DISTANCE ESTIMATION BASED ON PHASE DETECTION WITH ROBUST CHINESE REMAINDER THEOREM

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ABSTRACT

Distance estimation using multifrequency phases measurement is a common practice in many areas of engineering. This method brings in the issue of phase ambiguity. Some Chinese remainder theorem (CRT) based phase unwrapping algorithms have been suggested to solve the problem, however, these algorithms either have high complexity or lack precision. In this paper, we propose an efficient algorithm to reconstruct the unknown distance from the contaminated wrapped phases. The proposed method can be separated into two stages. The first stage is to obtain the optimal estimate of the common remainder which is significant to the estimation. In the second stage, the indefinite distance is estimated by using the extended CRT. Simulations test the validity of the proposed algorithm.

Index Terms— Distance estimation, Phase ambiguity, Chinese remainder theorem (CRT), Robustness

1. INTRODUCTION

The localization of nodes is very important in the application of wireless sensor network, such as environmental monitoring, health care, structural monitoring and military surveillance [1]-[2]. In most range-based localization methods, the phase detection based ranging methods have the advantage of high precision and long range simultaneously [3]-[9]. Since it locates the nodes which are based on the phase measurement, it inevitably introduces ambiguity. To be clear, the phase measured by the nodes is periodic, so the measured phase is the residue wrapped by 2π, and the integer information is ignored.

In order to eliminate the ambiguity, several methods have been developed which considered phase measurements noise. For instance, a Diophantine equation method is proposed in [10], which needs a series of phase remainders for different carrier frequencies. However, there is no efficient method proposed when the remainders have errors. In [3], a searching method is proposed to eliminate the ambiguity. But it is inefficient because the measurement accuracy is hard to ascetain. A least square phase unwrapping estimator algorithm is presented to estimate the original phase in [11]. Since the algorithm needs a special generator basis, it performs in polynomial time. A robust Chinese remainder theorem (CRT) method and its generalized version are proposed in [12]-[13], where the phase ambiguity is resolved by searching process. The computational complexity is still high when the distance to be estimated is large. The improved closed-form robust CRT is proposed in [14], which is more effective than the searching methods. Regardless of the fact that it has a closed form, the estimate is not the optimal one. In [15], a lattice based algorithm is proposed. Although it is efficient to estimate the unknown distance, it has a similar performance as the improved closed-form robust CRT.

In this paper, we propose a novel robust CRT method to deal with the above problem and this work is motivated by [14] and [16]. The ranging estimation problem based on multiple carrier frequencies is converted into the robust CRT. As common remainder is significant to the estimation, we discuss it firstly. After getting the optimal estimate, we give the estimate of the unknown distance by using the extended CRT. A sufficient condition of the ranging estimation is presented base on the proposed algorithm. Finally, the method is used to evaluate unknown distances. The effectiveness and robustness of the algorithm are demonstrated by simulations.

The remaining of this paper is organized as follows. In Section 2, we introduce the system model. In Section 3, we present a robust CRT method. Simulation results are presented in Section 4. Finally, in Section 5 this paper is ended.

2. SYSTEM MODEL

In localization applications, signal wavelength is much shorter than the distance to be measured, so distance ambiguity caused by signal phase wrapping is inevitable. In order to eliminate phase ambiguity, we use multiple carrier frequencies to measure distance. Suppose that the unknown distance to be estimated is d, and carrier wavelengths are $\lambda_1, \lambda_2, \ldots, \lambda_L$, then the distance d can be represented by the following congruence equations [4]

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$$d \equiv \frac{\phi_i}{2\pi}\lambda_i \mod \lambda_i,\ i = 1, 2, \ldots, L. \tag{1}$$

The congruence equations above can be resolved by CRT after all of the parameters are quantized to integer. Let quantization step be u. After quantization, we have

$$N \equiv r_i \mod M_i,\ i = 1, 2, \ldots, L, \tag{2}$$

where $N = \frac{d}{u}$, $r_i = \frac{\phi_i \lambda_i}{2\pi u}$, and $M_i = \frac{\lambda_i}{u}$.

According to CRT [18]-[19], N can be uniquely determined if N is less than the least common multiple of all moduli M_i. In this paper, we consider the extended CRT where all the moduli have the same common factor M ($M > 1$) and the remaining integers factorized by M are co-prime.

Let $M_i = M\Gamma_i$, $i = 1, 2, \ldots, L$, and $\Gamma_1 < \Gamma_2 < \cdots < \Gamma_L$ are co-prime integers. We denote $\Gamma = \Gamma_1\Gamma_2\cdots\Gamma_L$, $\gamma_i = \Gamma/\Gamma_i$, and $\bar{\gamma}_i$ is the modular multiplicative inverse of γ_i modulo Γ_i. Then the solution of the extended CRT is given by the following lemma.

For simplicity of presentation, the remainder of x modulo M is denoted by $\langle x \rangle_M$.

Lemma 1. *[18] If $N < M\Gamma$, then congruence equations (2) has a unique solution*

$$N = MN_0 + r^c, \tag{3}$$

where r^c and N_0 are

$$r^c = \langle r_i \rangle_M,\ i = 1, 2, \ldots, L, \tag{4}$$

and

$$N_0 = \langle \sum_{i=1}^{L} \gamma_i \bar{\gamma}_i q_i \rangle_\Gamma,\ q_i = (r_i - r^c)/M, \tag{5}$$

respectively.

Unfortunately, the phase measurements ϕ_i have errors in practice due to noise. In the presence of errors, traditional CRT is meaningless. Some searching methods were proposed in [3],[12]-[13]. However, these methods are impractical since the procedure is computationally inefficient. In the following, we give an efficient robust CRT algorithm to solve the problem.

3. ROBUST CHINESE REMAINDER THEOREM ALGORITHM

Suppose that the ith erroneous phase be

$$\hat{\phi}_i = \phi_i + \Delta\phi_i, \tag{6}$$

where $\Delta\phi_i$ is the error. Now, the question is how to robust estimate distance d from contaminated measurements $\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_L$. Equivalently, how to robust estimate N from the contaminated remainders $\hat{r}_i$, where

$$\hat{r}_i = \frac{\hat{\phi}_i \lambda_i}{2\pi u}, i = 1, 2, \ldots, L. \tag{7}$$

From Lemma 1 we can conclude that the common remainder is significant to the reconstruction. For the case of the remainders without errors, we can get it from any of them. However, this is not true when the remainders have errors. Putting $\hat{r}_i$ modulo M be $\hat{r}_i^c$, i.e.,

$$\hat{r}_i^c = \langle \hat{r}_i \rangle_M,\ i = 1, 2, \ldots, L.$$

then $\hat{r}_i^c$ may be different from each other due to the errors. Since these values are obtained by modular operation, the distances in Euclidean space are inappropriate for describing deviation of $\hat{r}_i^c$. To obtain the optimal estimate of the common remainder, we introduce a kind of circular distance as follows.

Definition 1. *For two angles α and β, the circular distance between the two angles is defined as*

$$d(\alpha, \beta) = 1 - \cos(\alpha - \beta). \tag{8}$$

It is clear that the circular distance has maximum 2 when $\alpha - \beta = 2k\pi + \pi$, while has minimum 0 when $\alpha - \beta = 2k\pi$, $k \in \mathbb{Z}$.

Putting a monotone increasing function $f(x)$ be

$$f(x) = \frac{2\pi}{M}x,\ x \in [0, M), \tag{9}$$

then we have

$$f(\hat{r}_i^c) \in [0, 2\pi),\ i = 1, 2, \ldots, L.$$

These values can be considered as angles of unit vectors. For convenience, we denote these angles as θ_i, i.e.,

$$\theta_i = \frac{2\pi}{M}\hat{r}_i^c,\ i = 1, 2, \ldots, L. \tag{10}$$

Based on the definition above, we can obtain that the summation of the deviation about variable θ is

$$D(\theta) = \sum_{i=1}^{L} \left[1 - \cos(\theta - \theta_i)\right]. \tag{11}$$

Let $\hat{\theta}$ be the angle which minimize $D(\theta)$, i.e.,

$$\hat{\theta} = \arg \min_{0 \le \theta < 2\pi} D(\theta). \tag{12}$$

Then the optimal estimate of the common remainder $\hat{r}^c$ can be estimated by

$$\hat{r}^c = \frac{\hat{\theta}}{2\pi}M. \tag{13}$$

Next, we give the optimal estimate of the common remainder from the contaminated remainders.

Theorem 1. *The optimal estimate of common remainder $\hat{r}^c$ in (13) is*

$$\hat{r}^c = \frac{M}{2\pi} Arg\left\{\sum_{i=1}^{L}\cos\theta_i + j\sum_{i=1}^{L}\sin\theta_i\right\}, \quad (14)$$

where Arg $\{\cdot\}$ denotes the principal value of the argument.

Proof. Denoting $R = \sqrt{(\sum_{i=1}^{L}\cos\theta_i)^2 + (\sum_{i=1}^{L}\sin\theta_i)^2}$, then (11) can be rewritten as

$$D(\theta) = L - R\cos(\theta - \hat{\theta}),$$

where $\hat{\theta}$ such that

$$\cos\hat{\theta} = \frac{1}{R}\sum_{i=1}^{L}\cos\theta_i, \ \sin\hat{\theta} = \frac{1}{R}\sum_{i=1}^{L}\sin\theta_i. \quad (15)$$

It follows that

$$D(\theta) = L - R + R\sin^2\left(\frac{\theta - \hat{\theta}}{2}\right).$$

Obviously, the minimum of $D(\theta)$ achieves at $\hat{\theta}$. Note that $\hat{\theta}$ in (15) equals

$$\hat{\theta} = \text{Arg}\left\{\sum_{i=1}^{L}\cos\theta_i + j\sum_{i=1}^{L}\sin\theta_i\right\}. \quad (16)$$

Combining (13) and (16), we can draw the conclusion. □

Theorem 1 gives an effective way to estimate common remainder. For a given erroneous remainders sequence $\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_L$, we can obtain the corresponding angles θ_i by (10). If we consider these angles as unit vectors $\overrightarrow{x_i}$, then the optimal estimate $\hat{\theta}$ is the angle of the resultant vector $\overrightarrow{x_1} + \overrightarrow{x_2} + \cdots + \overrightarrow{x_L}$. Thus, the optimal estimate of the common remainder $\hat{r}^c$ can be determined by (9). Consequently, we can obtain the estimate of N_0 and N by (5) and (3), respectively.

To sum up, we give the following robust CRT algorithm.

- **Step 1**: Calculate θ_i from $\hat{\phi}_i$:

$$\theta_i = \frac{2\pi}{M}\langle\frac{\hat{\phi}_i\lambda_i}{2\pi u}\rangle_M. \quad (17)$$

- **Step 2**: Calculate $\hat{r}^c$ by (14).
- **Step 3**: Calculate $\hat{N}_0$:

$$\hat{N}_0 = \langle\sum_{i=1}^{L}\gamma_i\bar{\gamma}_i\hat{q}_i\rangle_\Gamma, \quad (18)$$

  where $\hat{q}_i = \left[\frac{\hat{r}_i - \hat{r}^c}{M}\right]$, and $[\cdot]$ denotes the rounding integer operation.

- **Step 4**: Calculate $\hat{N}$ by (3).
- **Step 5**: Calculate distance $\hat{d}$:

$$\hat{d} = u\hat{N}. \quad (19)$$

Based on the robust CRT given above, we can draw the following conclusion.

Theorem 2. *Let $\tau = \max_{1\le i\le L}|\Delta\phi_i|$, and let $\lambda_{max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_L\}$. If $\tau < \frac{u\pi M}{2\lambda_{max}}$, then*

$$|\hat{d} - d| < \frac{uM}{4}, \quad (20)$$

Proof. Putting $\Delta\theta_i = \frac{2\pi}{M}\langle\frac{\Delta\phi_i\lambda_i}{2\pi u}\rangle_M$, then we have

$$|\Delta\theta_i| < \frac{\pi}{2}, \ i = 1, 2, \ldots, L. \quad (21)$$

Thus, the error between the optimal estimate $\hat{\theta}$ and the real value θ satisfies

$$\left|\hat{\theta} - \theta\right| < \frac{\pi}{2}. \quad (22)$$

It follows from (14) that

$$|\hat{r}^c - r^c| < \frac{M}{4}. \quad (23)$$

According to (7), we have

$$|\Delta r_i| = \left|\frac{\Delta\phi_i\lambda_i}{2\pi u}\right| < \frac{M}{4}. \quad (24)$$

Combining (23) and (24), we have

$$\begin{aligned} \hat{q}_i &= q_i + \left[\frac{r^c - \hat{r}^c + \Delta r_i}{M}\right] \\ &= q_i. \end{aligned} \quad (25)$$

Consequently, we obtain from (18) that $\hat{N}_0 = N_0$. Thus,

$$\begin{aligned} \left|\hat{d} - d\right| &= u\,|\hat{r}^c - r^c| \\ &< \frac{uM}{4}. \end{aligned} \quad (26)$$

□

4. SIMULATION AND ALGORITHM PERFORMANCE ANALYSIS

In the simulations, the carrier frequencies are in the range from 400 to 460 MHz. We set quantization step $u = 0.1$mm, and $M = 100$. We choose the pair-wise relative prime positive integers to be $67, 71, 73, 74$, and 75. The corresponding wavelength are $0.67, 0.71, 0.73, 0.74$, and 0.75 m, respectively. According to (19), we have the maximum unambiguous range $d_{max} = uM\Gamma_1\Gamma_2\cdots\Gamma_5 = 1.9273 \times 10^6$m. Assume that the unknown distance d is uniformly distributed in

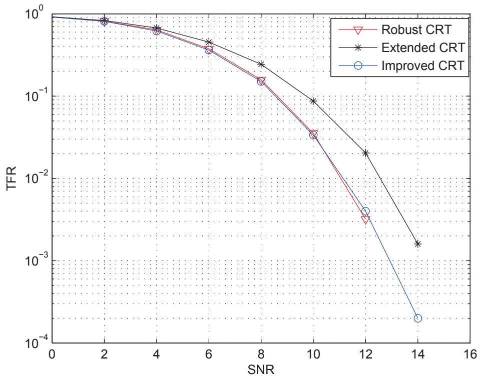

Fig. 1. TFR versus SNR.

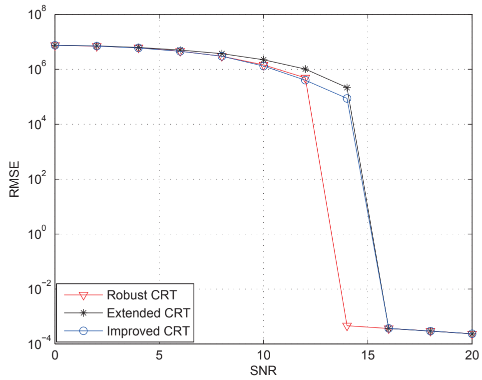

Fig. 2. RMSE versus SNR.

$(0, 1.9273 \times 10^6)$m. The number of the simulations is 10000 for each SNR.

Three algorithms are considered: the extended CRT [16], the improved closed-form robust CRT [14] and our proposed robust CRT algorithm. We take test fail rate (TFR) and root mean squared error (RMSE) as the performance measurements. In each trial, if the error of the estimate is within $\frac{uM}{4}$, the trial is passed, otherwise, the trial is failed. The RMSE is defined as

$$d_{RMSE} = \sqrt{\mathrm{E}\left\{|\hat{d} - d|^2\right\}}, \tag{27}$$

where $\mathrm{E}\{\cdot\}$ denotes the expectation.

Fig. 1 and Fig. 2 show that the proposed method has much better performance than the extended CRT. This is because our method has the optimal estimation of the common remainder, while the extended CRT estimates the common remainder on a randomly selected first remainder.

It also showns that our method has a little better performance than the improved closed-form robust CRT. When SNR is within $12 \sim 14$dB, our method has a lower threshold than the improved robust CRT. This is because the estimate of the common remainder is not optimal for the improved robust CRT algorithm. Note that our algorithm has much lower computational complexity than the improved robust CRT, which is the other main benefit of the proposed algorithm.

When TFR is down to zero, the RMSE performance is only determined by the noise level, which will not have anything to do with the rebuilding algorithm, that is why all these three algorithms share the same RMSE performance when SNR is high.

5. CONCLUSIONS

In this paper, we have proposed a robust CRT algorithm to estimate distance based on phase detection. We first convert the distance estimation problem into the robust CRT. We then give the optimal estimate of the common remainder and thus the distance to be estimated. We finally applied the proposed algorithm to estimate the unknown distances by using multiple frequencies. Simulation results demonstrate that it has much better performance than the extended CRT algorithm. It is also proved that it has a little better performance than the improved closed-form robust CRT algorithm. In addition, it has a great deal lower computational complexity.

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