

STATE-DEPENDENT OPINION DYNAMICS*

Daron Acemoglu

Mohamed Mostagir, Asuman Ozdaglar

MIT

Department of Economics

MIT

Laboratory for Information and Decision Systems

ABSTRACT

We study the simultaneous evolution of the opinion profile and network topology of a system of N agents. Based on the opinion profile at any given time, agents probabilistically decide which other agents to form links with. The probability of a link being formed with another agent depends on both similarity of their opinions and the popularity of that agent. Agents then average their opinion with the opinions of the agents they have formed links with, giving rise to a new opinion profile that determines—in a probabilistic fashion—the network topology for the next time step. Thus both opinions and network structure exhibit a strong correlation over time. Despite this correlation, we show that this system converges to a consensus in opinion. We provide simulations of convergence times and the limiting opinion profile as a function of the parameters of the system.

Index Terms— Social Learning, Opinion Dynamics

1. INTRODUCTION

Individuals form beliefs on various economic, political and social variables based on information they receive from their local network. A growing literature studies dynamics of these beliefs using plausible rules of thumb on how individuals combine the beliefs of their neighbors in the network with their own—e.g., by taking some weighted average—and a fixed structure of the underlying social network (for example, Golub and Jackson in [1], DeGroot's seminal work in [2]). This literature provides necessary and sufficient conditions for a consensus of opinion to emerge within a network. It is not clear, however, whether belief dynamics can be decoupled from the evolution of the network. For example, conformity of opinions between different agents may affect the likelihood that they will communicate in the future, and the popularity of individual might increase the weight that others attach to their opinion or whether they befriend them.

In this paper, we develop a tractable but fairly general model of the co-evolution of the topology of a social network

and the distribution of beliefs over the network. In our model, agents form and sever links based on the similarity of beliefs and the past popularity of others in the network. These two features enable a fairly general process for the co-evolution of the topology of the network and the belief distribution.

Both of these features have been studied in isolation and in somewhat different contexts in the existing literature. Hegselmann-Krause introduced a model of belief dynamics in which agents only communicate with others in the network who are no further than a certain distance from their opinion [3]. This model has been subsequently studied in [4] and [5], among others, and variants where the impact of belief differences on communication probabilities is smooth have been analyzed in [6]. Skyrms and Pemantle, on the other hand, proposed a friendship model where past interactions between agents reinforce the chance of future interactions [7]. In [8], Fazeli and Jadbabaie studied an opinion formation process for a special case of the Skyrms-Pemantle model where interactions are governed by an urn process and a visit increases the probability of a future repeat visit in the same agents. Fazeli and Jadbabaie exploit the properties of the urn process to show convergence to a consensus.

Our work differs from these existing papers in several dimensions. First, we combine two dimensions of the endogeneity of the network, allowing future communications to depend both on differences in opinion and on the history of past visits. Second, we allow a tractable and general formulation linking these variables to communication probabilities (e.g., as opposed to Hegselmann-Krause's model where the impact of belief differences on these probabilities is discontinuous). As a result, compared to Fazeli and Jadbabaie, the probability that agent i visits agent j can increase without i having ever visited j . This may happen, for example, because j has visited another agent that holds an opinion close to that of i and as a result j 's opinion has moved closer to that of i , increasing the probability of their future interaction. This is a plausible and quite realistic scenario, and one that is not captured by the Fazeli-Jadbabaie or Skyrms-Pemantle models. Compared to Hegselmann-Krause, communication is not precluded between any two agents because probabilities are continuous functions of belief differences and because

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the likelihood of communication with an agent holding a very different belief can be still very high if she is popular. These differences also imply that the mathematical structure of the evolution of the network topology and beliefs is different in our model than in existing work. In particular, the evolution of the network topology cannot be described by a stochastic process with independent and identically distributed increments. This necessitates a different mathematical approach, which we develop in this work.

More formally, in our model at each date agents start with a scalar belief, which they use to determine which others to visit. After visiting another agent, an agent updates his beliefs by simply averaging their pre-visit belief with the belief of the agent that he visited. The model we consider relies on two parameters, β and κ , which have direct interpretations in terms of the nature of the social network. In particular, in our model β measures how “open” a society is to opinions that are different from the norm within that society, with higher values of β indicating a more “conservative” society. In a conservative society, agents are unlikely to visit other agents whose opinions are different from their own. The second parameter, κ , describes the tendency of a society to follow the opinions of popular individuals. As we show, these two parameters have a strong effect on the behavior of the system.

Because at time $t+1$ agents use opinions at the end of time t to determine what connections they will form at time $t+1$, the system is correlated across time. We show that despite this correlation and despite the fact that parameter values that can make it highly unlikely for agents with different opinions to interact, the system still reaches consensus, though the time to do so strongly varies as a function of the system parameters. We provide simulations illustrating how the speed of convergence and the consensus value, which is a random convex combination of initial opinions, depends on the parameters of the system.

The paper is organized as follows. Section 2 introduces our model formally and discusses the criteria by which agents decide to connect to other agents. Section 3 discusses differences between our model and existing consensus results and provides our convergence theorem. Section 4 discusses how the system parameters affect the behavior of the system.

2. MODEL

Our model consists of a society of N agents with an initial opinion vector $x(0) \in [0, 1]^n$. Society is represented by a dynamic graph $G_t(N, E_t)$, where N is a fixed set of nodes representing the agents and E_t is the set of edges linking these agents at time t . The edges are formed by the agents in the following way. Each time step, all agents place weights on all other agents and use these weights to probabilistically select an agent to form a link with. We will refer to the link formation process as a ‘visit’. Thus in each period each agent visits exactly one other agent and has an out-degree equal to

one. We assume that the decision of whom to visit is independent across the agents. After each round, agents update their opinions. Agent i ’s opinion after a period where he visits agent j is the average opinion of i and j , while j ’s opinion remains unaffected by the visit. The weight that agents place on other agents has two components. The distance in opinion component, and the popularity component.

2.1. Distance in Opinion

The first component in the weight that agent i places on agent j at time t depends on how similar the opinions of i and j were at the end of the previous period, $t-1$, and is given explicitly by

$$e^{-\beta|x_i(t-1)-x_j(t-1)|}$$

where $\beta \geq 0$ is a constant. This expression is commonly used in the economics and physics literatures and lends itself naturally to our formulation, since agents with similar opinions put higher weights on each other. The probability with which agent i visits agent j at time t is then given by

$$\frac{e^{-\beta|x_i(t-1)-x_j(t-1)|}}{\sum_{k \neq i} e^{-\beta|x_i(t-1)-x_k(t-1)|}}$$

The parameter β can be agent specific, but for simplicity we take it to be the same for all agents. This parameter can be thought of as a quality that the society as a whole possesses. When β is high, agents are more inclined to visit only those agents who are similar to them in opinion, leading to a society that is more conservative on average than a society with lower values of β . When β is low, agents are more open to visit other agents who are different from them in opinion and society is more accepting of ideas that are different from the norm.

2.2. Popularity

Another aspect that goes into the weight that agents place on each other in our model is popularity. A popular agent at time t is an agent who has received many visits in time $t-1$. This means that popularity is equivalent to the in-degree of an agent, with agents with higher in-degrees being more popular. In particular, the weight that agent i assigns to agent j at time t based on popularity alone is equal to

$$e^{\kappa d_j(t-1)}$$

and is increasing in $d_j(t-1)$, the in-degree of agent j . Like before, the probability with which agent i visits j then is given by

$$\frac{e^{\kappa d_j(t-1)}}{\sum_{k \neq i} e^{\kappa d_k(t-1)}}$$

Similar to β , $\kappa \geq 0$ is a parameter that describes how important popularity is to agent when they assign weights to

other agents. Again, this parameter can be agent specific but we take it to be the same across the society. This means that a society with high values of κ is inclined to visit popular agents more than a society with a lower κ .

Combining these two components, we let $W(t)$ and $P(t)$ be the $n \times n$ weight and probability matrices at time t , respectively. Entry $w_{ij}(t)$ is the weight that agent i places on agent j at time t , while entry $p_{ij}(t)$ represents the probability with which agent i visits agent j at time t . Thus the i^{th} row in $P(t)$ represents the probability vector that agent i uses to decide whom to visit in period t . To reduce clutter, we drop the $t - 1$ argument with the understanding that all weights are computed from the values obtained at the end of the previous period. Thus $P(t)$ is a matrix with $p_{ii}(t) = 0$ and

$$p_{ij}(t) = \frac{w_{ij}(t)}{\sum_{k \neq i} w_{ik}(t)} = \frac{e^{-\beta|x_i - x_j| + \kappa d_j}}{\sum_{k \neq i} e^{-\beta|x_i - x_k| + \kappa d_k}}. \quad (1)$$

At each time t , agents use $P(t)$ to decide whom to visit. The visits give rise to the set of edges E_t and the adjacency matrix $A(t)$. Because each agent visits exactly one other agent, the matrix $A(t)$ has one entry equal to one in each row with the rest of the entries in that row equal to zero. Since agents update their opinions by taking the average of their opinion as well as the opinion of the agent they visited, we define the stochastic matrix $Z(t)$ as

$$Z(t) = \frac{A(t) + I_n}{2},$$

where I_n is the $n \times n$ Identity matrix. We use the matrix $Z(t)$ to describe the opinion profile at time $t + 1$ as a function of the opinion profile at time t through the relationship

$$x(t + 1) = Z(t)x(t), \quad (2)$$

and therefore the opinion profile at time $t + 1$ can be written as

$$Z(t)Z(t-1)\dots Z(0)x(0). \quad (3)$$

Let $\mathcal{Z} = \{Z : Z \in R^{n \times n}, Z_{ii} = \frac{1}{2} \text{ and } Z_{ij(i)} = \frac{1}{2} \text{ and } Z_{ik} = 0 \text{ for } k \neq i, j(i)\}$ be the set of all such matrices.

3. CONSENSUS

In this section we study the dynamics described by Eq. (2) and the backward product of the stochastic matrices $Z(t), \dots, Z(0)$ in Eq. (3). We show that the system in Eq. (2) reaches consensus despite the strong correlation exhibited by the matrices $Z(t)$. A general result on the stability (and consensus) of systems like the one in Eq. (2) is given by Lorenz in [5]. The model we consider in this paper however violates at least one of the three conditions that Lorenz posits on the matrices $Z(t)$. Namely, the 'confidence is mutual' condition which states that the matrices $Z(t)$ should be type-symmetric. Type-symmetric matrices require that $Z_{ji}(t) > 0$

whenever $Z_{ij}(t) > 0$, which translates in our model to insisting that at time t , if agent i visits agent j then agent j must visit agent i . This is a restrictive assumption that our model does away with. This is not very surprising considering that Lorenz's conditions are sufficient but not necessary.

On the other hand, the infinite flow and absolute infinite flow properties developed in the work of Touri and Nedić on backward products of stochastic matrices provide necessary but not sufficient conditions for the backward product of the sequence $\{Z(t)\}_{t=0}^{\infty}$ to converge to a consensus matrix, i.e. checking whether our matrices fulfill these properties or not does not automatically lead us to conclusions about convergence. The characterization in [9] gives an equivalence between the absolute infinite flow property and the convergence of doubly-stochastic matrices. Since our matrices are not doubly-stochastic, we can not make use of this equivalence for our results. Finally, other convergence results starting from Wolfowitz's theorem [10], and found in the works of Jadbabaie, Lin, and Morse [11], and Hedrickx and Blondel [12] are based on the assumption that the matrices $Z(t)$ are irreducible, which is again not the case in our model.

As mentioned earlier, the closest model to ours is that of Fazeli and Jadbabaie. The proof of convergence in their work utilizes the fact that the model is built on top of a Polya's urn process, where the probability that agent i visits agent j evolves in the same way as in the urn process.¹ This enables the authors to use the fact that the urn process has the property that sequences are exchangeable, meaning that the order in which the visits are made does not affect the joint probability of a sequence of visits. Our model is more complicated because $p_{ij}(t)$ is not just a function of $p_{ij}(t - 1)$, but also of who agent j visited in period $t - 1$, since that visit will change x_j and therefore has an effect on $w_{ij}(t)$ and consequently $p_{ij}(t)$.

For the purpose of stating the convergence theorem, we define a consensus matrix K as a matrix whose rows are all equal. We can then provide our result on convergence.

Theorem 3.1 *Denote by $Z(0, t)$ the matrix resulting from the multiplication $Z(t)Z(t-1)\dots Z(0)$ and let K be a consensus matrix, then $\lim_{t \rightarrow \infty} Z(0, t) = K$ with probability 1.*

The proof of the theorem relies on a modification of the Borell-Cantelli Lemma to apply to non-independent events, and then adapting the proof in [5].

4. NUMERICAL RESULTS

4.1. Speed of Convergence

We simulate a system of $N = 20$ agents. Figure 1 shows the speed of convergence for different values of β . When

¹In this model, agent i visiting agent j increases the probability that this visit is repeated in the next period, in the same way that choosing a red ball from an urn and then putting back two red balls increases the probability of picking a red ball in the next trial.

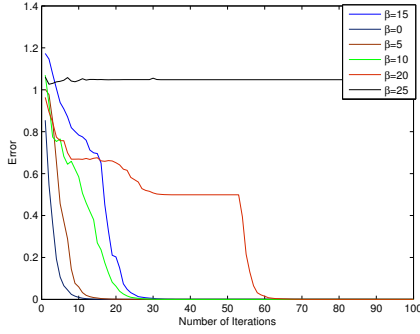


Fig. 1. Convergence Times For Different Values of β

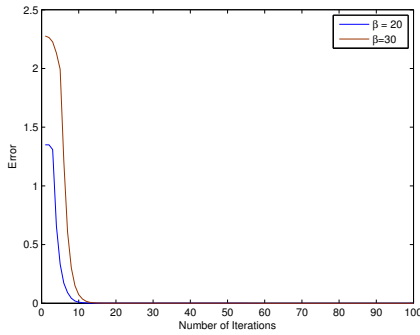


Fig. 2. Convergence Times for $\kappa = 2$ and $\beta = 20$ and 30

$\beta = 0$, agents in the society have no particular inclination to visit those who are similar to them, the system converges exponentially fast. As β grows larger convergence becomes slower. For β over 30, convergence was not obtained in any reasonable amount of time; instead, society clusters into two or three groups, with each group having a consensus amongst themselves.

Conversely, κ gives a strong boost to the speed of convergence, as society obtains most of its information from only a small group of agents. Figure 1 is revisited in Figure 2 with the addition of $\kappa = 2$, which shows how the previously non-convergent case with $\beta = 20$ now converges extremely fast, as does the case when $\beta = 30$. At these values, the popularity component takes over and the system converges even when β is quite high.

4.2. Limit Opinion

We simulate the process for a randomly generated opinion vector with mean 0.351 and we start with the case $\beta = \kappa = 0$. This baseline case is when agents visit each other in each period with the same probability, thus the Z matrices are iid. This baseline case is analyzed in [13], where the authors show that the limiting opinion profile is a random vector with expectation equal to the expectation of the initial opinion profile,

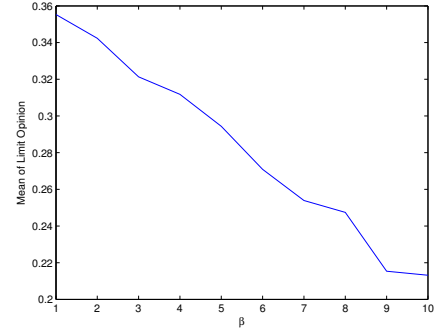


Fig. 3. Expectation of Limit Opinion as a Function of β

i.e. society is able to aggregate information (in this case, the average value) correctly. This is observed in the simulations.

When we keep κ equal to zero and increase β , the resulting expectation starts drifting away from the case $\beta = 0$. Figure 3 shows the expectation of the limit opinion falling steadily as β increases from 0 to 12. One explanation for this lies in the structure of the randomly generated opinion profile, which has five agents with $x(0)$ values that are less than 0.1, a 'majority' cluster at $t = 0$. This cluster can skew the resulting distribution by attracting the rest of the population closer to their position. Increasing β exacerbates this effect, leading to the conclusion that the initial distribution of opinions, not just in values but also in 'clusters' has an immediate effect on the limit distribution when β is high. A fringe majority cluster in the population can pull the rest of the network towards their opinion because they can practice their conservatism by visiting each other and ignoring the rest of the network, which given enough time, starts drifting towards the cluster. It is possible that this effect cancels out if there is a large number of agents, as groups of extreme agents on opposite sides neutralize each other, but the combination of a large number of agents and high β does not converge in a reasonable amount of time.

5. CONCLUSION

This paper presented a model of state-dependent opinion dynamics, where agents are more likely to communicate with others who carry a similar opinion to theirs or who are popular in the society. We have shown that the resulting dynamical system converges to a consensus and that the rate of convergence correlates negatively with the level of conservatism in society and positively with popularity. Preliminary simulation results suggest that society loses its ability to combine information correctly as the level of conservatism increases, and that the so-called Super Star behavior, where agents are inclined to follow agents who are popular in society leads to uncertain outcomes. Current work focuses on deriving an analytical expression for the limiting opinion distribution.

6. REFERENCES

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