

Compressive Circulant Matrix Based Analog to Information Conversion

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Abstract—Compressive Sampling is an attractive way implementing analog to information conversion (AIC), of which the most successful hardware architecture is modulated wideband converter (MWC). Unfortunately, the MWC has high hardware complexity owing to high degree of freedom of the random waveforms constructing the measurement matrix. To reduce the complexity, in this letter, we present a novel Compressive Circulant Matrix based AIC (CCM-AIC) generating random waveforms by cyclic shift of a special sequence with unit amplitude and random phase in frequency domain. Theoretical analysis shows this scheme is optimal for signals sparse in frequency. CCM-AIC outperforms MWC and is more robust. Simulations classify the above analysis.

Index Terms—Analog to information conversion, circulant matrix, compressive sampling, joint sparse recovery

I. INTRODUCTION

ANALOG TO INFORMATION CONVERSION (AIC) [1] is motivated by the desire to sample signals directly at its information rate. The emerging area compressive sampling (CS) [2], [3] is a nascent sampling theorem asserting that the number of measurements truly required to accurately reconstruct a signal, is, in fact, independent of dimensionality and instead solely proportional to the degree of underlying information. By exploiting the property of sparsity, CS indicates a way of implementing AIC. Until now, thousands of papers concerning with CS have been published, unfortunately, however only a few of them are related to the problem of implementing AIC by use of CS. As far as know there are several hardware architectures based on CS implementing AIC have been presented, such as random demodulation [4], [5], random filtering [6], [7], random convolution [8], compressive multiplexer [9], MWC [10] and its applications, such as radar sub-Nyquist sampling [11] and wideband spectrum sensing [12], successive approximation ADC architecture based compressive sampling [13], [14] which provides a novel strategy without introducing analog mixing, of which the most attractive technique is MWC, which has been extended in the framework of Xampling [15]. MWC consist of m parallel physical channels, each comprise an independent mixing function randomly generated, a low pass filter (LPF) and a low rate commercial analog to digital converter (ADC). However, owing to the independence of mixing functions between channels, the MWC exhibits high degree of freedom, which, results in high

complexity in realistic hardware design. Unfortunately, most of the recently presented papers concerning MWC focus on the problem of joint sparse signal recovery which resolves simultaneous signal recovery from multiple measurements [16], [17]. We furthermore fully investigate the design of novel architecture with low hardware complexity in this letter.

To reduce the complexity, new structure should be introduced to reduce the degree of freedom. The Subsampled Circulant Matrix based Analogue Compressed Sensing (SCM-ACS) [18] is a novel structure recently proposed to reduce the number of channels, which relies on the Zadoff-Chu sequence popular in communication technology. The SCM-ACS employs radio frequency accurate switches and shrinks multichannel into a single one. Unfortunately, SCM-ACS suffers from accurate timing constrains. Analog switches should be configured on and off from every hundreds of picoseconds when sampling signals whose Nyquist frequency exceeding 1GHz with bandwidth occupancy exceeding hundreds of megahertz, besides, which is more difficult, the input signal is implicitly assumed to be identical every time the switches are turned on, those requirements challenge most commercial devices and are unpractical. Besides, the Zadoff-Chu sequence used by SCM-ACS is a complex-valued mathematical sequence, from the engineering perspective, essentially stressed in this letter, the way of hardware implementation of the sequence, has not been demonstrated. However, the random waveform we choose in this letter is real-valued in time domain and can be effectively implemented.

The CCM-AIC this letter presented combines the idea of random convolution and MWC. CCM-AIC generates the mixing functions by cyclic shift of a single carefully designed sequence, which significantly reduces the degree of freedom of the system, and reuse the LPF, ADC and parallel structure of the MWC. We present detail analysis of CCM-AIC and then derives the coherence bound of the sensing matrix, and finally proves the sufficient condition to guarantee exact recovery in the term of minimal number of measurements needed. Simulation results show that the CCM-AIC scheme has better success recovery performance compared to MWC. Owing to the novel structure introduced by cyclic of the sequence, the sampling process of CCM-AIC is identical to the cyclic convolution proceeding by time shifted integration, which minimize the effects of the zero-mean Gauss random noise, therefore the CCM-AIC behaves more robust than MWC especially when the signal to noise ratio (SNR) is low.

II. THE ARCHITECTURE OF CCM-AIC

A. Signal Model

It is assumed that analog signal $x(t)$, termed as multiband signal, is continuous and bandlimited, whose spectrum is supported on N disjoint bands with band width no exceeding B .

Fig.1 depicts a typical multiband signal model. The receiver sees a three band signal with random carrier frequencies which are unknown in advance.

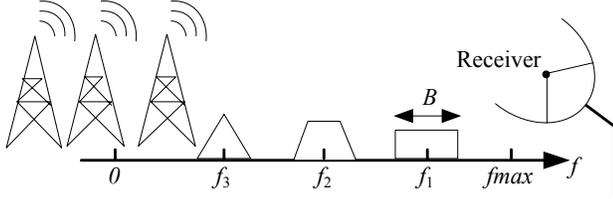


Fig. 1 Example of a typical multiband signal: three RF transmissions with different carrier frequencies

B. System Description

The architecture of CCM-AIC is presented in Fig. 2. The signal $x(t)$ enters m channels simultaneously. In the i th channel, $x(t)$ is multiplied by a mixing function which is a cyclic shift copy of $p(t)$. After mixing, the spectrum of the signal is truncated by a low pass filter $h(t)$ with cutoff frequency $f_s/2$ and then the filtered signal can be sampled at a rate f_s sufficiently low compared to the bandwidth of the signal, in which case existing commercial devices can be used.

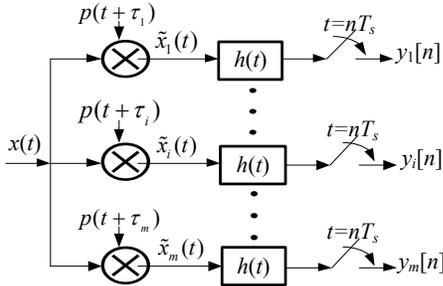


Fig 2. The system architecture of CCM-AIC

The mixing function $p(t)$ is random, T_p periodic, and real valued, whose energy spreads across the period.

$$p(t) = \alpha_k, kT_p / M \leq t \leq (k+1)T_p / M \quad (1)$$

where $0 \leq k \leq M-1$, $\alpha_{i,k} = p(t + \tau_i)$ is the i th cyclic shift copy of $p(t)$ and $1 \leq i \leq m$.

The alternating rate of the random waveform $p(t)$ should be no less than the signal's Nyquist frequency f_{NYQ} to guarantee no distortion of the original signal's spectrum and thus we have

$$M = \left\lceil \frac{f_{NYQ}}{f_p} \right\rceil = \lceil f_{NYQ} T_p \rceil \quad (2)$$

Operator $\lceil a \rceil$ denotes the minimal integer no less than a .

The time delay parameter τ_i is chosen randomly and independently and is defined by the following formula

$$\tau_i = c_i / T_p, 0 \leq c_i \leq M-1 \quad (3)$$

Denote

$$\Omega = \{c_i\}_{1 \leq i \leq m} \subset \{0, 1, \dots, M-1\} \quad (4)$$

as the support set of the time delay parameter.

C. Frequency domain analysis

According to the property of the Fourier transform, the spectrum of $\tilde{x}_i(t)$ is the f_p repetition of the spectrum of $x(t)$ so that the low pass filtered contains a mixture of the spectrum contents from the entire Nyquist frequency range of the original signal and each has unique signature that can be discerned.

So the discrete time Fourier transform of the measurements of the i th channel $y_i[n]$ can be expressed as

$$Y_i(e^{j2\pi f T_s}) = \sum_{l=-L_0}^{L_0} c_{il} X(f - l f_p) \quad (5)$$

where $X(f)$ is the Fourier transform of $x(t)$ and L_0 is selected as the minimal of

$$\frac{f_s}{2} + (L_0 + 1)f_p \geq \frac{f_{NYQ}}{2} \quad (6)$$

to make sure that all nonzero spectrum slices of $X(f)$ are contained in the summation. The number of spectrum slices is $L=2L_0+1$.

The Fourier expansion of the mixing function of the i th channel is

$$p(t + \tau_i) = \sum_{l=-\infty}^{\infty} c_{il} e^{j2\pi l t / T_p} \quad (7)$$

where the coefficient has the following form

$$\begin{aligned} c_{il} &= \frac{1}{T_p} \int_0^{T_p} p(t + \tau_i) e^{-j2\pi l t / T_p} dt \\ &= \frac{1}{T_p} \int_0^{T_p} \sum_{k=0}^{M-1} \alpha_{i,k} e^{-j2\pi l t / T_p} dt \\ &= \frac{1}{T_p} \sum_{k=0}^{M-1} \alpha_{i,k} \int_0^{T_p} e^{-j2\pi l t / T_p} dt \end{aligned} \quad (8)$$

Let $\beta = e^{-j2\pi / M}$ and

$$d_l = \frac{1}{T_p} \int_0^{T_p} e^{-j2\pi l t / T_p} dt = \begin{cases} \frac{1}{M}, l = 0 \\ \frac{1 - \beta^l}{2j\pi l}, l \neq 0 \end{cases} \quad (9)$$

Let \mathbf{A} be the discrete Fourier transform (DFT) matrix with dimension $M \times M$ and with i th column

$$\mathbf{A}_i = [\beta^{0-i}, \beta^{1-i}, \dots, \beta^{(M-1)-i}]^T \quad (10)$$

where $0 \leq i \leq M-1$. Let \mathbf{B} be a matrix with dimension $M \times L$ with columns $[\mathbf{A}_{-L_0}, \dots, \mathbf{A}_{L_0}]$, some kind of rearrangement of columns subset of matrix \mathbf{A} , \mathbf{R}_Ω be a $m \times M$ matrix selecting only rows of \mathbf{S} indexed by support set Ω , \mathbf{S} be the $M \times M$ circulant matrix with rows $\alpha_{i,k}$ and \mathbf{D} be the diagonal matrix with entries d_l . So we finally get

$$\Theta = \mathbf{R}_\Omega \mathbf{S} \mathbf{B} \mathbf{D} \quad (11)$$

Matrix Θ combines measurement matrix and representation matrix, which termed as sensing matrix. Matrix Θ has entries c_{il} . At this stage, formula (5) can be properly evaluated with known matrix Θ and measurements $y_i[n]$, and then the unknown

nonzero spectrum slices can be recovered.

The circulant matrix \mathbf{S} is defined as follows

$$\mathbf{S} = \frac{1}{M} \mathbf{A}^* \mathbf{P} \mathbf{A} \quad (12)$$

The factor $1/M$ is used to guarantee the columns of \mathbf{S} having the normalized norm. The nonzero entries of diagonal matrix \mathbf{P} are defined as follows

If M is even, we have

- 1) $q = 0, p_q = \{1, -1\}$ with equal probability
- 2) $q = M/2, p_q = \{1, -1\}$ with equal probability
- 3) $0 < q < M/2, p_q = e^{j\omega_q}$ where ω_q is the random phase, drawn uniformly from $[0, 2\pi]$.
- 4) $M/2 < q \leq M-1, p_q = p_{M-q}^*$ the conjugate of p_{M-q} .

If M is odd, we have

- 1) $q = 0, p_q = \{1, -1\}$ with equal probability
- 2) $0 < q \leq (M-1)/2, p_q = e^{j\omega_q}$ where ω_q is the random phase, drawn uniformly from $[0, 2\pi]$.
- 3) $(M-1)/2 < q \leq M-1, p_q = p_{M-q}^*$ the conjugate of p_{M-q} .

The sequence constituting the random function $p(t)$ corresponds to the first row of matrix \mathbf{S} . Instead of directly evaluating the performance of the specially designed sequence, we treat the matrix \mathbf{S} as a whole and incoherence commonly used in CS is then exploited to evaluate the performance of \mathbf{S} , equivalently, the performance of the sequence introduced by (1). This part is detailed in Section III.

We rewrite (5) as

$$y_i[n] = \sum_{l=-L_0}^{L_0} c_{il} x_l[n] \quad (13)$$

where $x_l[n]$ denotes the sampling sequence of the l th spectrum slice of $x(t)$ with rate f_s and c_{il} denotes the entries of $\mathbf{\Theta}$. Once $\mathbf{\Theta}$ and measurements are given, the unknown can be recovered with accordance to (13). For each value of n , (13) corresponds to the single measurement vector (SMV) problem aimed at recovering sparse vector from sub-samples. With multiple and consecutive values of n , (13) can be casted as the multiple measurement vector (MMV) problem, in which case the unknown sparse vectors can be jointly recovered.

III. THEORETICAL ANALYSIS

The mutual coherence is the maximum calculation of any columns. \mathbf{R}_Ω is an index matrix selecting only rows of \mathbf{S} indexed by support set Ω . \mathbf{D} is a dialog matrix and can be interpreted as multiplying each column of \mathbf{S} by some constant which will be normalized in the process of mutual coherence calculation. So \mathbf{R}_Ω and \mathbf{D} do not affect the mutual coherence of \mathbf{S} with any basis matrix either. As a result, it is sufficient to establish the signal recovery guarantee in the term of mutual coherence between \mathbf{S} and any orthogonal basis matrix.

Lemma 1 (incoherent measurement [19]): Let $\mathbf{\Phi} \in C^{m \times M}$ and $\mathbf{\Psi} \in C^{M \times M}$ be measurement matrix and representation matrix respectively. Then with high probability the measurements defined by $\mathbf{y} = \mathbf{\Phi} \mathbf{x}$ can uniquely determine the unknown K

sparse signal \mathbf{x} if

$$m \geq a\mu K \log M \quad (14)$$

where a is some positive constant and μ is the mutual coherence between $\mathbf{\Phi}$ and $\mathbf{\Psi}$. It is showed that [20]

$$1 \leq \mu \leq \sqrt{M} \quad (15)$$

Lemma 2 [8]: Let $\mathbf{\Psi}$ be an arbitrary orthogonal basis matrix, let \mathbf{S} be the measurement matrix defined by formula (12). Choose $0 < \eta < 1$, then with probability at least $1 - 2\eta$, the coherence μ will obey

$$\mu \leq \sqrt{2 \log(M^2/\eta)} \quad (16)$$

Remark: suppose $\mathbf{\Psi} = \mathbf{A}$, it is obviously that $\mu = 1$ and then

$$m \propto K \log M \quad (17)$$

Which means that the measurements guarantee reconstruction grows linearly with K and $\log M$. Formula (17) guarantees that this design of measurement matrix presented by this letter is optimal for signals sparse in frequency domain and achieves the minimal measurements.

IV. NUMERICAL SIMULATIONS

To evaluate the performance of CCM-AIC we simulate the system on test signals contaminated by zero-mean Gaussian noise. The test signal in the simulation is defined as $x(t) + n(t)$ while $x(t)$ is the multiband signal expressed as follows

$$x(t) = \sum_{i=1}^N E_i \text{sinc}(B(t - u_i)) \cos(2\pi f_i(t - u_i)) \quad (18)$$

and $n(t)$ is the Gaussian noise.

f_{NYQ} is fixed to be 1GHz. B denotes the maximum bandwidth and is of value 10MHz. The sampling frequency $f_s = f_p$, while $f_p = 10.2\text{MHz}$ is chosen a little more than the signal's maximum bandwidth B to avoid spectrum aliasing, in which case the random waveform alternates at a rate $M=197$. Variable N represents the number of active bands. E_i denotes the amplitude of each band. f_i is the carrier frequency which is chosen randomly and uniformly. u_i is the time offset. The exact values $X(f)$ takes on the support do not affect the simulation result and thus E_i and u_i are fixed in all our simulations.

Simultaneous orthogonal matching pursuit (SOMP) [21] proved a potential choice for engineers is adopted to implement signal recovery in the simulations because it is fast and easy to implement. Success is reported when all nonzero $x_l[n]$ are identified. Success rate, the same to empirical recovery rate in [10], is defined as the ratio of the number of successes and total trials. The greater the value, the better the performance. For each experiment, 500 trials are performed.

We demonstrate the results for two different cases: noise-free (where $n(t)$ equals zero) and noisy. In noise-free case, signals with different values of N are evaluated, specially, $N=2, 3, 4$. We want to compare the performance with input signals of different sparsity. In noisy case, the signal is fixed with $N=3$. We want to compare robustness when fixed input signal contaminated by Gaussian noise. The results are presented in Fig. 3 and Fig. 4.

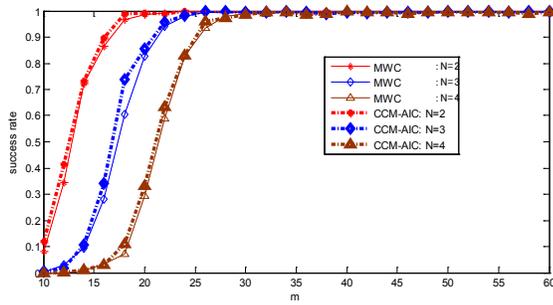


Fig. 3. Performance comparison between CCM-AIC and MWC for noise free case

It is observed that the recovery performance of the CCM-AIC is not affected by the reduction of the degree of freedom, while on the contrary this scheme has competitive recovery performance to MWC and even outperforms MWC especially when m is small. Full random measurement matrix is sufficient for the signal recovery but not necessary. The results show that the measurements CCM-AIC needed to guarantee the recovery is comparative to the MWC and even less in some cases, which provides a profile that the decrease of degree of freedom is achieved without sacrifice of recovery accuracy.

The setup for noisy simulation is: The number of measurement channels is fixed to be 24, 34 and 44 separately, in each case the value of SNR varies from 5dB to 35dB with a step 2dB. Fig. 4 depicts the results.

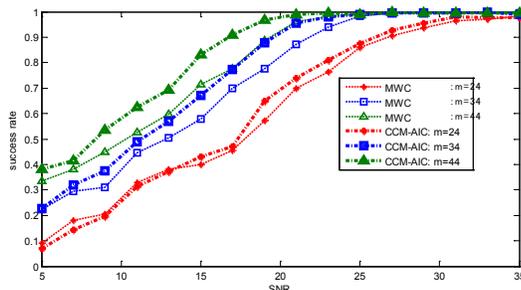


Fig. 4. Performance comparison between CCM-AIC and MWC: success rate versus SNR

Owing to the circulant structure of CCM-AIC, the measurements taken by CCM-AIC can be interpreted as the random convolution between the original signal and the mixing function. This process is a kind of integral, which minimizes the effect of the zero-mean Gaussian noise. This architecture enables more robustness against Gaussian noise which is familiar in typical applications, such as quantitative error of the analog to digital converter.

It is observed that for various measurement channels, the CCM-AIC behaves more robust than MWC. This difference in performance enlarges with the increase of the number of channels. It is conjectured that this trend will continue with larger value of the number of channels. This is based on the fact that the increase of the number of channels can be interpreted as the increase of the measurements by convolution. In reality, the sampling process inevitably involves some certain noise introduced by the low pass filter, the ADCs and so on. Fortunately, the circulant structure presented by CCM-AIC

naturally bears with the capacity of noise suppression, which enables a more broad application prospects.

V. CONCLUSION

The proposed CCM-AIC architecture reuses a single random waveform to generate mixing functions between channels by cyclic shift, which significantly reduce the degree of freedom whereas preserves better recovery performance especially when the measurements are highly contaminated with Gaussian noise commonly inevitable. Future work will elaborate on the general design principle of this kind of matrix and analysis of its robustness theoretically.

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