A STUDY ON THE STATISTICAL MODELING OF FADING AND ITS EFFECTS ON SYSTEM PERFORMANCE USING SIRP AND SDP METHODS

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ABSTRACT

Many statistical models (mainly due to physical propagation studies and field measurements) have been proposed to characterize the statistics of the NLOS flat-fading wireless communication channel. There appears to have no known statistical theory on the formation of many of these fading envelope statistics. We offer the statistical modeling of fading envelope statistics under the NLOS assumption based on the SIRP representation. In practice, these channel statistics cannot be modeled with certainty. We advocated the use of moment-bound theory to obtain bounds on the system performance metrics. In this paper, we combine the theory of moment bounds with that of SIRP, and the use of the SDP optimization method to show how sharper bounds of the error rate and ergodic channel capacity of a communication system can be computed.

Index Terms— Fading distribution; Spherically-invariant random process; Semi-definite programming; Moment bound theory; performance bounds of BPSK systems.

1. INTRODUCTION

Most wireless transmission systems are operating in Non-Line-Of-Sight (NLOS) scenarios, where the received waveforms do not have direct path components from base stations (BS) to receivers. The simplest fading model for the NLOS scenario follows from the Central Limit Theorem, where large number of small-valued multipath signals in both the I-component and the Q-component of the received waveform are assumed to follow a Gaussian distribution. Thus, the envelope of the received fading signal becomes Rayleigh distributed. In reality, extensive field measured data for outdoor and indoor scenarios have shown the NLOS fading signal envelopes have diverse statistical characterizations such as Weibull, Nakagami-m, Gamma, etc. distributions (which are all examples of a generalized Gamma distribution). These statistical distributions can lead to communication system performances significantly worse than those provided by the simple Rayleigh envelope model. To analyze a set of distributions which are far more general than the Rayleigh, a model is proposed based on the replacement of the Gaussian-process model with that of Spherically Invariant Random Processes (SIRPs). The combination of SIRPs with the Fox *H*-function yields an exceedingly general *parametric* model which encompasses, as special cases, the most commonly used fading-envelope generalized Gamma distributions. In Section 2, we consider a concise summary of the Spherically Invariant Random Process (SIRP) method [1, 2] for modeling the envelope of the fading signals. Specifically, we show the distribution of the fading envelope denoted by the random variable (r.v.) X can be modeled as the product of a nonnegative-valued r.v. V (with a pdf $f_V(v)$) and a Rayleigh distributed r.v. R. In Section 3, we formulate the performance evaluation of a wireless communication system with uncertain fading distribution as a moment-bounds problem. Using the results in Section 2, we further transform the problem into a semi-definite program (SDP), which can be efficiently solved. In Section 4, we consider the minimum and maximum of the bit error rate (BER) P_b of a binary phase-shift-keyed (BPSK) wireless fading communication system under different SIRP fading scenarios. Explicit solutions to these minimum and maximum values of BER's are compared to the explicit evaluations of BER's for different SIRP cases in Section 2. A brief conclusion is given in Section 5.

2. SPHERICALLY INVARIANT RANDOM PROCESS

The SIRP $\{X(t), -\infty < t < \infty\}$'s nth order pdf is

$$p_{\boldsymbol{X}}(\boldsymbol{x}) = C_n \int_0^\infty \frac{1}{v} \mathrm{e}^{-(1/2)(\boldsymbol{x}-\boldsymbol{\mu})^T (v^2 \boldsymbol{\Sigma})^{-1} (\boldsymbol{x}-\boldsymbol{\mu})} f_V(v)\, dv, \quad (1)$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $C_n = (2\pi)^{-n/2} |\boldsymbol{\Sigma}|^{-1/2}$ is a normalization constant, $\boldsymbol{\mu}$ is the mean vector, and $\boldsymbol{\Sigma}$ is the positive definite covariance matrix. Eq.(1) shows that the nth order pdf of an

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SIRP is the statistical average of the nth order pdf of a Gaussian process with an arbitrary nonnegative-valued univariate r.v. V whose pdf is $f_V(v)$. This means that SIRP processes X have the simple interpretation of being equivalent to $\{X(t) = VY(t), -\infty < t < \infty\}$, where $\{Y(t), -\infty < t < \infty\}$ is a Gaussian process independent of V. Thus, SIRP processes generalize Gaussian processes.

2.1. SIRP and NLOS Fading Statistics

Consider now narrowband processes and their envelopes. A narrowband Gaussian process can be expressed as $Y(t) = Y_I(t)\cos(2\pi ft) - Y_Q(t)\sin(2\pi ft)$, where $Y_I(t)$ and $Y_Q(t)$ are two zero-mean independent low-pass Gaussian processes. Its envelope $R_Y(t) = (Y_I(t)^2 + Y_Q(t)^2)^{1/2}$, has a Rayleigh pdf (without loss of generality, throughout this paper we assume $\mathbf{E}(R_Y^2) = 1$). For SIRPs, we have $\{X(t) = VY(t), -\infty < t < \infty\}$. Thus, $X(t)$ is also a zero-mean narrowband process whose envelope is $R_X(t) = (X_I(t)^2 + X_Q(t)^2)^{1/2} = ((VY_I(t))^2 + (VY_Q(t))^2)^{1/2} = VR_Y(t)$, where $R_Y(t)$, the envelope of a Gaussian process, has a Rayleigh pdf. To simplify our notation, we suppress the variable t and denote the original envelope of the Gaussian process by R, and the fading SIRP envelope by X. Thus,

$$X = VR, \tag{2}$$

where V is the same nonnegative-valued univariate r.v. as above having a pdf $f_V(v)$.

From elementary probability theory, if V and R in (2) are two independent nonegative-valued univariate r.v.'s, then the pdf of X satisfies $f_X(x) = \int_0^\infty (1/v) f_R(x/v) f_V(v)\,dv, \quad 0 < x < \infty$, showing that the pdf of X is the random mixture of $f_R(r)$ with mixing distribution $f_V(v)$.

2.2. Parametrizing the pdf of X: Fox H-function

For the SIRP model to be useful to characterize the fading channel statistic for the NLOS scenario, we must be able to show that for any of the known $f_X(x)$ fading envelope pdfs (e.g., Weibull, Nakagami-m, etc.) and $f_R(r)$ being a Rayleigh pdf, there must be $f_V(v)$ pdfs satisfying (2).

The Mellin transform [3] $F(s)$ of any $f(x)$ univariate pdf defined on $(0, \infty)$ is given by

$$F(s) = \mathcal{M}\{f(x)\} = \int_0^\infty f(x) x^{s-1} dx = E\{X^{s-1}\}. \tag{3}$$

Raising both sides of (2) to the $(s-1)$ power and taking expectations after noting that V and R are independent, we see that (3) yields

$$\mathcal{M}\{f_X(x)\} = \mathcal{M}\{f_V(x)\}\mathcal{M}\{f_R(x)\}. \tag{4}$$

We denote the Mellin transform of $f_X(x)$, $f_V(x)$, and $f_R(x)$ by $F_X(s)$, $F_V(s)$, and $F_R(s)$, respectively. Then

$$F_V(s) = \mathcal{M}\{f_V(x)\} = \frac{\mathcal{M}\{f_X(x)\}}{\mathcal{M}\{f_R(x)\}} = \frac{F_X(s)}{F_R(s)}, \tag{5}$$

and hence

$$f_V(x) = \mathcal{M}^{-1}\left\{\frac{\mathcal{M}\{f_X(x)\}}{\mathcal{M}\{f_R(x)\}}\right\} = \mathcal{M}^{-1}\left\{\frac{F_X(s)}{F_R(s)}\right\}. \tag{6}$$

The Mellin transform of $f_R(x)$ is well-known, and Mellin transforms of $f_X(x)$ are also known for many classes of NLOS envelope pdfs. However, the inverse Mellin transform of the ratio on the two terms on the RHS of (6) may not be readily obtainable in closed form. However, by using the Fox H-function representation [4] of all these pdfs, $f_V(x)$ can be found explicitly [2, 5].

2.3. System Performance Under SIRP Fading

In wireless communications, it is well known that the fading statistics can significantly affect the performance of the systems. Consider coherent detection of a BPSK system over a frequency-flat slow SIRP fading channel modeled by

$$Z = Xs + n, \tag{7}$$

where $s \in \{\pm\sqrt{E_b}\}$ is the transmitted symbol with energy E_b, the noise $n \overset{d}{=} \mathcal{N}(0, N_0/2)$ is Gaussian with variance $N_0/2$, and X is the envelope of the SIRP fading channel. Let $X = VR$ be the SIRP decomposition. The bit error probability (BER) given V can be written as

$$P_{b_\text{Ray}}(V) = \frac{1}{2}\left(1 - \sqrt{\frac{V^2\overline{\gamma}}{1 + V^2\overline{\gamma}}}\right), \tag{8}$$

where $\overline{\gamma} = E_b/N_0$ is the average signal-to-noise ratio (SNR). The average BER is given by

$$P_b = \int_0^\infty P_{b_\text{Ray}}(v^2\overline{\gamma}) f_V(v)\,dv. \tag{9}$$

3. GENERALIZED MOMENT PROBLEM

Consider a performance metric $\phi(X)$ such as BER or channel capacity of a wireless communication system, which depends on the fading channel envelope X. When the distribution $f_X(x)$ of X is completely known, we can evaluate the expected performance $\mathbf{E}(\phi(X))$ by numerical integration or Monte Carlo simulation. However, in many situations the distribution of X is unknown and we only have limited knowledge about it. Since we cannot evaluate the expected value of $\phi(X)$ directly, we seek the bounds $L \le \mathbf{E}(\phi(X)) \le U$ so that they are consistent with our prior knowledge of X. The problem of finding the sharpest bounds is called the Generalized Moment Problem (GMP).

The tightness of the bounds depends on how much we know about X. The more information we have, the tighter the bounds will be. We may model the pdf $f_X(x) \in \mathcal{P}$ comes from some "uncertainty set" $\mathcal{P}$ of distributions. We may also provide the range of the moments of X. In this paper, we will consider $\mathcal{P}$ to be the class of SIRP fading envelopes, as it provides a unified framework for all well-known fading distributions. Mathematically, the GMP has the form

$$\inf_{X \sim \mathcal{P}} \quad \mathbf{E}(\phi(X)) \tag{10}$$

$$\text{s.t.} \quad \mathbf{E}(\boldsymbol{f}(X)) = \boldsymbol{b} \tag{11}$$

$$\mathbf{E}(\boldsymbol{g}(X)) \leq \boldsymbol{c}, \tag{12}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector-valued functions known as the generalized moment functions, and $\boldsymbol{b}$, $\boldsymbol{c}$ are constant vectors. The notation $X \sim \mathcal{P}$ means $f_X(x) \in \mathcal{P}$.

3.1. GMP with SIRP Fading

We cannot solve (10) directly because the constraint $X \sim \mathcal{P}$ is in an intractable form. Instead, we define $\Phi(V) = \mathbf{E}(\phi(X) \mid V)$, $\boldsymbol{F}(V) = \mathbf{E}(\boldsymbol{f}(X) \mid V)$, $\boldsymbol{G}(V) = \mathbf{E}(\boldsymbol{g}(X) \mid V)$ and consider the following GMP

$$\inf_{V \geq 0} \quad \mathbf{E}(\Phi(V)) \tag{13}$$

$$\text{s.t.} \quad \mathbf{E}(\boldsymbol{F}(V)) = \boldsymbol{b}, \tag{14}$$

$$\mathbf{E}(\boldsymbol{G}(V)) \leq \boldsymbol{c}. \tag{15}$$

Because of the SIRP decomposition (2), it is not difficult to see that (10) and (13) are equivalent. With this technique, we transformed the difficult constraint $X \sim \mathcal{P}$ into a Stieltjes type problem and is efficiently solvable by SDP solvers [6].

4. PERFORMANCE BOUNDS OF BPSK

Arguably the most important performance metrics for wireless communication systems are the bit error rate and the channel capacity. In this section, we use the BPSK system to demonstrate how to find sharp performance bounds using the GMP reformulation (13).

4.1. Bounds of BER with SIRP and SNR constraints

The instantaneous bit error rate of a BPSK system is given by

$$P_{\text{e}} = Q(\sqrt{2X^2\overline{\gamma}}). \tag{16}$$

Condition on V, the BER is given by (8). Using the technique in section 3.1, we can reformulate the GMP as

$$\inf_{V \sim \mathcal{M}^+(\mathbb{R}_+)} \quad P_{b_{\text{Ray}}}(V) \tag{17}$$

$$\text{s.t.} \quad \overline{\gamma}\,\mathbf{E}(V^2) = \text{SNR}. \tag{18}$$

The problem can be transformed into an SDP and hence efficiently solvable by the freely available tools like cvx [7] and SeDuMi [8].

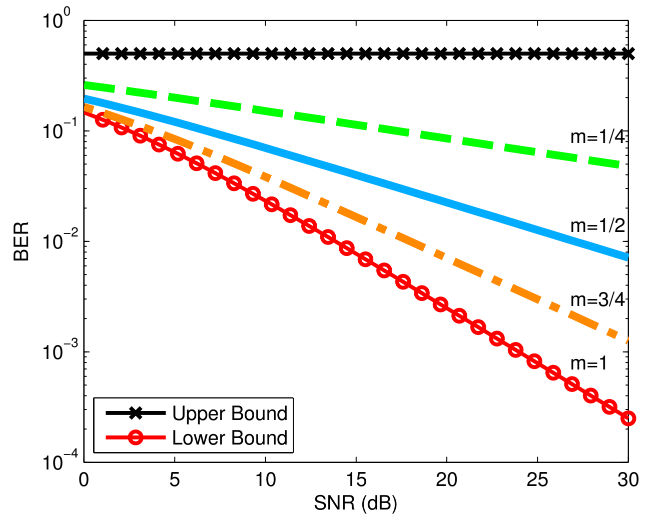

Fig. 1: The BER performance bounds for BPSK systems under uncertain SIRP fading statistic. The value m refers to the corresponding Nakagami-m fading case. Note that for $m = 1$ Nakagami-m becomes Rayleigh, and it achieves the BER lower bound.

We compare the bounds with Nakagami-m fading channel in Fig. 1. The pdf of Nakagami-m is given by

$$f_X(x) = \frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right). \tag{19}$$

Using Fox H-function technique in section 2.2, the mixing distribution of Nakagami-m fading envelope is found to be

$$f_V(v) = \frac{2(\frac{m}{\Omega})^m}{\Gamma(m)\Gamma(1-m)} v^{2m-1}(1 - \frac{m}{\Omega}v^2)^{-m}, \tag{20}$$

for $0 < v < \sqrt{\Omega/m}$ and $0 < m \leq 1$. See Fig. 2. Note that Nakagami-m distribution is defined for $m \geq 1/2$. Nevertheless, the pdf and its mixing distribution f_V is still well-defined for $m = 1/4$.

Observe that as m decrease, the corresponding mixing distribution will have more mass sitting around $V = 0$. This coincides with the general interpretation of the m parameter as the "severity" of Nakagami-m fading. The smaller m is, the worse the fading will be. In fact, as $m \to 0$, almost all the mass will be concentrated at 0 and an infinitesimal mass will be located at infinity to satisfy the SNR constraint.

On the other hand, the best SIRP fading case is when V is deterministic, namely no mixing. The fading envelope becomes Rayleigh.

4.2. Capacity bound with non-Rayleigh constraint

Another important performance metric for wireless communication systems is the channel capacity. The ergodic channel

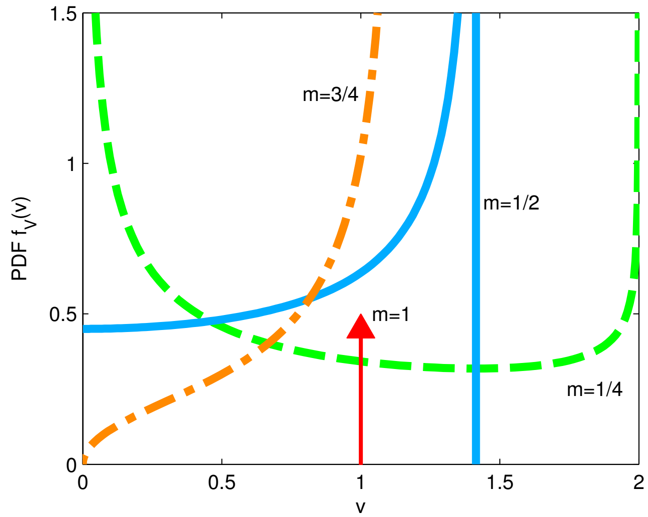

Fig. 2: The mixing distribution $f_V(v)$ of Nakagami-m fading envelope. When $m = 1$ Nakagami-m becomes Rayleigh, hence $f_V(v)$ becomes a delta function, indicating that there is no mixing. The case $m = 1/4$ does not belong to the Nakagami-m family. Nevertheless it has well-defined mixing distribution and envelope.

capacity of the system (7) is given by

$$\mathbf{E}(C(X)) = \mathbf{E}(\log(1 + X^2\bar{\gamma})) \quad \text{bits/sec/Hz.} \tag{21}$$

We can find the capacity bounds by solving the GMP

$$\inf_{V\sim\mathcal{M}^+(\mathbb{R}_+)} \quad \mathbf{E}(\mathbf{E}(C(X) \mid V)) \tag{22}$$

$$\text{s.t.} \quad \bar{\gamma}\,\mathbf{E}(V^2) = \text{SNR}, \tag{23}$$

$$\text{Var}\,(V) = \sigma_V^2. \tag{24}$$

Note that to further tighten the bound, we impose a variance constraint (24) on V. This prevents the mixing distribution from degenerating to a delta function, and the resulting SIRP fading statistic will be non-Rayleigh. Hence we refer to this constraint as the non-Rayleigh constraint.

We compare the capacity bounds with Nakagami-m fading model with $m = 1/4$ in Fig. 3. We select the variance σ_V^2 in (24) according to the mixing distribution (20) so that it is consistent with the case $m = 1/4$.

Without fading, the capacity is given by $\log(1 + \text{SNR})$ (marked by AWGN). Without the non-Rayleigh constraint (24), the capacity upper bound will match the Rayleigh curve.

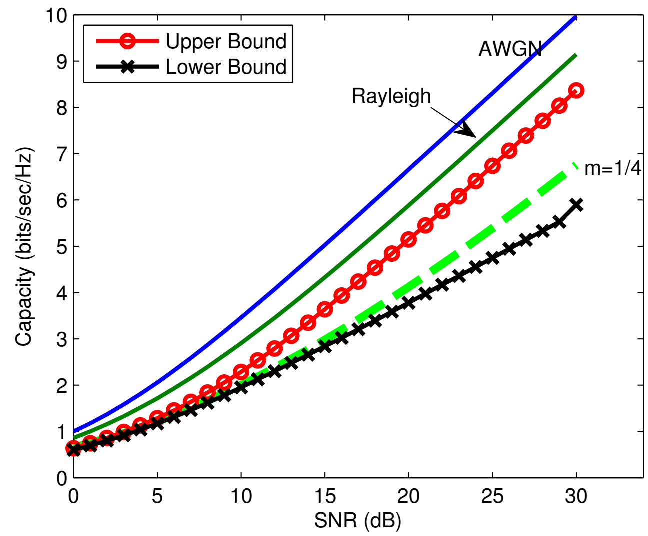

Fig. 3: The ergodic channel capacity of (7). With the non-Rayleigh constraint (24) we obtained tighter bounds that include the Nakagami-m fading model with $m = 1/4$ as a special case.

5. CONCLUSION

In this paper, we first introduced the SIRP model to characterize a broad class of fading channel envelope statistics. The resulting representation allows us to explicitly evaluate the upper and lower bounds of the BER and ergodic capacity using the SDP method. Explicit evaluation of four cases of the Nakagami-m fading envelope scenarios are shown to be consistent with the bounds. From the above (and extensive prior known) results, severe fading can cause significant loss of system performance for a single transmitter/receiver system. Thus, diversity, MIMO, etc., schemes have been proposed to mitigate these severe fading problems.

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