Minimum Fourier Measurements for Stable Recovery of Block Sparse Signal

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Abstract— **Model based sparse signal recovery requires fewer measurements and has attracted lots of attention recently. One prototypical sparsity model is block sparsity whose stability is guaranteed from block restricted isometry property (RIP). However, the existing block RIP methods in the ℓ_2 norm space only consider Gaussian measurement case. In this paper, we extend the block RIP to the Fourier measurement case and demonstrate that the minimum number of measurements satisfying block RIP is as low as $O\left(sd\log q\log(sd\log q)\log^2 s\right)$, where d is the block size, s represents the block sparsity, and N is the length of unknowns satisfying $N = qd$ for some integer q.**

I. Introduction

Compressive Sensing (CS) has attracted great amount of attention from various applications, such as wireless channel estimation, sensor networks, and optical imaging, etc. Since signals in these applications often possess certain structures, researchers have proposed many model based compressive sensing [1]–[6], where the number of essential measurements required for stable recovery can be greatly reduced by utilizing the structure information.

One popular prototypical signal model is the block sparsity, which has been extensively studied in resent years [3]–[5]. In [3] and [4], the authors applied group norm method to recover block sparse signal, and showed better recovery performance compared to traditional methods. In [5], the authors generalized [3] and [4] further by introducing the concept of block restricted isometry property (RIP) to guarantee the stable recovery of block sparse signal, where the tight Gaussian measurements bound for the block RIP is much smaller than that for standard RIP.

Though block RIP is a general framework that can adapt to any block sparse signal, the existing results mainly focus on Gaussian measurement case [1], [5]. On the other side, Fourier measurement is also widely used and certain signals in Fourier measurement system show block sparse structure too. In [6], the group sparsity method was applied to recover block sparse signal from Fourier measurements and could achieve better performance compared to traditional method.

In this paper, we target at deriving the minimum number of Fourier measurements that could guarantee the stable recovery of block sparse signal by extending the block RIP to Fourier measurement system. The minimum number of measurements is demonstrated to be $O\left(sd\log q\log(sd\log q)\log^2 s\right)$, which is much lower than the traditionally required measurements for standard RIP, say $O((\frac{rlogN}{\varepsilon^2})log(\frac{rlogN}{\varepsilon^2})log^2r)$, where d, s and r represent the block size, the block sparsity and the general sparsity respectively, ε is a certain small constant and N is the length of unknowns satisfying $N = qd$ for some integer q.

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II. Review and Problem Formulation

Restricted isometry property was first introduced by Candes and Tao in [9], which is used to offer a guarantee for stable signal recovery.

Definition 1: Let $\mathbf{\Psi}$ be an $M \times N$ measurement matrix, W be a subset of $\{1, \ldots, N\}$, and $\mathbf{\Psi}_W$ denote the $M \times |W|$ matrix that is composed of the columns of $\mathbf{\Psi}$ indexed by W. Then $\mathbf{\Psi}$ is said to have the RIP of order s if there is the smallest positive number δ_s satisfying:

$$C(1-\delta_s)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Psi}_W\mathbf{v}\|_2^2 \leq C(1+\delta_s)\|\mathbf{v}\|_2^2, \qquad (1)$$

for all set W, $|W| \leq s$, any $\mathbf{v} \in \mathbb{C}^W$ and some $C > 0$.

In [5], Eldar and Mishli extended the RIP to block sparse case, i.e., block RIP. Before we give its definition, lets us provide the definition of block sparsity model first:

Definition 2: (block sparsity) Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_q | \mathcal{L}(\mathbf{b}_1) = d_1, \ldots, \mathcal{L}(\mathbf{b}_q) = d_q\}$ be the set that divides $\{1, \ldots, N\}$ into q blocks

$$B = \{\underbrace{1\ldots d_1}_{\mathbf{b}_1}\ldots\underbrace{N-d_q+1,\ldots,N}_{\mathbf{b}_q}\}, \qquad (2)$$

where $\mathbf{b}_i$'s are defined as the corresponding items and $\mathcal{L}(\mathbf{b}_i)$ denotes the length of $\mathbf{b}_i$. Then $\mathbf{u} \in \mathbb{C}^N$ is said to be s-block sparse over B, if there exists s out of the q vectors $\{\mathbf{u}(\mathbf{b}_1), \ldots, \mathbf{u}(\mathbf{b}_q)\}$ satisfying that each of those s vectors contains at least one nonzero element, where $\mathbf{u}(\mathbf{b}_i)$ denotes the subvector of $\mathbf{u}$ indexed by block $\mathbf{b}_i$.

Remark 1: In [4] the authors introduced the concept of group norm, which is often used to solve block sparse problem [6]. Let $\mathbf{x} \in \mathbb{C}^N$ be an arbitrary vector, then the (i,p)-group norm of $\mathbf{x}$ over block $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_q | \mathcal{L}(\mathbf{b}_1) = d_1, \ldots, \mathcal{L}(\mathbf{b}_q) = d_q\}$ is defined as

$$\|\mathbf{x}\|_{i,p|B} = \sum_{j=1}^{j=q} \|\mathbf{x}(\mathbf{b}_j)\|_p^i. \qquad (3)$$

To simplify discussion, in the rest of the paper we assume every block in B has the same size d, i.e., $\mathcal{L}(\mathbf{b}_1) = \dots \mathcal{L}(\mathbf{b}_q) = d$ and assume that $N = qd$ for some integer q. Moreover, we denote the set B under this condition as $B = \{\mathbf{b}_1, \dots, \mathbf{b}_q | N = qd, \mathcal{L}(\mathbf{b}_i) = d\}$.

Definition 3: (block RIP [5]) Let $\mathbf{\Psi}$ be an $M \times N$ measurement matrix, R be a subset of $B = \{\mathbf{b}_1, \dots, \mathbf{b}_q | N = qd, \mathcal{L}(\mathbf{b}_i) = d\}$, and $\mathbf{\Psi}_R$ denote the $M \times |R|$ matrix that is composed of the columns of $\mathbf{\Psi}$ indexed by R. Then $\mathbf{\Psi}$ is said to have the block-RIP of order s, if for some number $C > 0$ there exists the smallest positive number $\delta_{s|d}$ satisfying

$$C(1 - \delta_{s|d})\|\mathbf{v}\|_2^2 \le \|\mathbf{\Psi}_R \mathbf{v}\|_2^2 \le C(1 + \delta_{s|d})\|\mathbf{v}\|_2^2, \quad (4)$$

for all $\mathbf{v}$ and for all subset $|R| \le s|d$, where $s|d$ denotes certain s blocks from the block set $\{\mathbf{b}_1, \dots, \mathbf{b}_q\}$.

In this paper, we will use the group norm based algorithm proposed in [5] to recover block sparse signal. Hence, the following second order cone programming (SOCP) problem is formulated:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_{1,2|B}, \quad \text{s.t.} \quad \mathbf{y} = \mathbf{\Psi x}. \quad \Longrightarrow \text{SOCP:} \quad (5)$$

$$\min_{\mathbf{x}, t_i} \sum_{i=1}^{q} t_i \quad \text{s.t.} \begin{cases} \mathbf{y} = \mathbf{\Psi x} \\ \|x(\mathbf{b}_i)\|_2 \le t_i, t_i \ge 0 \end{cases} \quad 1 \le i \le q, \quad (6)$$

where $\mathbf{x}$, $\mathbf{x}(\mathbf{b}_j)$ and B has the same definition as in (2) and (3) respectively, and $\mathbf{\Psi}$ is the $M \times N$ measurements as in Definition 3.

We learn from [8] that when RIP constant of $\mathbf{\Psi}$ satisfies $\delta_{3s} + 3\delta_{4s} < 2$, any signal $\mathbf{x}$ with sparsity level no bigger than s can be stably recovered from measurements $\mathbf{y} = \mathbf{\Psi x} + \mathbf{e}$, where $\mathbf{e}$ is measurement noise. For simplicity, we always replace the condition $\delta_{3s} + 3\delta_{4s} < 2$ with $\delta_{4s} \le 1/2$, since $\delta_{3s} \le \delta_{4s}$. Therefore, our main problem can be reformulated as deriving the minimum Fourier measurements satisfying $\delta_{4s|d} < 0.5$.

III. Block RIP Guarantees for Fourier Measurements

Our following results and the corresponding proofs mainly rely on the paper [7], but we have made necessary alterations to fit our block sparse case better.

Lemma 1: Suppose $\{\mathbf{x}_i\}_{1 \le i \le l}$, $l \le N$, are l vectors of length N, each of which having uniformly bounded entries, $||\mathbf{x}_i||_\infty < K$. Let R, $|R| \le s|d$, denote a subset of $B = \{\mathbf{b}_1, \dots, \mathbf{b}_q | N = qd, \mathcal{L}(\mathbf{b}_i) = d\}$ of size at most s, and $\{\epsilon_i\}_{1 \le i \le l}$ denote independent, symmetric, {-1,1}-valued random variables. Then there is

$$E \sup_{|R| \le s|d} \left\| \sum_{i=1}^{l} \epsilon_i \mathbf{x}_i^R \otimes \mathbf{x}_i^R \right\|_p \le w(l) \cdot \sup_{|R| \le s|d} \left\| \sum_{i}^{l} \mathbf{x}_i^R \otimes \mathbf{x}_i^R \right\|_p^{\frac{1}{2}}, \quad (7)$$

where $|| \cdot ||_p$ is the operator norm, and

$$w(l) = cK\sqrt{sd}\log(s)\sqrt{\log l}\sqrt{\log q} \quad (8)$$

for some number c.

Proof: Let $\beta_p^{q|d}$ denote the unit ball of the (i,p)-group norm $\|\cdot\|_{i,p|B}$ on $\mathbb{C}^N$ over q blocks $B = \{\mathbf{b}_1, \dots, \mathbf{b}_q | N = qd, \mathcal{L}(\mathbf{b}_i) = d\}$. Suppose R is the subset of B that contains $|R|$ out of q blocks of B. Further assume that β_p^R is a subset of $\beta_p^{q|d}$ whose vectors have identical support R.

Denote the expression in the left-hand side of (7) as G, and let $\{g_i\}_{1 \le i \le l}$ be the standard independent normal random variables. If we replace ϵ_i in (7) by g_i, according to [7] (see inequality (4.8) in [10] for more details) we have:

$$\begin{aligned} G &\le c_1 E \sup_{|R| \le s|d} \left\| \sum_{i=1}^{l} g_i \mathbf{x}_i^R \otimes \mathbf{x}_i^R \right\|_p \\ &= c_1 E \sup_{\substack{|R| \le s|d \\ \mathbf{x} \in \beta_2^{|R|}}} \left| \sum_{i=1}^{l} g_i < \mathbf{x}_i, \mathbf{x} >^2 \right| \\ &= c_1 E \sup_{\substack{|R| \le s|d \\ \mathbf{x} \in \beta_2^{R}}} \left| \sum_{i=1}^{l} g_i \left(\sum_{j=1}^{|R|} < \mathbf{x}_i(R_j), \mathbf{x}(R_j) > \right)^2 \right| \\ &\le c_1 E \sup_{\substack{|R| \le s|d \\ \mathbf{x} \in \beta_2^{R}}} \left| \sum_{i=1}^{l} g_i \left(\sum_{j=1}^{|R|} \|\mathbf{x}_i(R_j)\|_2 \|\mathbf{x}(R_j)\|_2 \right)^2 \right| \end{aligned} \quad (9)$$

where $< \cdot >$ denotes inner product operation, R_j represents the i-th element of set R, and $\mathbf{x}_i(R_j)$ and $\mathbf{x}(R_j)$ represent the subvector of $\mathbf{x}_i$ and $\mathbf{x}$ indexed by block R_j respectively. The last inequality in (9) results from Cauchy-Schwarz inequality.

Denote $\mathcal{B}_p^q$ and $\mathcal{B}_p^{T,q}$ as unit balls of the norm $\|\cdot\|_2$ on $\mathbb{C}^q$, where T is any subset of $\{1, \dots, q\}$ of size at most s, and each vector in $\mathcal{B}_p^{T,q}$ has the same support T. Let us define two types of vectors:

$$\hat{\mathbf{x}}_i = \{\|\mathbf{x}_i(\mathbf{b}_1)\|_2, \dots, \|x_i(\mathbf{b}_q)\|_2\}, \hat{\mathbf{x}}_i \in \mathbb{C}^q, 1 \le i \le l, \quad (10)$$

$$\hat{\mathbf{x}} = \{\|\mathbf{x}(\mathbf{b}_1)\|_2, \dots, \|\mathbf{x}(\mathbf{b}_q)\|_2\}, \mathbf{x} \in \beta_2^R, \hat{\mathbf{x}} \in \mathcal{B}_2^{T,q}. \quad (11)$$

If we denote the expression of (9) as G_1, then according to (11) and (10) we have

$$G_1 = c_1 E \sup_{\substack{|T| \le s \\ \mathbf{x} \in \mathcal{B}_2^{T,q}}} \left| \sum_{i=1}^{l} g_i < \hat{\mathbf{x}}_i, \hat{\mathbf{x}} >^2 \right|. \quad (12)$$

Similar to [7], we use Dudley's inequality [11] to bound (12) as:

$$G_1 \le c_2 \int_0^\infty \log^{0.5} N \left(\bigcup_{|T| \le s} \mathcal{B}_2^{T,q}, \sigma, u \right) du, \quad (13)$$

where $N(Z, \sigma, u)$ has the same definition as in [7], namely the minimal number of balls of radius u in pseudometric σ centered in points of z that are required to cover the set Z. The rest of proof essentially follows the step of [7], so we present them for our proof integrity. Nevertheless, the following proof is more explicit. According to [7], σ in our case has the form:

$$\sigma = [\sum_{\substack{i=1,|T|\leq s\\ \hat{\mathbf{x}},\hat{\mathbf{y}}\in\mathcal{B}_2^{T,q}}}^{l} (<\hat{\mathbf{x}}_i,\hat{\mathbf{x}}>^2 - <\hat{\mathbf{x}}_i,\hat{\mathbf{y}}>^2)]^{0.5}$$

$$\leq [\sum_{\substack{i=1,|T|\leq s\\ \hat{\mathbf{x}},\hat{\mathbf{y}}\in\mathcal{B}_2^{T,q}}}^{l} <\hat{\mathbf{x}}_i,\hat{\mathbf{x}}+\hat{\mathbf{y}}>^2 <\hat{\mathbf{x}}_i,\hat{\mathbf{x}}-\hat{\mathbf{y}}>^2]^{0.5}$$

$$\leq 2 \max_{\substack{|T|\leq s\\ \hat{\mathbf{z}}\in\mathcal{B}_2^{T,q}}} (\sum_{i=1}^{l} <\hat{\mathbf{x}}_i,\hat{\mathbf{z}}>^2)^{0.5} \max_{i\leq l} <\hat{\mathbf{x}}_i,\hat{\mathbf{x}}-\hat{\mathbf{y}}> . \quad (14)$$

Let us denote $\max_{\substack{|T|\leq s\\ \hat{\mathbf{z}}\in\mathcal{B}_2^{T,q}}} (\sum_{i=1}^{l} < \hat{\mathbf{x}}_i,\hat{\mathbf{z}} >^2)^{0.5}$ by $\mathcal{C}$. Moreover in (13), let us replace the pseudometric σ with $\|\cdot\|_X$ and replace $\bigcup_{|T|\leq s}\mathcal{B}_2^{T,q}$ with $\frac{1}{\sqrt{s}}D_2^{s,q}$, where $D_p^{s,q} = \bigcup_{|T|\leq s}\mathcal{B}_p^{T,q}$, and $\|\mathbf{x}\|_X = \max_{i\leq l}|<\hat{\mathbf{x}}_i,\hat{\mathbf{z}}>|$. Then:

$$G_1 \leq c_3\mathcal{C}\sqrt{s}\int_0^\infty log^{0.5} N\left(\frac{1}{\sqrt{s}}D_2^{s,q},\|\cdot\|,u\right)du. \quad (15)$$

Since $\frac{1}{\sqrt{r}}D_2^{s,q} \subseteq D_1^{s,q}$, we can replace $\frac{1}{\sqrt{r}}D_2^{s,q}$ in (15) with $D_1^{s,q}$. Hence, we can complete our proof if

$$\int_0^\infty \log^{\frac{1}{2}} N\left(D_1^{s,q},\|\cdot\|_X,u\right)du \leq c_4\log s\sqrt{\log q\log l} \quad (16)$$

is true.

Since, for any $q \leq i \leq l$, $\|\hat{\mathbf{x}}_i\|_\infty = \max_{1\leq i\leq q}\|x_i(d_i)\|_2 \leq \sqrt{K^2 d} = K\sqrt{d}$, the upper bound of the integral in (16) can be set as $K\sqrt{d}$. According to Lemma 3.9 in [7], for big u, $N\left(D_1^{s,q},\|\cdot\|_X,u\right)$ in our case has the follow form:

$$\text{big } u\text{:}\quad N\left(D_1^{s,q},\|\cdot\|_X,u\right) \leq (4q)^{c_5K^2d\log(l)/u^2}. \quad (17)$$

For small u, we directly quote the result of equation (3.10) in [7] as:

$$\text{small } u\text{:}\quad N\left(D_1^{s,q},\|\cdot\|_X,u\right) \leq L(s,q)\left(\frac{u+K\sqrt{d}}{u}\right)^s, \quad (18)$$

where $L(s,q) = \sum_{j=1}^{s}\binom{q}{j} \leq (c_6\frac{q}{s})^s$. Suppose A is the boundary between big u and small u and denote $K\sqrt{d}$ by J. Then there is:

$$\begin{aligned}
&\int_0^{k\sqrt{d}} \log^{0.5} N(D_1^{s,q},\|\cdot\|_X,u)du\\
&\leq \int_0^A c_6\log^{\frac{1}{2}}\left(\frac{q}{s}\frac{u+J}{u}\right)^s + \int_A^J \log^{\frac{1}{2}}(4q)^{\frac{c_5J^2\log(l)}{u^2}}du\\
&\leq c_7\int_0^A \sqrt{s\log\frac{q}{s}} + \sqrt{s\log\frac{J}{u}} + \int_A^J \frac{J\sqrt{\log l\log q}}{u}du\\
&\leq c_8\left[\sqrt{s}A(\sqrt{\log\frac{q}{s}} + \log\frac{A+J}{A}) + J\log\frac{J}{A}\sqrt{\log l}\sqrt{\log q}\right]
\end{aligned} \quad (19)$$

If we choose A as $\frac{K\sqrt{d}}{\sqrt{s}}$, the right side of the last inequality in (19) can be approximated by

$$c_8K\sqrt{d}\left(\sqrt{\log\frac{q}{s}} + \log\sqrt{s} + \log(s)\sqrt{\log l}\sqrt{\log q}\right).$$

Obviously, both $\sqrt{\log\frac{q}{s}}$ and $\log\sqrt{s}$ are ignorable compared to $\log(s)\sqrt{\log l}\sqrt{\log q}$. Thus the above expression can be reduced to

$$c_9K\sqrt{d}\log(s)\sqrt{\log l}\sqrt{\log q}. \quad (20)$$

Combining (20) with (13) and (9), we can conclude

$$w(l) = O(K\sqrt{sd}\log(s)\sqrt{\log l}\sqrt{\log q}),$$

which completes the proof of Lemma.1. ■

Let $\mathbf{\Phi}$ be the $N\times N$ discrete Fourier transform (DFT) matrix, $\{\tau_i\}_{1\leq i\leq N}$ be the sequence of Bernoulli random variables where each τ_i takes the value 1 with probability M/N, and Ω denote the set

$$\Omega = \{i \in 1,\ldots,N|\tau_i = 1\}. \quad (21)$$

Theorem 1: Let Ω, $E|\Omega| = M$, be the random set as defined in (21), and $\mathbf{\Phi}$ be the $N\times N$ DFT matrix. Suppose that we construct our measurement matrix $\mathbf{F}$ as $\mathbf{F} = \mathbf{\Phi}^\Omega$, where $\mathbf{\Phi}^\Omega$ denotes the $|\Omega|\times N$ matrix that consists of the rows of $\mathbf{\Phi}$ indexed by Ω. Further assume that $\delta_{s|B}$ is the block RIP constant of the order s over $B = \{\mathbf{b}_1,\ldots,\mathbf{b}_q|N = qd, \mathcal{L}(\mathbf{b}_i) = d\}$. Then if

$$M \geq C\frac{sd\log q}{\varepsilon^2}\log\frac{sd\log q}{\varepsilon^2}\log^2 s \quad (22)$$

for some constant $\varepsilon \leq 1$ and C, $\delta_{4s|B}$ will be less than 0.5 with probability being greater than $1-5e^{-ct}$, where $t = 1/\varepsilon^2$ and c is some constant.

Proof: [1] According to Definition 3, it is easy to perform the following formulations:

$$\begin{aligned}
\delta_{s|B} &= \sup_{|R|\leq s|B}\|\mathbf{v}\|_2\left|\frac{1}{C}\|\mathbf{F}_R\mathbf{v}\|_2 - \|\mathbf{v}\|_2\right|\\
&= \sup_{|R|\leq s|B}\|\mathbf{v}\|_2\left|C_1 <\mathbf{F}_R'\mathbf{F}_R\mathbf{v},\mathbf{v}> - <\mathbf{v},\mathbf{v}>\right|\\
&= \sup_{|R|\leq s|B}\|\mathbf{v}\|_2\left|<(C_1\mathbf{F}_R'\mathbf{F}_R - \mathbf{I}_R)\mathbf{v},\mathbf{v}>\right|
\end{aligned} \quad (23)$$

where $C_1 = 1/C$, $\mathbf{F}'$ is the transpose of $\mathbf{F}$, $\mathbf{F}_R$ is the matrix that compose the columns of $\mathbf{F}$ indexed by R, and $\mathbf{I}_R$ represents the identity operator which has the same dimension as $\mathbf{F}_R'\mathbf{F}_R$. According to Cauchy-Schwarz inequality, we have

$$|<(C_1\mathbf{F}_R'\mathbf{F}_R - \mathbf{I}_R)\mathbf{v},\mathbf{v}>| \leq \|C_1\mathbf{F}_R'\mathbf{F}_R - \mathbf{I}_R\|_p\|\mathbf{v}\|_2. \quad (24)$$

Substituting (24) into (23), we conclude

$$\delta_{s|B} = \inf_{C_1>0}\sup_{|R|\leq s|B}\|C_1\mathbf{F}_R'\mathbf{F}_R - \mathbf{I}_R\|_p. \quad (25)$$

[1]The following proof also relies on [7], but we made many modifications to fit our block sparse framework better.

Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_l\}$ and $l = E\Omega$ be the columns of $\mathbf{F}'$. Then $\mathbf{f}_i^R$ is the i-th column of $\mathbf{F}'_R$ and we have

$$\mathbf{F}'_R\mathbf{F}_R = [\mathbf{f}_1^R, \ldots, \mathbf{f}_l^R] \begin{bmatrix} (\mathbf{f}'_1)^R \\ \vdots \\ (\mathbf{f}'_l)^R \end{bmatrix}$$

$$= [\mathbf{f}_1^R, \mathbf{0}, \ldots, \mathbf{0}] \begin{bmatrix} (\mathbf{f}'_1)^R \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + \ldots + [\mathbf{0}, \ldots, \mathbf{0}, \mathbf{f}_l^R] \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ (\mathbf{f}'_l)^R \end{bmatrix}$$

$$= \sum_{i=1}^{i=l} \mathbf{f}_i^R \otimes \mathbf{f}_i^R, \tag{26}$$

where the symbol $\otimes$ denotes tensor operator and $\mathbf{f}'_i$ represents the transpose of $\mathbf{f}_i$. Substituting (26) into (25), we have

$$\delta_{s|B} = \inf_{C_1>0} \sup_{|R|\le s|B} \|C_1 \sum_{i=1}^{i=l} \mathbf{f}_i^R \otimes \mathbf{f}_i^R - \mathbf{I}_R\|. \tag{27}$$

Let $\{\mathbf{y}_i\}_{1\le i\le N}$ be the rows of $\mathbf{\Phi}$ and $\mathbf{x}_i = \sqrt{N}\mathbf{y}_i$. Since DFT matrix $\mathbf{\Phi}$ is itself an orthogonal matrix, we have

$$\mathbf{I} = \sum_{i=1}^{N} \mathbf{y}_i \otimes \mathbf{y}_i = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i \otimes \mathbf{x}_i \Longrightarrow \mathbf{I}_R = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i^R \otimes \mathbf{x}_i^R. \tag{28}$$

Combining (27) with (21) and according to Lemma 6.3 in [12], we have

$$\begin{aligned} E\delta_{s|B} &= E \inf_{C_1>0} \sup_{|R|\le s|B} \|C_1 \sum_{i\in\Omega} \mathbf{y}_i^R \otimes \mathbf{y}_i^R - \mathbf{I}_R\| \\ &\le E \sup_{|R|\le s|B} \|\mathbf{I}_R - \frac{1}{M}\sum_{i\in\Omega} \mathbf{x}_i^R \otimes \mathbf{x}_i^R\| \qquad (29) \\ &\le 2E \sup_{|R|\le s|B} \|\frac{1}{M}\sum_{i\in\Omega} \epsilon_i \mathbf{x}_i^R \otimes \mathbf{x}_i^R\|, \qquad (30) \end{aligned}$$

where ϵ_i has the same definition as in Lemma 1. Denote $E\sup_{|R|\le s|B} \|\frac{1}{M}\sum_{i\in\Omega} \epsilon_i \mathbf{x}_i^R \otimes \mathbf{x}_i^R\|$ in (30) by Q and apply (7) to (30). Then we can directly use the result of [7] (see Proof of Theorem 3.6 in [7]) to obtain

$$Q \le \frac{2w(M)}{\sqrt{M}}, \tag{31}$$

provided that $w(M)/\sqrt{M} \le 1/2$. Suppose the right side of (29) is bounded by ε. According to (31), we can solve M from $\frac{2w(M)}{\sqrt{M}} \le \varepsilon$ and obtain $M \ge C_1 \frac{sd\log q}{\varepsilon^2} \log \frac{sd\log q}{\varepsilon^2} \log^2 s$, which proves (22).

Moreover, the proof of the probability $P(\delta_{4s|B} \le 0.5) \ge 1 - 5e^{-c/\varepsilon^2}$ is completely identical to Theorem 3.10 and Theorem 3.11 of [7], and are omitted here. Therefore, we have completed the proof of Theorem 1. ■

Remark 2: Obviously, (28) holds for all unit orthogonal matrices. Therefore, according to Lemma 6.3 in [12], inequality (30) also holds for any other unit orthogonal matrix. This implies that if we replace $\mathbf{\Phi}$ in Theorem 1 by other unit orthogonal matrix, the theorem still holds.

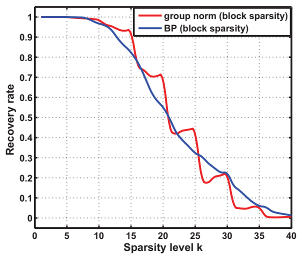

Fig. 1. Recovery rate of block sparse signal using group norm method and BP method.

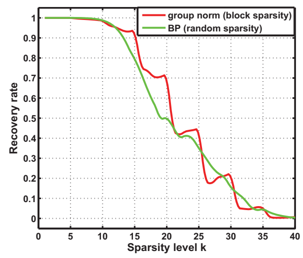

Fig. 2. Recovery rate of block sparse signal using group norm method and random sparse signal using BP method.

IV. Simulations

In this section, we will present two groups of comparative experiments. In the first group, we compare the recovery rate of group norm method (6) and basis pursuit(BP) method [13] for the same set of block sparse signals. In the second group, we compare the recovery rate of group norm method (6) and BP method for a set of block sparse signals and a set of random sparse signals respectively. In the experiments, we let $M = 45$, $N = 100$ and generate measurement matrix $\mathbf{F}$ as that in Theorem 1. For block sparse signal, we let block size $d = 5$ and draw $1 \le k \le 40$ nonzero entries from zero-mean Gaussian distribution and divide them into blocks which are chosen uniformly within set $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_q | N = qd, \mathcal{L}(\mathbf{b}_i) = d\}$, where $q = N/d = 20$. For random sparse signal, we choose its support uniformly at random within set $\{1, \cdots, N\}$ with the support size k ranging from 1 to 40, and let the nonzero entries in the support be randomly chosen from zero-mean Gaussian distributions. We repeat each group of experiments for each sparsity level k ranging from 1 to 40 over 2000 independent trials. Simulation results are showed in Fig. 1 and Fig. 2, where the red curve and blue curve in Fig. 1 represent group norm method and BP method respectively, while red curve and green curve in Fig. 2 correspond to recovery of block sparse signal and random sparse signal respectively.

In Fig. 1, We can easily see that the red curve is roughly constant over the block size d, therefore the performance of group norm method is better than BP method when recovery rate is higher than approximately 0.3. In Fig. 2, the results show that the performance of recovering block sparse signal is obviously better than recovering the signal without any structure.

V. Conclusions

In this paper, we extended the block RIP from Gaussian measurements to Fourier measurements and proved that the minimum number of Fourier measurements can be as low as $O\left(sd\log q \log sd \log q \log^2 s\right)$. In contrast, under the same condition, the minimum number of measurements for standard RIP is $O\left(sd\log N \log sd \log N \log^2 sd\right)$, which is obviously greater than block RIP case.

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