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THE NATURAL SCALE OF SIGNALS: PULSE DURATION AND SUPEROSCILLATIONS

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ABSTRACT

Superoscillations are oscillations at frequencies above the maximum frequency in the signal spectrum. Signals of very small bandwidth can indeed oscillate at arbitrarily high frequencies, over arbitrarily long intervals. This work addresses the matter from a different angle, emphasizing scale and discussing the following question: can an arbitrarily narrow pulse be constructed by linearly combining arbitrarily wider pulses? The connection with superoscillations and approximation theory is also discussed.

Index Terms— Pulse width, scale, superoscillations, approximation theory, linear combinations, bandwidth

1. INTRODUCTION

Some of the heuristic arguments that are often given to explain sampling theorems, or at least to render them intuitively acceptable, are open to criticism. Consider, for example: "A bandlimited signal contains no frequencies above a certain limit, hence it cannot change to substantially new values in a time less than one half-cycle of its highest frequency" — or the following one: "a bandlimited signal has a maximum frequency component, hence it cannot oscillate faster than that".

Both statements are false. There are finite-energy signals of any given bandwidth that show arbitrarily high slew rates, or that oscillate arbitrarily fast over arbitrarily long intervals. The cost of these superoscillations in terms of signal energy and some other consequences are treated in detail in [1]. Yield-optimization is the subject of [2].

Berry [3] attributes the idea of superoscillations to Aharonov, who had told him how he had constructed (using quantum-mechanical arguments) signals that could "oscillate faster than any of their Fourier components" [4].

Despite its relatively recent history, superoscillations have already found several applications. Kempf [5] discussed them in reference to transplanckian frequencies in black hole radiation and Berry [6] in the context of a quantum billiards problem. The article [7] considers superoscillations in quantum mechanical wave functions and some unusual associated phenomena. Applications to superresolution and subwavelength imaging have been given [8, 9, 10, 11]. The connection between superdirectivity and superoscillations are explored in [12, 13] to obtain subwavelength focusing schemes capable of focusing down to 0.6 times the diffraction limit, five wavelengths away from the source. The work [14] claims the formation of a focus at 75% the spatial width of the diffraction limited sinc pulse, 4.8 wavelengths away from the source distributions. The arbitrary compression of a temporal pulse is studied in [15], which reports the design of a class of superoscillatory electromagnetic waveforms and claims a pulse compression improvement of 47% beyond the Fourier transform limit. The article [16] shows that superpositions of plane waves with random complex amplitudes and directions show naturally superoscillatory behavior. The connections between information theory and spectral geometry are explored in [17]. The article [18] deals with superoscillations in monochromatic waves in several dimensions. Other applications to physics include [19, 20] and [21], the latter on backflow, a phenomenon related to superoscillation.

Given the interest and range of the applications, it is worthwhile to look at superoscillations in depth and from more than one point of view. The goal of this paper is to discuss constructions similar to superoscillations but related to scale rather than frequency. Obviously, if $f(x)$ oscillates at a certain rate, $f(ax)$ will oscillate at a different rate or scale, determined by a. Superoscillations show that it is possible to locally approximate the behavior of $f(ax)$ but using frequencies $1/a$ smaller, in a linear way. We consider the following question: can a pulse $f(ax)$ be built by linearly combining pulses of widths at least $1/a$ greater?

2. PULSE APPROXIMATION

Superoscillations are oscillations at a rate that seems to be ruled out by the bandwidth of the signal (which imposes only an average rate, by a classical result of Tichmarsh). The average scale or frequency is determined by the bandwidth. Lo-

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cally, however, over finite but arbitrarily large intervals, the signal may behave very differently. In a superoscillating signal, the bandwidth imposes an average behavior (zero density, for example) on $f(x)$, but locally the signal may behave like $f(ax)$, with a much greather than unity. Are there ways of synthesizing pulses that behave like $f(ax)$ by linearly combining pulses like $f(x)$?

To pose a well defined question, let $f(x)$ be a Gaussian function. Is it possible to approximate $f(ax)$ by linear combinations of shifted, wider Gaussians? The equivalent question for superoscillations would be the following: is it possible to approximate a sinusoid $f(ax)$ based on linear combinations of sinusoids of frequencies smaller at least by $1/a$, and possibly phase shifted?

The question about pulses suggests the consideration of linear combinations of the translates of a single function ϕ:

$$\sum_{i=1}^{N} a_i\, \phi(t - t_i).$$

A theorem of Norbert Wiener asserts that any function $f \in L^1$ can be arbitrarily well approximated in the L^1 norm by such expressions if and only if the Fourier transform of ϕ, given by

$$\widehat{\phi}(\omega) = \int_{-\infty}^{+\infty} \phi(t) e^{-i2\pi\omega t} dt,$$

has no zeros. Wiener also showed that a similar result holds in L^2 if and only if the set of zeros of the Fourier transform of ϕ has zero measure. Proofs of these results can be found in [22, 23], for example.

The Gaussian function $g(x) = e^{-at^2}$ satisfies the hypotheses of Wiener's theorems. Therefore (picking for example the L^2 case), given any $f \in L^2$ and $\epsilon > 0$, there exists an integer N and constants $(c_i)_{1\le i\le N}$ and $(t_i)_{1\le i\le N}$ such that

$$\int_{-\infty}^{+\infty} \left| f(t) - \sum_{k=1}^{N} c_k\, g(t - t_k) \right|^2 dt < \epsilon^2.$$

The following is therefore true:

Theorem 1 *Any function f belonging to L^1 (or L^2) can be approximated arbitrarily well in the L^1 (or L^2) norm by linear combinations of translates of a Gaussian function:*

$$f \sim \sum_{i=1}^{N} a_i\, g(t - t_i).$$

This approximation property, already noted in [24], is independent of the value of the parameters of the approximating Gaussian (thus independent of a in $g(x) = e^{-at^2}$). This is somewhat surprising, given the possible extreme situations: very rapidly varying functions being approximated by very wide Gaussian functions (with a close to zero), or extremely slowly varying functions, being approximated by very narrow Gaussians (with a very large).

Corollary 1 *Any scaled version of a Gaussian function $g(ax)$ can be arbitrarily well approximated in the L^1 (or L^2) norms by linear combinations of translates of the original Gaussian $g(x)$.*

The methods used by Wiener are not constructive, and do not offer any hints on how to pick N, $(a_i)_{1\le i\le N}$ and $(t_i)_{1\le i\le N}$, in order to approximate a given function. We take a more constructive approach to Wiener's theorem for Gaussian functions, in the L^2 case. Our method also shows that it is enough to consider uniformly spaced $(t_i)_{1\le i\le N}$.

Given $f(t) = e^{-at^2}$, we consider approximations of the form

$$\sum_{k=-N}^{N} c_k\, g\left(t - \frac{k}{A}\right),$$

where $g(t) = e^{-bt^2}$, $a > b$ and A is a positive real number. This corresponds to one of the two situations mentioned above (one can be reduced to the other by taking Fourier transforms).

The square of the L^2 norm of the difference is

$$\begin{aligned} \xi &= \|f - \sum_{k=-N}^{N} c_k\, g(t - k/A)\|^2 \\ &= \left\| \widehat{f}(\omega) - \widehat{g}(\omega) \sum_{k=-N}^{N} c_k\, e^{-\mathrm{i}\frac{2\pi}{A}\omega k} \right\|^2 \\ &= \left\| \sqrt{\frac{\pi}{a}} e^{-a'\omega^2} - \sqrt{\frac{\pi}{b}} e^{-b'\omega^2} P_N(\omega) \right\|^2, \end{aligned} \tag{1}$$

where

$$a' = \frac{\pi^2}{a}, \quad b' = \frac{\pi^2}{b}, \quad P_N(\omega) = \sum_{k=-N}^{N} c_k\, e^{-\mathrm{i}\frac{2\pi}{A}\omega k}.$$

Because $a' < b'$, we may set $a' = b' - c'$, with $c' > 0$. Thus,

$$\begin{aligned} \xi &= \left\| e^{-b'\omega^2} \left[\sqrt{\frac{\pi}{a}} e^{c'\omega^2} - \sqrt{\frac{\pi}{b}} P_N(\omega) \right] \right\|^2 \\ &= \left(\int_I + \int_{\overline{I}} \right) e^{-2b'\omega^2} \left| \sqrt{\frac{\pi}{a}} e^{c'\omega^2} - \sqrt{\frac{\pi}{b}} P_N(\omega) \right|^2 d\omega, \end{aligned} \tag{2}$$

where $I = [-A/2, A/2]$. We now take $P_N(\omega)$ to be the partial sum of the Fourier series of

$$e(\omega) = \sqrt{\frac{b}{a}} e^{c'\omega^2}$$

in the interval $I = [-A/2, A/2]$:

$$P_N(\omega) = \sum_{k=-N}^{N} c_k\, e^{-\mathrm{i}\frac{2\pi}{A}\omega k} \sim \sqrt{\frac{b}{a}} e^{c'\omega^2}.$$

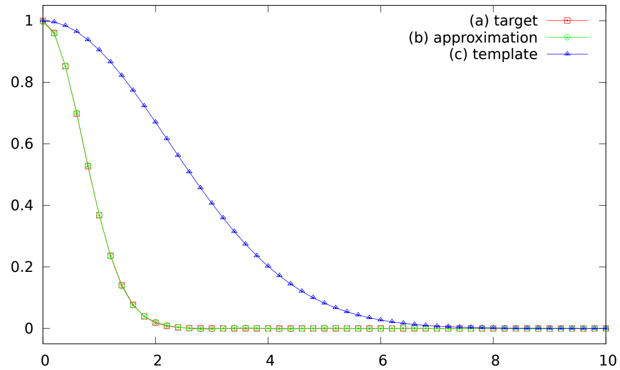

Fig. 1. The desired pulse e^{-t^2} (a) and its approximation (b), which looks almost identical at the scale of the figure (the approximation error is shown in Fig. 2). The approximation was built using linear combinations of translates of a template, the wider pulse (c), given by $e^{-0.1t^2}$.

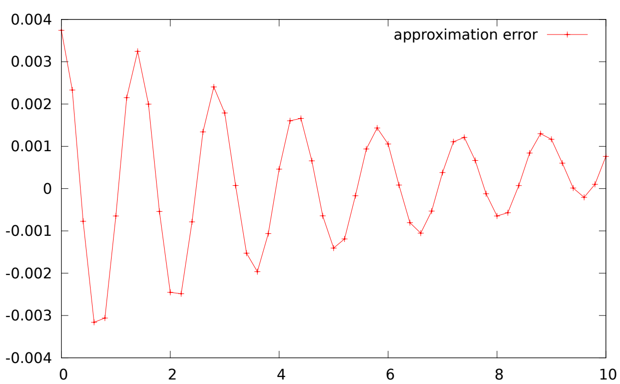

Fig. 2. The approximation error corresponding to the example given in Fig. 1.

This determines the value of the coefficients c_k:

$$c_k = \frac{1}{A}\sqrt{\frac{b}{a}} \int_I e^{c'\omega^2} e^{-\mathrm{i}\frac{2\pi}{A}k\omega} d\omega.$$

Returning to (2), we see that for fixed A the integral over I can be made small by selecting a sufficiently large N. This is possible because the partial sums $P_N(\omega)$ converge to $e(\omega)$ in the L^2 norm on the interval $I = [-A/2, A/2]$. The integral over $\overline{I}$ can also be made small due to the presence of the factor $e^{-2b'\omega^2}$, which outweights the remaining terms.

Figures 1 and 2 show the result of approximating e^{-t^2} using translates of the template $e^{-0.1t^2}$ and the corresponding error. The number of translates used was 50 and $A = 1.3$.

3. DISCUSSION AND CONSEQUENCES

No linear combination of translates of a bandlimited function can result in a wider spectrum. Thus, the approximation of a narrow pulse by translates of a wider pulse seems impossible. Gaussians have the approximation properties discussed because they are not bandlimited[1]. Their Fourier transform is nonzero for any frequency, no matter how high. This is critical. Without it, it would have been impossible to construct the example in Figs. 1–2.

The spectrum of a linear combination of the translates of a Gaussian is given by a product, as seen in equation (1). One of the factors is the Fourier transform of the original Gaussian, which is simply another Gaussian. The other factor is a function resembling a trigonometric polynomial, which is completely determined by the coefficients of the linear combination and the positions of the translates. Since the first term of the product is never zero, it is theoretically possible to shape the product as desired, using the trigonometric polynomial as a shaping function.

Numerical problems are to be expected in extreme situations, due to the size of the Fourier coefficients and the magnitude of the signals involved. The construction depends on cancellation, which is fragile. Exactly the same can be said about superoscillations. Outside the superoscillatory segment, the amplitude of the signal increases rapidly [1]. Yield-optimization can be considered [2], but as the frequency and number of superoscillations increase the numerical problems will undoubtedly be felt.

Superoscillations seem to allow the encoding of arbitrary amounts of information into an arbitrarily short segment of a low-bandwidth signal, but there is no contradiction with information theory. The amplitude of the superoscillations decreases exponentially with the length of the superoscillating segment, and the L^2 norm of a maximally superoscillatory wave function grows exponentially with the number of superoscillations. This is consistent with Shannon's capacity formula $B \log_2(1 + S/N)$, where S/N is the signal-to-noise ratio. The formula demands that the power must grow exponentially with the amount of compressed information [1].

Similarly, the synthesis of a narrow Gaussian using much wider Gaussians, which seem to require much less bandwidth, cannot be used to beat Shannon's capacity formula. Gaussians are not bandlimited, and the slight distortion imposed by a bandlimited channel would distort the construction.

Note that there is no need to restrict the templates to the Gaussian shape. We have based our discussion on Gaussians simply as a matter of convenience. Other functions (with non-vanishing Fourier transforms) would work.

The procedure described has interest even if the Fourier transform of the pulse vanishes on sets of finite or even infinite measure. In these cases, Wiener's non-constructive theorems no longer apply, and the closure of the translates cannot be all of L^2 or L^1. However, it can be a subspace (the subspace of functions with Fourier transforms vanishing on those sets).

[1] They are essentially bandlimited, though. One reviewer pointed out [25], which describes an approach to sampling that considers "soft" prefilters and includes functions such as Gaussians.

For example, the closure of the translates of the sinc function is a subspace of L^2 containing bandlimited functions.

Finally, note that the construction given above also implies that linear combinations of arbitrarily wide Gaussian functions can approximate *any* function in L^2, and not just other Gaussians. This is because convolution with a sufficiently narrow Gaussian produces an arbitrarily good approximation to the original function. The discretization of that convolution and the expression of the narrow Gaussian in terms of wider ones would lead to the result.

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