-1$, generated by the powers of some nonzero $\tilde{a} \in \mathbb{F}_q$. Thus, if $\omega_- = e^{\frac{2\pi i}{q-1}}$, then χ will be the function $\chi(\tilde{a}^\ell) = \omega_-^{k\ell}$, for some $k \in \{1, 2, ..., q-2\}$. (It is required that $k \not\equiv 0 \mod q-1$ for this representation to be irreducible).

The fourth and final type of character is that of a degree-$(q-1)$ *cuspidal representation*, denoted π_η. This is actually constructed from a degree-1 representation η of the group of norm-1 elements of $\mathbb{F}_{q^2}$, which is a multiplicative group of size $q+1$. This group is again cyclic. If we let $\tilde{s}$ be a generator, so that every norm-1 element is written as a power of $\tilde{s}$, then $\eta(\tilde{s}^\ell) = \omega_+^{h\ell}$, where $\omega_+ = e^{\frac{2\pi i}{q+1}}$ and $h \in \{1, 2, ..., q\}$. (Again, we require $h \not\equiv 0 \mod q+1$ for irreducibility).

4. FRAMES FROM INDUCED AND CUSPIDAL REPRESENTATIONS

We can now use our previous results from Theorem 2 to construct frames in the form of $\mathcal{F}$ in (7) with low coherence using the representations of $SL_2(\mathbb{F}_q)$ for q even. We begin by using only the induced representations. For convenience, we will write ρ_k for the representation ρ_χ where $\chi(\tilde{a}) = \omega_-^k$. Now consider choosing q such that $n = q-1$ is a prime, and take m to be a divisor of $n-1$. As before, we would intuitively like to choose the set $K = \{k_1, ..., k_m\}$ to be the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ of size m, and then use the representations ρ_{k_i} to construct $\mathcal{F}$. However, notice from Table 1 that the characters corresponding to ρ_k and ρ_{-k} are the same (where $-k$ is taken modulo n), indicating that they are equivalent representations. This follows from the fact that if $\chi_k(\tilde{a}) = \omega_-^k$ and $\chi_{-k}(\tilde{a}) = \omega_-^{-k}$, then

$$\chi_k(\tilde{a}^\ell) + \chi_k(\tilde{a}^{-\ell}) = \omega_-^{k\ell} + \omega_-^{-k\ell} = \chi_{-k}(\tilde{a}^{-\ell}) + \chi_{-k}(\tilde{a}^\ell).$$

Therefore, if -1 is contained in the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ of size m, we must choose K slightly differently.

Lemma 3 *For n prime, m dividing $n-1$, the unique size-m subgroup K_m of $(\mathbb{Z}/n\mathbb{Z})^\times$ contains -1 if and only if m is even. In this case, $m/2$ is odd, and $K_m = K_{m/2} \cup -K_{m/2}$ where $K_{m/2}$ is the unique size-$\frac{m}{2}$ subgroup.*

Theorem 3 *Take q a power of 2 such that $n = q-1$ is prime, and let m be an odd divisor of $n-1$ and $r = \frac{n-1}{2m}$. Let $K = \{k_1, ..., k_m\}$ be the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ of size m, and form $\mathcal{F}$ from the induced representations ρ_{k_i}. Then the coherence $\mu_\mathcal{F}$ of $\mathcal{F}$ is bounded by*

$$\frac{1}{q+1}\max\left(1,\ \frac{1}{r}\left((r-1)\sqrt{\frac{1}{2m}\left(r-\frac{1}{2m}\right)}+\frac{1}{2m}\right)\right). \tag{10}$$

Proof: From (9) and Table 1, we see that the only nontrivial inner products are those corresponding to the conjugacy classes represented by $u := \begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}$ and $w_\ell := \begin{bmatrix} \tilde{a}^\ell & 0 \\ 0 & \tilde{a}^{-\ell}\end{bmatrix}$ for $\ell \in \{1, ..., q-2\}$. These inner products are

$$\sum_{i=1}^m d_i \chi_{\rho_{k_i}}(u) = m(q+1), \tag{11}$$

$$\sum_{i=1}^m d_i \chi_{\rho_{k_i}}(w_\ell) = \sum_{i=1}^m (q+1)\cdot(\chi_{k_i}(\tilde{a}^\ell) + \chi_{k_i}(\tilde{a}^{-\ell})) \tag{12}$$

$$= (q+1)\sum_{i=1}^m (\omega_-^{\ell k_i} + \omega_-^{-\ell k_i}). \tag{13}$$

From Lemma 3, we can bound (13) using Theorem 2. The final result is then scaled after normalizing the columns by $\sqrt{m(q+1)^2}$, which we obtain from Lemma 2. □

We remark that we can obtain similar results using a parallel construction of $\mathcal{F}$ with only cuspidal representations, which works when $q+1$ is prime.

5. CONCLUSION

For fixed parameter r in Theorem 3, our frames' coherences asymptotically approach the Welch bound (2). Table 2 shows that even for relatively small dimensions our frames tend to have significantly lower coherence than random Gaussian frames. We are now exploring our framework in new groups.

Table 2. $SL_2(\mathbb{F}_q)$ vs. Gaussian Frame Coherences

Frame Dimensions	$SL_2(\mathbb{F}_q)$	Random Gaussian
25×60	.2000	.5214
81×504	.2002	.3482
243×504	.1111	.2274

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