Modified Adaptive Basis Pursuits for Recovery of Correlated Sparse Signals

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Abstract-In Distributed Compressive Sensing (DCS), correlated sparse signals stand for an ensemble of signals characterized by presenting a sparse correlation. If one signal is known apriori, the remaining signals in the ensemble can be reconstructed using l_1 -minimization with far fewer measurements compared to separate CS reconstruction. Reconstruction of such correlated signals is possible via Modified-CS and Regularized-Modified-BP. However, these methods are greatly influenced by the support set of the known signal that includes locations irrelevant to the target signal. While recovering each signal, prior to Modified-CS or Regularized-Modified-BP, we propose an adaptation step to retain only the sparse locations significant to that signal. We call our proposed methods as Modified-Adaptive-BP and Regularized-Modified-Adaptive-BP. Theoretical guarantees and experimental results show that our proposed methods provide efficient recovery compared to that of the Modified-CS and its regularized version.

Index Terms—Distributed Compressive Sensing, Correlated sparse signals, Adaptation, Modified-Adaptive-BP, Regularized-Modified-Adaptive-BP.

I. INTRODUCTION

Compressive Sensing (CS) ensures the recovery of a sparse signal $x \in \mathbb{R}^n$ using a small number of linear observations of the form $y = Ax \in \mathbb{R}^m$, where $A \in \mathbb{R}^{m \times n}$ is a known matrix with $m \ll n$. If the signal x is S-sparse, in the sense that there are S non-zero entries in x, then exact recovery is possible through l_1 -minimization given below provided the number of measurements $m = O(S \log(n/S))$ [1] [2]:

$$\min_{\beta} \|\beta\|_{\ell_1} \text{ s.t. } A\beta = y.$$
(1)

Robustness of CS can be studied using the Restricted Isometry Property (RIP) of the sensing matrix. For all S-sparse x, a sensing matrix A is said to follow RIP if

$$(1 - \delta_S) \|x\|_{\ell_2}^2 \le \|Ax\|_{\ell_2}^2 \le (1 + \delta_S) \|x\|_{\ell_2}^2$$
(2)

Matrix A is said to be obeying RIP if the restricted isometry constant δ_S is not close to one [3]. RIP implies that all subsets of S columns of A will be nearly orthogonal to each other.

A. Motivation and Relation to Prior Work

Distributed Compressive Sensing (DCS) exploits both intrasignal and inter-signal correlation structures. DCS encodes each signal individually by projecting it onto another, incoherent, random basis and then transmits just a few of the resulting coefficients to the decoder. Therefore, a decoder

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can reconstruct all the correlated signals precisely. As the signals are sparse, one could encode and decode each of them separately using the CS framework. If one signal is known apriori, motivated by the idea of using side information in DCS, the remaining signals in the ensemble can be reconstructed using l_1 -minimization with far fewer measurements compared to seperate CS reconstruction. Reconstruction of such correlated signals is possible via Modified-CS (MOD-CS) [4] and Regularized-Modified-BP (Reg-MOD-BP) [5]. However, these methods are greatly influenced by the support set of the known signal which also includes locations irrelevant to the target signal. Therefore, we propose an adaptation step, prior to MOD-CS or Reg-MOD-BP, which tries to retain only those locations that are relevant to the target signal. We call our proposed methods as Modified-Adaptive-BP (MABP) and Regularized-Modified-Adaptive-BP (RMABP).

B. Paper Outline

The rest of this paper is organized as follows. In section 2, we discuss the correlated sparse signal model and two existing BP based recovery techniques. In section 3, we propose our modified adaptive basis pursuit methods for correlated sparse signals and the theoretical guarantees are given in section 4. In section 5, we present the simulation results of our proposed methods and compare its performance to that of the existing methods. Section 6 concludes the paper.

II. RECOVERY OF CORRELATED SPARSE SIGNALS

In this section, we introduce the signal model of correlated sparse signals and discuss two BP based CS reconstruction algorithms. In the correlated sparse signal ensemble, each signal consists of two components: a common sparse component that is present in all of the signals, and a sparse innovation component that is unique to each signal. This is similar to the Joint Sparse Model (JSM) analyzed in [6]. Let us denote the J signals in the ensemble by $x^{(t)}$, $t \in 1, 2, ..., J$. Assume that $x^{(t)} \in \mathbb{R}^n$ and it has a sparse representation in basis Ψ . The correlated sparse signal model is,

$$x^{(t)} = z + z^{(t)} \quad t \in 1, 2, ..., J$$
(3)

with $z = \Psi \Theta_z$, $\|\Theta_z\|_0 = K$ and $z^{(t)} = \Psi \Theta^{(t)}$, $\|\Theta^{(t)}\|_0 = K^{(t)}$. Let A be the i.i.d. Gaussian measurement matrix for signal $x^{(t)}$ such that

$$y^{(t)} = Ax^{(t)} \tag{4}$$

gives $m \ll n$ incoherent measurements of $x^{(t)}$. In the reconstruction part, we need to estimate every *n*-length sparse signal $x^{(t)}$ from its *m*-length measurement vector. The support of $x^{(t)}$ is denoted as *N* and it can be split as $N = T \cup \Delta \setminus \Delta_e$ where *T* is the known part of the support, Δ_e is the error in *T* and Δ is the unknown part to be estimated. The known part is either available from prior knowledge (as in static problems) or an estimate of support obtained from the known signal in the ensemble (as in time sequence problems). The error in the known part Δ_e comes from those non-zero locations in the signal with support set *T* that becomes zero in the target signal. The sparsity patterns change slowly and therefore, Δ and Δ_e are assumed to be much smaller than |T|.

A. Modified Compressive Sensing (MOD-CS)

For problems with partially known support, a CS reconstruction procedure called MOD-CS was proposed in [4]. MOD-CS aimed at estimating the sparsest possible signal estimate whose support contains T and which satisfies the data constraint (5). The convex optimization problem is formulated as

$$\min_{\beta} \|\beta_{T^c}\|_{\ell_1} \text{ s.t. } A\beta = y \tag{5}$$

where $T^c := [1:n] \setminus T$ is the complement of T. [4] applied MOD-CS for time sequence reconstruction problem where $y \equiv y^{(t)}$ and $x \equiv x^{(t)}$ with support $N \equiv N^{(t)}$. At time t = 1, CS reconstruction is applied with enough measurements to give exact recovery. At each time t > 1, the known part of the support is estimated as $T = \hat{N}^{(t-1)}$ and the signal $\hat{x}^{(t)}$ is obtained using (5). The support at time t, $\hat{N}^{(t)}$ is computed by thresholding $\hat{x}^{(t)}$ with a small threshold μ_0 . The two sufficient conditions for MOD-CS to give exact reconstruction is given in terms of the restricted isometry constant δ as,

$$\delta_{|T|+|\Delta|} < 1 \tag{6}$$

$$2\delta_{2|\Delta|} + \delta_{3|\Delta|} + \delta_{|T|} + \delta_{|T|+|\Delta|}^2 + 2\delta_{|T|+2|\Delta|}^2 < 1.$$
(7)

The conditions hold if $|\Delta|<|T|$ and $\delta_{|T|+2|\Delta|}<0.5.$ The requirement for CS is

$$2\delta_{2(|T|+|\Delta|)} + \delta_{3(|T|+|\Delta|)} < 1 \tag{8}$$

which holds when $\delta_{3(|T|+|\Delta|)} < \frac{1}{3}$. The requirements for MOD-CS is much weaker compared to that of CS [4].

B. Regularized Modified Basis Pursuit (Reg-MOD-BP)

MOD-CS puts no constraint on x_T , and therefore, if the available measurements m is small, x_T might result in a very bad reconstruction. In [5], a method was proposed to constrain x_T by bounding $(||\beta_T - \hat{x}_T||)_{\infty}$, i.e.

$$\min_{\beta} \|\beta_{T^c}\|_{\ell_1} \text{ s.t. } A\beta = y , \ (||\beta_T - \hat{x}_T||)_{\infty} \le \mu \qquad (9)$$

where \hat{x}_T is the signal estimate on *T*. Therefore, Reg-MOD-BP includes a data constraint used in CS and a second constraint to impose the closeness of *x* to \hat{x} . The sufficient conditions for Reg-MOD-BP to give exact reconstruction includes the two conditions corresponding to MOD-CS and another condition on the closeness of \hat{x}_T to x_T . Their conditions on \hat{x} seemed restrictive but their simulation results showed smaller error bound compared to that of MOD-CS.

III. PROPOSED MODIFIED ADAPTIVE BASIS PURSUITS FOR CORRELATED SPARSE SIGNALS

Reconstruction of correlated sparse signals using MOD-CS or Reg-MOD-BP with the help of known partial support has a drawback: The estimated support set T of the known signal, used for recovering a signal $x^{(t)}$ in the ensemble, not only contains the common support K but also the wrong locations. Solving (5) or (9) might lead to complication and inaccuracy (especially in the case K being closer to $K^{(t)}$). To overcome this drawback, for each signal $x^{(t)}$, we propose to include a adaptation step to drop the atoms irrelevant to $x^{(t)}$ in order to improve the speed and accuracy of the signal approximations through MOD-CS and Reg-MOD-BP. We term these methods as MABP and RMABP respectively.

For each signal $x^{(t)}$, the known part of the support is estimated as

$$T = \{ i = [1:n] : |\hat{x}^{(t-1)}(i)|^2 > \mu_0 \}.$$
 (10)

After estimation of the known support T, an adaptation procedure follows: First, the approximate coefficients ($\hat{x}_T = \Phi_T^{\dagger}.y^{(t)}$ are computed and the atoms with approximate coefficients smaller than $\mu_a. \max |\hat{x}_T|$ are deleted in order to initialize the support set more effectively. μ_a is the adaptation parameter controlling the number of atoms in the delete set Δ_a . Then the estimated support set is initialized as $\Gamma = T \setminus \Delta_a$. Then, approximate coefficients \hat{x}_{Γ} of the estimated support set are computed. Then the MABP problem is formulated as,

$$\min_{\beta} \left\| \beta_{\Gamma^c} \right\|_{\ell_1} \text{ s.t. } A\beta = y.$$
(11)

where $\Gamma^c := [1:n] \setminus \Gamma$ is the complement of Γ .

In [5], we observed that Reg-MOD-BP is a restricted framework compared to MOD-CS. Though it gives lesser reconstruction error than MOD-CS for certain range of measurements, it cannot guarantee exact reconstruction. Due to the restricted framework, it is expected that the inclusion of the adaptation step will not improve the guarantees for exact reconstruction of Reg-MOD-BP. However, for the comparison purpose, the RMABP is formulated as,

$$\min_{\beta} \|\beta_{\Gamma^c}\|_{\ell_1} \text{ s.t. } A\beta = y \text{ , } (||\beta_{\Gamma} - \hat{x}_{\Gamma}||)_{\infty} \le \mu.$$
 (12)

There are some intuitions for proposing this kind of modifications to the MOD-CS and Reg-MOD-BP. In the known support estimation step, when α is small, it chooses more atoms. Since a final refined selection step follows, a coarse selection of known support will not affect the reconstruction performance. Care should be taken while deciding the adaptation parameter. If μ_a is too close to zero, then MABP behaves like MOD-CS. On the other hand, if μ_a is too close to one, it behaves like traditional BP. The adaptation procedure has a signal approximation step using Modified Gram-Schmidt algorithm requiring [7] $O(T^2m)$ computations which is insignificant compared to the complexity of BP. Therefore, inclusion of adaptation step will not affect the overall complexity of MABP and RMABP. Algorithm 1 shows the step-by-step procedure involved in our proposed adaptation based basis pursuits for reconstructing correlated sources having partially disjoint support.

Algorithm 1 Proposed Adaptive Reconstruction of Correlated Sparse Signals

Input:

 Φ - Measurement matrix $y^{(t)}$ - Measurement vector of $x^{(t)}$ $\hat{x}^{(t-1)}$ - Previous signal (known signal) μ_0 - selection threshold μ_a - adaptation parameter **Initialization**: Estimate the known support as $T = \{i = [1:n] : |\hat{x}^{(t-1)}(i)|^2 > \mu_0\}.$ Adaptation:

Step 1: The approximate coefficients $(\hat{x}_T = \Phi_T^{\dagger}.y^{(t)})$ are computed and the atoms with approximate coefficients smaller than $\mu_a \cdot \max |\hat{x}_T|$ are dropped.

Step 2: The known part T is replaced with Γ and the signal estimate \hat{x}_{Γ}^{0} of the estimated support set Γ are estimated. **Basis Pursuit**:

Solve the convex optimization problem to obtain an unique minimizer of $x^{(t)}$

Modified Adaptive BP: $\min_{\beta} \|\beta_{\Gamma^{c}}\|_{\ell_{1}} \text{ s.t. } A\beta = y$ OR Regularized Modified Adaptive BP: $\min_{\beta} \|\beta_{\Gamma^{c}}\|_{\ell_{1}} \text{ s.t. } A\beta = y , \ (||\beta_{\Gamma} - \hat{x}_{\Gamma}||)_{\infty} \leq \mu$ End **Output:** $\hat{x}^{(t)} \text{ - Estimated coefficients}$

IV. THEORETICAL GUARANTEES FOR EXACT RECONSTRUCTION

A. Exact Reconstruction: MABP versus MOD-CS

Theorem 1: If T is the support set from known signal such that $N = T \cup \Delta \setminus \Delta_e$ and Γ is the retained support set after adaptation, then MABP recovers the target signal with a higher probability compared to the MOD-CS.

Due to adaptation, we have the retained support set $\Gamma = T \setminus \Delta_a$ such that,

$$|\Gamma| = |T| - |\Delta_a| \tag{13}$$

where Δ_a is the set of atoms dropped and $\Delta_a \subset T$. $x^{(t)}$ is the unique minimizer of MABP if,

$$\delta_{|\Gamma|+|\Delta|} < 1 \tag{14}$$

$$2\delta_{2|\Delta|} + \delta_{3|\Delta|} + \delta_{|\Gamma|} + \delta_{|\Gamma|+|\Delta|}^2 + 2\delta_{|\Gamma|+2|\Delta|}^2 < 1$$
(15)

Considering the first condition, $\delta_{|\Gamma|+|\Delta|} = \delta_{|T|-|\Delta_a|+|\Delta|}$ which implies $\delta_{|\Gamma|+|\Delta|} < \delta_{|T|+|\Delta|}$. Therefore, the first condition for exact recovery using MABP is weaker compared to that of MOD-CS. Similarly, in the second condition, replacing $|\Gamma|$ with $|T| - |\Delta_a|$ reveals the fact that the second condition also is weak compared to that of MOD-CS. Therefore, the MABP has a higher probability of exact reconstruction compared to that of the MOD-CS.

B. Exact Reconstruction: MABP versus Reg-Mod-BP

In ([5], Theorem 1), three conditions for $x^{(t)}$ being the unique minimizer of (9) is given. Along with the two conditions for $x^{(t)}$ given in (6) and (7), Reg-MOD-BP requires one additional conditioning on \hat{x}_T . Though the conditioning on \hat{x}_T seemed to be restrictive, they have a much better error bound compared to that of MOD-CS. The closeness of \hat{x}_T to $x^{(t)}$ is taken care by the second constraint in Reg-MOD-BP, whereas in MABP, the adaptation step enforces it. Simulation results in the following section show that MABP has a better probability of exact reconstruction compared to that of Reg-MOD-BP.

V. SIMULATION RESULTS

For our experiments, we generated two correlated sparse signals (as in JSM-1) of length n=256. The total sparsity of each signal is fixed as 25 with the sparsities of common and innovation parts being 20 and 5 respectively. By choosing different measurement values we performed signal recovery using MOD-CS, Reg-MOD-BP, MABP and RMABP. For each value of m, 250 independent trials are performed to obtain the average results. In each trial, an $m \times n$ Gaussian random measurement matrix is generated. For Reg-Mod-BP and RMABP, μ is fixed as 0.1. For all four methods, the initial selection threshold μ_0 is fixed as 0.0001.

A. Reconstruction Performance

First, we present the probability of exact reconstruction as a function of the number of measurements. Number of measurements were chosen from m=60 to m=100 in steps of 5. For MABP and RMABP, the adaptation parameter is set to be 0.2. If the maximum magnitude difference between the original signal and the reconstructed signal is smaller than 10^{-3} , the reconstruction is considered to be perfect. Fig.1 shows that our proposed MABP has the best probability of exact reconstruction among all four methods. For example, at m=85, MABP gives 98% probability of exact recovery whereas the other three methods recover with less than 80% probability. Recovery performance of RMABP is same as that of the Reg-MOD-BP.

Next, we present the average Mean Square Error (MSE) as a function of number of measurements. As discussed in section IV, the Reg-MOD-BP and our proposed RMABP have the best error performance but pronounced only in the less measurements region. Fig.2 shows the average MSE for different measurement values. In the region where the reconstruction occurs with higher probability, the average MSE is same for



Fig. 1. Exact reconstruction versus number of measurements



Fig. 2. Average MSE versus number of measurements

all four methods (of the order of 10^{-9} . Even in the region where Reg-MOD-BP and RMABP have the least error, MABP has much lesser MSE compared to MOD-CS.

B. Choice of the adaptation parameter μ_a

In this experiment, we show the probability of exact reconstruction as a function of number of measurements for different values of adaptation parameter μ_a . Fig.3 shows the reconstruction probability curves of MABP with μ_a ranging from 0.05 to 0.25. As can be seen, MABP gives better reconstruction when the adaptation parameter is fixed as 0.15 or 0.2. At m=85, reconstruction probability corresponding to μ_a =0.15 is 1. When μ_a is reduced from 0.25 to 0.1 in steps of 0.05, the performance of MABP improved. When μ_a is reduced below 0.1 (for μ_a =0.05), the performance started getting worser. Fig.4 shows the reconstruction probability curves of RMABP. The adaptation parameter μ_a is varied from 0.05 to 0.25. It is evident from the figure that the adaptation parameter has no significant effect on the recovery



Fig. 3. Reconstruction probability versus number of measurements for different values of μ_a - MABP



Fig. 4. Reconstruction probability versus number of measurements for different values of μ_a - RMABP

performance of RMABP. Between m=80 and m=90, RMABP gives its best reconstruction when $\mu_a=0.15$.

VI. CONCLUSION

In this work, we proposed two basis pursuits for reconstructing correlated sparse signals. Our proposed MABP method gave better probability of success compared to MOD-CS, Reg-MOD-BP and our proposed RMABP. Therefore, in the case of correlated sparse signals, the adaptation step improves the reconstruction performance of MOD-CS. As expected, our RMABP performed similar to Reg-MOD-BP. Both Reg-MOD-BP and our RMABP gave lesser reconstruction error compared to MABP only in the region where the probability of exact reconstruction is less than 0.1. We are currently trying to replace the basis pursuit in (11) with a greedy pursuit to give a faster recovery.

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