REDUCED-RANK NEURAL ACTIVITY INDEX FOR EEG/MEG MULTI-SOURCE LOCALIZATION

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ABSTRACT

We consider the problem of electroencephalography (EEG) and magnetoencephalography (MEG) source localization using beamforming techniques. Specifically, we propose a reduced-rank extension of the recently derived multi-source activity index (MAI), which itself is an extension of the classical neural activity index to the multi-source case. We show that, for uncorrelated dipole sources and any nonzero rank constraint, the proposed reduced-rank multi-source activity index (RR-MAI) achieves the global maximum when evaluated at the true source positions. Therefore, the RR-MAI can be used to localize multiple sources simultaneously. Furthermore, we propose another version of the RR-MAI which can be seen as a natural generalization of the proposed index to arbitrarily correlated sources. We present a series of numerical simulations showing that the RR-MAI can achieve a more precise source localization than the full-rank MAI in the case when the EEG/MEG forward model becomes ill-conditioned, which in our settings corresponds to the case of closely positioned sources and low signal-to-noise ratio.

Index Terms— Electroencephalography, magnetoencephalography, beamforming, reduced-rank signal processing, MV-PURE estimator, dipole source localization

1. INTRODUCTION

The linearly constrained minimum variance beamformer (LCMV) [1] remains a backbone of spatial filtering applications, which includes the case of dipole source localization and dipole signal estimation from electroencephalographic (EEG) and magnetoencephalographic (MEG) measurements [2–5]. Extensions of the classic LCMV beamformer to the case of simultaneous multiple source localization have been recently introduced in [5, 6]. Such multisource extensions showed an improved performance in the localization of sources of brain activity from EEG or MEG measurements in the case of correlated sources. In particular, the multi-source activity index (MAI) is proposed in [5] as an extension of the classic neural activity index introduced in [2] for the single-source case. However, it has already been demonstrated in [7] that the LCMV beamformer is inadequate if the EEG/MEG forward model becomes illconditioned, e.g., as result of closely positioned sources and high background activity. This inherent limitation of the LCMV beamformer may also affect the performance of the MAI by limiting its ability to distinguish between closely positioned sources of brain electrical activity. On the other hand, it has also been shown that the minimum-variance pseudo-unbiased reduced-rank estimator (MV-PURE) offers a drastic gain in performance compared to the LCMV beamformer when estimating dipole source signals using EEG in ill-conditioned settings [7]. The MV-PURE was introduced in [8,9] and can be naturally interpreted as the reduced-rank extension of the LCMV beamformer. Thus, it is expected that a reduced-rank extension of the MAI should provide greater spatial resolution than the full-rank MAI in such settings.

In this paper, we introduce the reduced-rank multi-source activity index (RR-MAI) as a reduced-rank extension of the MAI that achieves its maximum value when evaluated at the true position of the sources, for the case of uncorrelated sources and any nonzero rank constraint. Such maximum is obtained by replacing the covariance matrix in the MAI of the output of the LCMV by the covariance matrix of the output of the MV-PURE. In addition, we propose another version of the RR-MAI which can be seen as a natural generalization of the proposed index to arbitrarily correlated sources.

In practice, the covariance matrix of the background activity and the one corresponding to the sources of interest are unknown, thus they are estimated from the measurements. This estimation introduces a perturbation in the values of the activity indices, which in turn makes the choice of optimal rank a key issue for reduced-rank beamformers. For this reason, in this paper we evaluate the performance of the proposed RR-MAI as a function not only of the SNR, but of the rank as well. We show that reducing the rank of the RR-MAI yields significantly smaller localization error compared with the full-rank MAI for a range of rank constraints.

The paper is organized as follows: in Section 2 we introduce the EEG/MEG measurement model considered, the LCMV beamformer, and the single- and multi-source neural activity indices. Then, in Section 3 we introduce the proposed RR-MAI, and in Section 4 we show through a series of numerical examples its applicability in dipole source localization under ill-conditioned settings using realistically simulated MEG data. We close with Section 5 where conclusions are drawn and areas of future research are discussed.

Due to lack of space, all proofs are omitted.

2. PRELIMINARIES

We consider the case of EEG or MEG measurements produced by l dipole sources using an array of m EEG or MEG sensors. We assume that the sources' activity change in time, but their positions $\theta = (\theta_1, \dots, \theta_l)$ remain the same during the measurement period t = 1, 2, ..., n, where θ_i denotes the position of the *i*-th source. Then, the $m \times n$ spatio-temporal data matrix Y for a given trial can be modeled as

$$Y = A(\boldsymbol{\theta})Q + N,\tag{1}$$

where $A(\theta)$ is the $m \times 3l$ array response matrix representing the material and geometrical properties of the medium in which the sources are submerged relevant for EEG or MEG measurements, Q is the $3l \times n$ matrix of dipole moments, and N is the noise matrix representing spontaneous background brain activity.

In terms of (1), spatial filtering allows for estimation of dipole source moments Q (when source locations are known), and for source localization by defining a neural activity index as a function of positions θ , with the assumption that the maximum value of the activity index corresponds with the true location of the sources. In terms of the reduced-rank framework proposed by us, the first problem has been tackled initially in [7], and it is the latter source localization problem in which we focus on this paper.

To our knowledge, the idea of using a neural activity index as a localizer originates in [2], where the following activity index (AI) has been proposed for the single-source, i.e., l = 1in (1):

$$\operatorname{AI}(\boldsymbol{\theta}) := \frac{\operatorname{tr}\{S(\boldsymbol{\theta})^{-1}\}}{\operatorname{tr}\{G(\boldsymbol{\theta})^{-1}\}},$$
(2)

where $tr\{\cdot\}$ indicates the trace of the matrix,

$$S(\boldsymbol{\theta}) := A(\boldsymbol{\theta})^T R_{\mathbf{Y}}^{-1} A(\boldsymbol{\theta}), \qquad (3)$$

and

$$G(\boldsymbol{\theta}) := A(\boldsymbol{\theta})^T R_{\mathbf{N}}^{-1} A(\boldsymbol{\theta}), \qquad (4)$$

where $R_{\mathbf{Y}}$ and $R_{\mathbf{N}}$ are covariance matrices of Y and N, respectively. The properties of AI(θ) have been throughly investigated in [2] and derived works, where the main drawback of AI(θ) was found to be its sensitivity to correlated source cancellation and its poor performance under low SNR conditions.

To circumvent this difficulty, a multi-source extension of $AI(\theta)$ has been recently proposed in [5], among other localizers. Namely, the following multi-source activity index (MAI) has been proposed in [5] for the general case $l \ge 1$:

$$MAI(\boldsymbol{\theta}) := tr\{G(\boldsymbol{\theta})S(\boldsymbol{\theta})^{-1}\} - 3l_x,$$
(5)

where l_x is the unknown number of concurrently active sources. The key properties of MAI(θ) are summed up in the following theorem [5, p. 485]:

Theorem 1 The MAI(θ) in (5) is a non-negative and bounded function of θ , which reaches its global maximum for $\theta = \theta_0$, where θ_0 are the true source locations, in which case $l_x = l$, where l is the number of active sources. The global maximum is of the form:

$$\max_{\boldsymbol{\theta}} \operatorname{MAI}(\boldsymbol{\theta}) = \operatorname{MAI}(\boldsymbol{\theta}_0) = \operatorname{tr}\{G(\boldsymbol{\theta}_0)S(\boldsymbol{\theta}_0)^{-1}\} - 3l = \operatorname{tr}\{R_{\boldsymbol{\theta}}G(\boldsymbol{\theta}_0)\}, \quad (6)$$

where R_0 is the covariance matrix of Q.

The applicability of the MAI(θ) has been already demonstrated in [5]. In particular, a practical method for determining the number of active sources can be deducted

from Theorem 1 as follows: initiate the search with $l_x = 1$, find $\max_{\theta} \text{MAI}(\theta)$, and increase l_x until the value of $\max_{\theta} \text{MAI}(\theta)$ saturates. Thus, from now on we assume that the number of active sources l has been identified. On the other hand, it shall be noted that the maximum of $\text{MAI}(\theta)$ is achieved for $S(\theta_0)^{-1}$, which is the covariance matrix of the output of the LCMV filter [8–11] (denoted by Φ_{LCMV}) evaluated at the true source locations:

$$\Phi_{\text{LCMV}} := S(\boldsymbol{\theta}_0)^{-1} A(\boldsymbol{\theta}_0)^T R_{\mathbf{Y}}^{-1}.$$
 (7)

However, small changes in $S(\theta_0)$ may cause huge changes in $S(\theta_0)^{-1}$ if the former is ill-conditioned [12]. In our case, this could be the result of closely positioned sources and high background activity. In such situations, $G(\theta_0)$ will be illconditioned as well. Furthermore, ill-conditioning can also be the result of using the estimates \widehat{R}_Y and \widehat{R}_N instead of R_Y and R_N , respectively.

Let us now introduce the following estimate:

$$\widehat{\mathrm{MAI}}(\boldsymbol{\theta}) := \mathrm{tr}\{\widehat{G(\boldsymbol{\theta})}\widehat{S(\boldsymbol{\theta})}^{-1}\} - 3l_x, \qquad (8)$$

where

and

$$\widehat{G(\boldsymbol{\theta})} := A(\boldsymbol{\theta})^T \widehat{R_{\mathbf{N}}}^{-1} A(\boldsymbol{\theta}), \qquad (9)$$

$$\widehat{S(\boldsymbol{\theta})} := A(\boldsymbol{\theta})^T \widehat{R_{\mathbf{Y}}}^{-1} A(\boldsymbol{\theta}).$$
(10)

Obviously, the value of (8) will differ from (5) for a given θ . More crucially, its maximum value is likely to change from $\theta = \theta_0$ due to sensitivity of $S(\theta_0)^{-1}$ to even smallest changes of ill-conditioned $S(\theta_0)$. This is a fundamental problem as it makes source localization using MAI(θ) very prone to errors in ill-conditioned settings. In the next section, we provide a solution that allows for robust source localization in ill-conditioned settings by introducing reduced-rank extension of the MAI(θ).

3. RR-MAI ACTIVITY INDEX

In order to alleviate the aforementioned shortcomings of the multi-source activity index defined in (5), we introduce its reduced-rank extension as follows:

$$\operatorname{RR-MAI}_{\operatorname{TI}}(\boldsymbol{\theta}, r) := \operatorname{tr}\{G(\boldsymbol{\theta})P_{\mathcal{R}(G(\boldsymbol{\theta})_r)}S(\boldsymbol{\theta})^{-1}\} - r, \quad (11)$$

where r is a natural number such that $1 \leq r \leq 3l$, and $P_{\mathcal{R}(G(\theta)_r)}$ is the orthogonal projection matrix onto $\mathcal{R}(G(\theta)_r)$, i.e., the subspace spanned by the r eigenvectors corresponding to the r largest eigenvalues of $G(\theta)$. The following theorem establishes the key properties of the RR-MAI_{TI}(θ, r):

Theorem 2 Let us fix a rank constraint r such that $1 \le r \le 3l$, and consider uncorrelated sources such that

$$R_{\boldsymbol{\varrho}} = I_{3l}.\tag{12}$$

Then RR-MAI_{TI}(θ , r) in (11) is a non-negative and bounded function of θ , which reaches its global maximum for $\theta = \theta_0$, where θ_0 are the true source locations. The global maximum is of the form:

$$\max_{\boldsymbol{\theta}} \operatorname{RR-MAI}_{TI}(\boldsymbol{\theta}, r) = \operatorname{RR-MAI}_{TI}(\boldsymbol{\theta}_0, r) = \operatorname{tr}\{G(\boldsymbol{\theta}_0)P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)}S(\boldsymbol{\theta}_0)^{-1}\} - r = \operatorname{tr}\{P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)}G(\boldsymbol{\theta}_0)\}, \quad (13)$$

where $P_{\mathcal{R}(S(\theta_0)_r)}$ is the orthogonal projection matrix onto $\mathcal{R}(S(\theta_0)_r)$, i.e., the subspace spanned by the r eigenvectors corresponding to the r largest eigenvalues of $S(\theta_0)$.

It should also be noted that $\mathcal{R}(S(\boldsymbol{\theta}_0)_r) = \mathcal{R}(G(\boldsymbol{\theta}_0)_r)$ for $1 \leq r \leq 3l$ under the assumption in (12). Due to lack of space, we omit the proof of this fact.

Theorem 2 shows that the RR-MAI_{T1}($\boldsymbol{\theta}, r$) achieves its maximum when the covariance matrix $S(\boldsymbol{\theta}_0)^{-1}$ of the output of the LCMV is replaced by

$$P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)}S(\boldsymbol{\theta}_0)^{-1} = P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)}S(\boldsymbol{\theta}_0)^{-1}P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)},$$
(14)

which corresponds to the covariance matrix of the output of the MV-PURE estimator [8,9] (denoted by Φ_{MVP}) evaluated at the true source locations:

$$\Phi_{\text{MVP}} := P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)} \Phi_{\text{LCMV}}.$$
(15)

The MV-PURE has been recently recognized as a natural reduced-rank extension of the LCMV beamformer [7]. In particular, if we consider an eigenvalue decomposition (EVD) of $S(\theta_0)$ as

$$EVD(S(\boldsymbol{\theta}_0)) := U_0 \Sigma_0 U_0^T, \qquad (16)$$

with eigenvalues organized in nonincreasing order, then it is simple to verify that

$$P_{\mathcal{R}(S(\boldsymbol{\theta}_0)_r)}S(\boldsymbol{\theta}_0)^{-1} = U_0 \Sigma_{0_r}^{\dagger} U_0^T, \qquad (17)$$

where Σ_{0_r} is obtained from Σ_0 by replacing its $r + 1, \ldots, 3l$ diagonal entries by zeros, and $\Sigma_{0_r}^{\dagger}$ is the Moore-Penrose pseudo-inverse of Σ_{0_r} [13]. Therefore, by targeting (14) in the place of $S(\theta_0)^{-1}$, and by choosing an appropriate rank constraint, we ensure that (11) is fully robust to an illconditioned $S(\theta_0)$ possessing vanishingly small singular values $\sigma_{0_{r+1}}, \ldots, \sigma_{0_{3l}}$, unlike the full-rank MAI(θ) described in Section 2.

It is important to find out how Theorem 2 can be generalized for any positive definite $R_{\mathbf{Q}}$. Such condition is presented in the following proposition:

Proposition 1 Let us fix a rank constraint r such that $1 \le r \le 3l$, and let R_Q be any positive definite matrix. Furthermore, let us define

$$K(\boldsymbol{\theta}) := A(\boldsymbol{\theta})^T R_N^{-1} A(\boldsymbol{\theta}_0), \qquad (18)$$

and

$$Y_r(\boldsymbol{\theta}) := K(\boldsymbol{\theta})^T P_{\mathcal{R}(G(\boldsymbol{\theta})_r)} G(\boldsymbol{\theta})^{-1} K(\boldsymbol{\theta}).$$
(19)

Then,

$$\max_{\boldsymbol{\theta}} \left\{ \operatorname{tr} \{ Y_r(\boldsymbol{\theta}) \} \right\} = \operatorname{tr} \{ Y_r(\boldsymbol{\theta}_0) \} = \operatorname{tr} \{ P_{\mathcal{R}(G(\boldsymbol{\theta}_0)_r)} G(\boldsymbol{\theta}_0) \},$$
(20)

where θ_0 are the true source locations. Moreover, we have

$$\max_{\boldsymbol{\sigma}} \operatorname{RR-MAI}_{\tau_l}(\boldsymbol{\theta}, r) = \operatorname{RR-MAI}_{\tau_l}(\boldsymbol{\theta}_0, r) \qquad (21)$$

if

$$\max_{\boldsymbol{\theta}} \{ \operatorname{tr} \{ R_{\boldsymbol{\varrho}} Y_r(\boldsymbol{\theta}) \} \} = \operatorname{tr} \{ R_{\boldsymbol{\varrho}} P_{\mathcal{R}(G(\boldsymbol{\theta}_0)_r)} G(\boldsymbol{\theta}_0) \}.$$
(22)

A few comments regarding Proposition 1 are in place here. Firstly, it should be noted that $Y_r(\theta)$ in (19) cannot be used as a source localizer function on its own, as it explicitly depends on the unknown θ_0 via $K(\theta)$ in (18). Secondly, we note that, for $R_{\mathbf{Q}}$ of the form (12), the fact that RR-MAI_{T1}(θ, r) achieves its maximum for $\theta = \theta_0$ follows from Proposition 1, as in such a case (22) is satisfied immediately from (20). However, this result does not extend directly to an arbitrary positive definite $R_{\mathbf{Q}}$, due to the fact that $P_{\mathcal{R}(G(\theta_0)_r)}G(\theta_0) - Y_r(\theta)$ is not in general positive semidefinite, even if it corresponds to the difference of two positive semidefinite matrices. Nevertheless, the simulations we have conducted to date indicate that the less correlated the sources are, the lower the rank r_0 may be chosen such that (22) holds for all rank constraints $r \in \{r_0, \ldots, l\}$. This suggests a natural direction towards the generalization of Theorem 2 for arbitrary positive definite $R_{\mathbf{Q}}$.

Furthermore, based on the above considerations, we take the liberty of proposing another reduced-rank activity index:

RR-MAI₁₂(
$$\boldsymbol{\theta}, r$$
) := tr{ $G(\boldsymbol{\theta})P_{\mathcal{R}(S(\boldsymbol{\theta})r)}S(\boldsymbol{\theta})^{-1}$ } - r, (23)

for $1 \le r \le 3l$. This alternative index takes advantage of the result in (14) at $\theta = \theta_0$ for any positive definite matrix $R_{\mathbf{Q}}$. In this sense, (23) can be seen as a natural generalization of (11). Note that a similar proof of Theorem 2 for the RR-MAI_{T2}(θ, r) in (23) is currently being developed.

4. NUMERICAL EXAMPLES

We consider the case of estimating the position of l = 3 dipole sources through the MAI and RR-MAI indices previously described. Let us assume that the dipole source components are allowed to change in time as $q_i(t) = \sin(i\pi(t/150 - 1))$, for $i = 1, 2, \ldots, 9$, and $t = 0, 1, \ldots, 300$. Then, Q is defined as

$$Q = \begin{bmatrix} q_1(0) & \dots & q_1(300) \\ \vdots & & \vdots \\ q_9(0) & \dots & q_9(300) \end{bmatrix}.$$
 (24)

Note that, under these conditions, $R_{\mathbf{Q}} = I$.

MEG measurements were generated using the Helsinki BEM library [14] with a head model composed by three tessellated meshes which were nested one inside the other in order to approximate the geometry of the scalp, skull, and brain. Such head model was created based on the anatomical information of "Subject # 1" of the MEG-SIM portal, which is a repository that contains an extensive series of real and simulated MEG measurements for testing purposes [15]. In particular, the volume modeling the brain was constructed with 11520 triangles, and it is shown in Figure 1. There, we show the centroids of the triangles under which the dipole sources were located, as well as the centroids of 100 neighboring triangles which define a region-of-interest (ROI) around the sources. The large number of triangles used to approximate the head's anatomy in BEM guarantees that the modeling errors are negligible.

Next, the three dipole sources were located 2 mm below their corresponding triangle (i.e., going downward in the normal direction to the surface). Since the distances from the dipoles to the surface are not much larger than the length of the triangles sides, the MEG data generated through the Helsinki BEM library can be considered to be a very close approximation of real MEG measurements. In



Fig. 1. Tessellated mesh of the brain showing the dipoles' positions (in red) and the ROI around them (in blue).

particular, the MEG data corresponding to the simultaneous activation of the three sources were generated using an array of m = 275 magnetometers with the spatial distribution of the VSM MedTech MEG system considered in the MEG-SIM portal. Finally, uncorrelated (in time and space) random noise $N \sim \mathcal{N}(0, \sigma^2)$ was added to the measurements, and the signal-to-noise ratio (SNR) was defined as $\text{SNR} = 10 \log_{10} || AQ ||_F / || N ||_F$ in decibels (dB), where $|| \cdot ||_F$ denotes the Frobenius norm of the matrix. Different values of σ^2 were set in order to obtain SNR levels of 3, 6, and 9 dB.

Under these conditions, we computed the indices $MAI(\theta)$, RR-MAI₁₁($\boldsymbol{\theta}, r$), and RR-MAI₁₂($\boldsymbol{\theta}, r$), for $\boldsymbol{\theta}$ corresponding to a combination of three positions chosen from the centroids in the ROI, but actually located 2, 4, and 6 mm be-low the surface. Therefore, we evaluated each index for $\binom{100}{3} \times 3 = 485100$ possible values of θ . Furthermore, we tested our proposed reduced-rank indices for different rank values. In all cases, the estimated positions (denoted by θ) were taken as the value of θ for which the maximum value of the corresponding index was achieved. Then, the total error in the estimation was computed as the sum of the minimum distances from each of the real to the estimated positions. All calculations were repeated for 100 independent realizations of the noise, then the mean error was obtained. The results of this exhaustive evaluation process are shown in Figure 2. In addition, we performed two-sample t-tests between the errors of MAI and RR-MAI for different rank values (but same SNR) in order to establish if an improvement in the performance was indeed achieved. Our results show that significant improvement can be obtained through the RR-MAI estimates with a considerable rank reduction, specially for the case of low SNR. Note that the results only show a range of rank values for each of the two different types of the RR-MAI, but other values were explored as well.

5. CONCLUDING REMARKS

We proposed a novel reduced-rank multi-source activity index for robust source localization in the case of ill-conditioned EEG/MEG forward models. Such ill-conditioning may arise as a result of closely positioned sources and low SNR. Our



(a) MAI($\boldsymbol{\theta}$) compared against RR-MAI_{T1}($\boldsymbol{\theta}, r$)



(b) MAI($\boldsymbol{\theta}$) compared against RR-MAI_{T2}($\boldsymbol{\theta}, r$)

Fig. 2. Mean localization errors as a function of SNR and rank. The vertical lines on top of the bars indicate the (\pm) standard error, while horizontal lines indicate those groups with significant differences (*p < 0.05, **p < 0.01).

numerical examples showed that the reduced-rank neural indices offered significant improvement in the performance in comparison to the corresponding full-rank index, which itself allows for numerically efficient procedure to determine the number of active sources. Future work will focus on providing further theoretical insight for the proposed activity indices, as well as deriving an automated rank-selection criterion, and introducing a computationally efficient method for iterative source localization. The proposed method shall be also validated against a range of other source localization methods on a real EEG and MEG data.

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