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GAUSSIAN PROCESS PARAMETER ESTIMATION USING ZERO CROSSING DATA FROM WIRELESS SENSORS

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ABSTRACT

The parameters in a general Gaussian process, including the parameters in an additive Gaussian noise process, are estimated based on zero crossing data for the total process and arbitrarily filtered versions thereof. A nonlinear weighted least squares estimate is considered and an analysis of the asymptotic covariance matrix of the estimated parameter vector is made. The proposed estimator and the use of zero crossing data are suitable when information of a process is sent from wireless sensors to a node center for further processing due to an efficient use of available bandwidth.

Index Terms— Gaussian process, estimation, zero crossings, wireless sensors, accuracy analysis

1. INTRODUCTION

Wireless sensors are frequently used for registering and monitoring various processes. Many cheap sensors without additional hardware that allows computations are often strategically placed in the space of interest, and the information collected by the sensors are sent wirelessly to a node center for further processing. This puts requirements on the available bandwidth that can not always be fulfilled. Alternatives where reduced amounts of information are sent from the sensors to the node center could therefore be interesting. However, this requires that the result of the information processing at the node center based on a reduced quantity of information will be of sufficiently high quality for the application at hand. One set of information that makes efficient use of available bandwidth is based on level crossings of the process of interest and filtered versions thereof. In fact, the only information to be sent from a sensor to the node center is the number of level crossings of a process during a certain time interval. The aim of the paper is therefore to consider information processing of level crossing data for a general Gaussian process and filtered versions thereof with the purpose of estimating the process parameters.

Some of the first results on the theory of zero crossings are found in the information theory literature [1–4]. The survey paper [5] and the book [6] present many important results and are interesting from a signal processing perspective. Additional results on zero crossing rates of functions of Gaussian processes are presented in [7, 8] and a survey of results for Gaussian processes are given in [9]. Furthermore, some recent results on the theory of level crossings are found in [10, 11].

Parameter estimation based on level crossings of the process of interest and filtered versions thereof is considered in [6, 12–15]. In [6], a relation between the number of zero crossings and the correlation function is explained and it is described how the relation can be used for parameter estimation purposes. The problem of estimating the parameters in autoregressive processes based on zero crossing data is studied in [12]. Estimation of the fractal index and the fractal dimension of a Gaussian process based on level crossing data is considered in [13]. In [14], the poles of autoregressive moving average processes are estimated based on information of higher order crossings. Time delay estimation based on zero crossing data is considered in [15].

Here, estimation of the parameters in general Gaussian processes based on information of the number of zero crossings for the process itself and for filtered versions of the process. This is done by using a relation between the number of zero crossings and the correlation function of a process. The relation is considered for the process itself as well as for filtered versions thereof. As the number of zero crossings can be registered and the correlation function is a possibly nonlinear function in the unknown parameters, a nonlinear weighted least squares criterion is defined and minimized. Thereafter, the asymptotic covariance matrix of the estimated parameter vector is analyzed. In the analysis, the covariance matrix of the vector containing the registered number of zero crossings for the process and its filtered versions must be evaluated. Unfortunately, the evaluation of the covariance between the registered number of zero crossings for different processes is nontrivial. As an example of a Gaussian process for illustrating the material in the paper numerically, a diffusion process is considered. The rest of the paper is organized as follows. In Section 2, the relation to prior work is given. The parameter estimation is considered in Section 3, the asymptotic normalized covariance matrix of the estimated parameter vector is given in Section 4, and second order statistics of zero crossings is considered in Section 5. The numerical examples are presented in Section 6, some discussions are found in Section 7, and conclusions are given in Section 8.

2. RELATION TO PRIOR WORK

The paper is devoted to estimation of parameters in general Gaussian processes based on information on the number of zero crossings for the process itself and for filtered versions of the process. Prior work on parameter estimation based on zero crossing data includes [6, 12–15], but the approach for estimating the parameters in general Gaussian processes taken here has not been considered before. This includes, in addition to considering a general structure for Gaussian processes, the estimation of the parameters characterizing a possible additive Gaussian measurement noise process, and the use of zero crossing data for arbitrarily filtered versions of the process. For example, special cases of Gaussian processes are considered in terms of autoregressive processes in [12] and autoregressive moving average processes in [14], whereas general Gaussian processes are considered in [13], but with the special aim of estimating the fractal index and the fractal dimension. Here, an analysis of the asymptotic covariance matrix of the estimated parameter vector for the general Gaussian case, not previously considered in the literature, is also made. In the analysis, expressions for zero crossing covariances recently published in [16] are used for evaluating variances of zero

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crossing rates.

3. PARAMETER ESTIMATION

Consider a Gaussian stationary process u_k with correlation function ρ_L. The indicator of a zero crossing at time k is

$$Z_k = \begin{cases} 1, & \text{if } u_k > 0 \text{ and } u_{k+1} < 0 \\ 1, & \text{if } u_k < 0 \text{ and } u_{k+1} > 0 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

and the total number of zero crossings for $k = 0, \ldots, N-1$ is $z = \sum_{k=0}^{N-1} Z_k$, with $z_N = z/N$ being the zero crossing rate. Due to stationarity of the process u_k, it holds that [16] $\mathbb{E}\{z_N\} = \mathcal{P}(Z_k = 1)$, where $\mathcal{P}$ denotes probability. Furthermore, the relation [6]

$$\mathbb{E}\{z\} = \frac{N-1}{\pi} \arccos(\rho_1) \tag{2}$$

holds between the expected value of the number of zero crossings and the correlation element ρ_1 of the process.

Consider a general situation in which a parameter vector $\boldsymbol{\theta}_0 = [\theta_{0_1} \ \cdots \ \theta_{0_n}]^T$ is needed for describing the properties of u_k. For example, if u_k is an autoregressive process observed in white noise, $\boldsymbol{\theta}_0$ contains the autoregressive parameters and the noise variance. Consequently, the correlation element ρ_1 is dependent on $\boldsymbol{\theta}_0$ and the right hand side in (2) can be described as a function $f_1(\boldsymbol{\theta}_0)$. If the registered number of zero crossings is denoted as $\hat{s}_1$, the relation $\hat{s}_1 = f_1(\boldsymbol{\theta}) + \delta_1$, where δ_1 is an error, is motivated from (2). Note that for $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, $\delta_1 = 0$ and $\hat{s}_1 = \mathbb{E}\{z\}$. An analoguous reasoning for $M-1$ different filtered versions of u_k gives the relations $\hat{s}_i = f_i(\boldsymbol{\theta}) + \delta_i$, $i = 2, \ldots, M$. With the aim of estimating $\boldsymbol{\theta}_0$ based on zero crossing data, the loss function

$$V(\boldsymbol{\theta}) = ||\boldsymbol{\delta}||^2_{\mathbf{Q}} = ||\hat{\mathbf{s}} - \mathbf{f}(\boldsymbol{\theta})||^2_{\mathbf{Q}},$$

where $\boldsymbol{\theta} = [\theta_1 \ \cdots \ \theta_n]^T$, $\boldsymbol{\delta} = [\delta_1 \ \cdots \ \delta_M]^T$, $\hat{\mathbf{s}} = [\hat{s}_1 \ \cdots \ \hat{s}_M]^T$, and $\mathbf{f}(\boldsymbol{\theta}) = [f_1(\boldsymbol{\theta}) \ \cdots \ f_M(\boldsymbol{\theta})]^T$ is now considered. Furthermore, $M \geqslant n$, $\hat{\mathbf{s}}$ contains the registered number of zero crossings for the process u_k and filtered versions thereof, $\mathbf{f}(\boldsymbol{\theta})$ is a nonlinear function in $\boldsymbol{\theta}$, and $\mathbf{Q}$ is a positive definite and symmetric weighting matrix. Define the estimate

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} V(\boldsymbol{\theta}). \tag{3}$$

In the next section, the asymptotic normalized covariance matrix of nonlinear weighted least squares estimate $\hat{\boldsymbol{\theta}}$ is considered.

4. ASYMPTOTIC COVARIANCE MATRIX

Provided that the loss function $V(\boldsymbol{\theta})$ is twice continuous differentiable in $\boldsymbol{\theta}$, the asymptotic normalized covariance matrix $\mathbf{C}_{\hat{\boldsymbol{\theta}}}$ of $\hat{\boldsymbol{\theta}}$ is given by

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \lim_{N\to\infty} N \cdot \mathbb{E}\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^T\} = \mathbf{H}^{-1}\mathbf{G}\mathbf{H}^{-1}, \tag{4}$$

where

$$\mathbf{G} = \lim_{N\to\infty} N \cdot \mathbb{E}\{V^{'}(\boldsymbol{\theta}_0)\big(V^{'}(\boldsymbol{\theta}_0)\big)^T\}, \tag{5}$$

$$\mathbf{H} = \lim_{N\to\infty} V^{''}(\boldsymbol{\theta}_0), \tag{6}$$

with $V^{'}(\boldsymbol{\theta}_0) = \frac{\mathrm{d}V(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$ and $V^{''}(\boldsymbol{\theta}_0) = \frac{\mathrm{d}^2 V(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}^T}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$. Straightforward differentiation gives

$$V^{'}(\boldsymbol{\theta}) = 2\sum_{i=1}^{M}\sum_{j=1}^{M}\{f_i^{'}(\boldsymbol{\theta})Q_{ij}f_j(\boldsymbol{\theta}) - \hat{s}_i Q_{ij} f_j^{'}(\boldsymbol{\theta})\}, \tag{7}$$

$$\begin{aligned} V^{''}(\boldsymbol{\theta}) = 2\sum_{i=1}^{M}\sum_{j=1}^{M}\{&f_i^{''}(\boldsymbol{\theta})Q_{ij}f_j(\boldsymbol{\theta}) + f_i^{'}(\boldsymbol{\theta})Q_{ij}\big(f_j^{'}(\boldsymbol{\theta})\big)^T \\ &- \hat{s}_i Q_{ij} f_j^{''}(\boldsymbol{\theta})\}, \end{aligned} \tag{8}$$

where $f_i^{'}(\boldsymbol{\theta}) = \frac{\mathrm{d}f_i(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}} \in \mathbb{R}^{m\times 1}$, $f_i^{''}(\boldsymbol{\theta}) = \frac{\mathrm{d}^2 f_i(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}^T} \in \mathbb{R}^{m\times m}$, and Q_{ij} denotes element ij of $\mathbf{Q}$. Furthermore, using (7) in (5) results in

$$\begin{aligned} \mathbf{G} = 4\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{\ell=1}^{M}\{&Q_{ij}Q_{kl}f_j^{'}(\boldsymbol{\theta})\big(f_\ell^{'}(\boldsymbol{\theta})\big)^T \\ &\times \big(\mathbb{E}\{\hat{s}_i\hat{s}_k\} - 2f_i(\boldsymbol{\theta})\mathbb{E}\{\hat{s}_k\} + f_i(\boldsymbol{\theta})f_k(\boldsymbol{\theta})\big)\}. \end{aligned} \tag{9}$$

From

$$\hat{s}_i = f_i(\boldsymbol{\theta}_0) + \varepsilon_i, \tag{10}$$

where ε_i is an error, it follows that

$$\mathbb{E}\{\hat{s}_i\} = f_i(\boldsymbol{\theta}_0), \tag{11}$$

$$\mathbb{E}\{\hat{s}_i\hat{s}_j\} = f_i(\boldsymbol{\theta}_0)f_j(\boldsymbol{\theta}_0) + \mathbb{E}\{\varepsilon_i\varepsilon_j\}. \tag{12}$$

Using (11) in (9) gives

$$\begin{aligned} \mathbf{G} = 4\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{\ell=1}^{M}\{&Q_{ij}Q_{kl}f_j^{'}(\boldsymbol{\theta})\big(f_\ell^{'}(\boldsymbol{\theta})\big)^T \\ &\times \big(\mathbb{E}\{\hat{s}_i\hat{s}_k\} - f_i(\boldsymbol{\theta})f_k(\boldsymbol{\theta})\big)\}, \end{aligned} \tag{13}$$

and using (12) in (13) gives the final expression

$$\mathbf{G} = 4\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{\ell=1}^{M} Q_{ij}Q_{kl}f_j^{'}(\boldsymbol{\theta}_0)\big(f_\ell^{'}(\boldsymbol{\theta}_0)\big)^T\mathbb{E}\{\varepsilon_i\varepsilon_k\},$$

with $f_i^{'}(\boldsymbol{\theta}_0) = \frac{\mathrm{d}f_i(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$. The evaluation of $\mathbb{E}\{\varepsilon_i\varepsilon_j\}$, i.e., the covariance between $\hat{s}_i$ and $\hat{s}_j$, is considered in Section 5. To find $\mathbf{H}$, it is noted that from (10), $\lim_{N\to\infty}\hat{s}_i = f_i(\boldsymbol{\theta}_0)$, which together with (8) in (6) give

$$\mathbf{H} = 2\sum_{i=1}^{M}\sum_{j=1}^{M} Q_{ij}f_i^{'}(\boldsymbol{\theta}_0)\big(f_j^{'}(\boldsymbol{\theta}_0)\big)^T.$$

As M is a small number, equal to or slightly larger than the number of unknown parameters, it is not a computationally intensive task to compute the quadruple and double sums in the expressions for $\mathbf{G}$ and $\mathbf{H}$, respectively. Also note that due to symmetry, only the upper diagonal matrix has to be calculated in both cases. A possibility is however to consider a notation based entirely on vectors and matrices for expressing $\mathbf{G}$ and $\mathbf{H}$, but the current expressions are likely easier to interpret. Finally, note that in the special case of no weighting, i.e., $\mathbf{Q} = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, the expressions for $\mathbf{G}$ and $\mathbf{H}$ simplify to

$$\mathbf{G} = 4\sum_{i=1}^{M}\sum_{j=1}^{M} f_i^{'}(\boldsymbol{\theta}_0)\big(f_j^{'}(\boldsymbol{\theta}_0)\big)^T\mathbb{E}\{\varepsilon_i\varepsilon_j\},$$

$$\mathbf{H} = 2\sum_{i=1}^{M} f_i^{'}(\boldsymbol{\theta}_0)\big(f_i^{'}(\boldsymbol{\theta}_0)\big)^T.$$

5. SECOND ORDER STATISTICS OF ZERO CROSSINGS

The variance of a zero crossing rate and the covariance between zero crossing rates are considered in Sections 5.1 and 5.2, respectively.

5.1. The variance of a zero crossing rate

The variance of the zero crossing rate z_N, defined in Section 3, is given as [16]

$$\mathrm{var}(z_N) = \frac{1}{N^2}\left(N\gamma_0 + 2\sum_{k=1}^{N-1}(N-k)\gamma_k\right), \tag{14}$$

where $\gamma_k = \mathrm{cov}(Z_0, Z_k)$. Furthermore, it holds that [16]

$$\gamma_0 = \frac{1}{4} - \frac{1}{\pi^2}\big(\arcsin(\rho_1)\big)^2, \tag{15}$$

$$\gamma_1 = \frac{1}{2\pi}\arcsin(\rho_2) - \frac{1}{\pi^2}\big(\arcsin(\rho_1)\big)^2$$

and that

$$\gamma_k \approx \frac{2(1-\rho_1)}{\pi^2(1+\rho_1)}\rho_k^2.$$

The covariance between the zero crossing rates of two different processes is considered next.

5.2. The covariance between zero crossing rates

The variance of the zero crossing rate of a certain process can be evaluated as described in Section 5.1. However, the problem of finding the covariance $\mathrm{cov}(z_N, v_N)$ between the zero crossing rates z_N and v_N of two different processes u_k^z and u_k^v is more difficult. To the best of the authors knowledge, no direct way for evaluating $\mathrm{cov}(z_N, v_N)$ exists.

As $\mathrm{var}(z_N)$ and $\mathrm{var}(v_N)$ can be expressed, an attempt to find $\mathrm{cov}(z_N, v_N)$ is to use the well known formula

$$\mathrm{cov}(z_N, v_N) = \big(\mathrm{var}(z_N + v_N) - \mathrm{var}(z_N) - \mathrm{var}(v_N)\big)/2, \tag{16}$$

provided that $\mathrm{var}(z_N + v_N)$ can be found. The latter variance can be expressed if the correlation function for the underlying process u_k^{z+v} with a zero crossing rate of $z_N + v_N$ is known. The problem is that such a process is not unique; two processes can indeed have the same expected zero crossing rate but the variances of the zero crossing rates for the processes can be different. For example, the first order autoregressive process $\eta_k = \alpha\eta_{k-1} + \psi_k$ for $k = 0, \dots, N-1$, where $\alpha = \cos(\pi(z+v)/(N-1))$ and ψ_k is zero mean white Gaussian noise, has approximately an expected zero crossing rate of $z_N + v_N$ [6]. Unfortunately, the variance of the zero crossing rate of this autoregressive process is not necessarily similar to the variance of the zero crossing rate for the particular underlying process u_k^{z+v} that is sought. What can be said though about the correlation function ρ_L^{z+v} for the particular underlying process u_k^{z+v} that is sought is that for $L = 1$,

$$\rho_1^{z+v} = \cos\big(\arccos(\rho_1^z) + \arccos(\rho_1^v)\big),$$

which follows from (2), where ρ_1^z and ρ_1^v denote the correlation functions for u_k^z and u_k^v, respectively, evaluated for $L = 1$. Based on ρ_1^{z+v}, a rough estimate of $\mathrm{var}(z_N + v_N)$ is obtained as

$$\mathrm{var}(z_N + v_N) \approx \frac{1}{N}\Big(\frac{1}{4} - \frac{1}{\pi^2}\left(\arcsin(\rho_1^{z+v})\right)^2\Big), \tag{17}$$

using (14) and (15). A possibility is then to consider the estimate (17) together with the estimates

$$\mathrm{var}(z_N) \approx \frac{1}{N}\Big(\frac{1}{4} - \frac{1}{\pi^2}\left(\arcsin(\rho_1^z)\right)^2\Big), \tag{18}$$

$$\mathrm{var}(v_N) \approx \frac{1}{N}\Big(\frac{1}{4} - \frac{1}{\pi^2}\left(\arcsin(\rho_1^v)\right)^2\Big) \tag{19}$$

of $\mathrm{var}(z_N)$ and $\mathrm{var}(v_N)$, respectively, in (16) to get an estimate of $\mathrm{cov}(z_N, v_N)$. The reason for chosing approximations as (18) and (19) based on only one covariance element $\gamma_0^z = \mathrm{cov}(Z_0, Z_0)$ and $\gamma_0^v = \mathrm{cov}(V_0, V_0)$, respectively, and not an arbitrary number of covariance elements is to "match" the approximation (17) that is based on only one covariance element $\gamma_0^{z+v} = \mathrm{cov}(S_0, S_0)$. Here, Z_0, V_0, and S_0 are the indicators of a zero crossing at time 0 for the processes u_k^z, u_k^v, and u_k^{z+v}, respectively, defined as in (1).

6. NUMERICAL EXAMPLES

As an example of a Gaussian process, the diffusion process

$$\mathrm{d}x(t) = a_0 x(t)\mathrm{d}t + \mathrm{d}w(t)$$

is considered, where $\mathrm{d}w(t)$ is the increment of a Wiener process $w(t)$ with unit incremental variance. The correlation function of $x(t)$ is given as [17, 18] $\rho^x(\tau) = \mathrm{e}^{-a_0|\tau|}$ and the correlation function of the corresponding sampled process x_k with sampling interval h is $\rho_L^x = \rho^x(Lh)$. It is assumed that x_k is observed in zero mean white Gaussian noise e_k of variance λ_0^2 as $y_k = x_k + e_k$.

In the first example, the noise-free case is considered, and based on (2), $\boldsymbol{\theta}_0 = a_0$ is estimated from the registered number of zero crossings $\hat{s}_1$ for the process x_k according to (3), where $f_1(\boldsymbol{\theta}) = ((N-1)/\pi)\arccos(\rho_1^x(\boldsymbol{\theta}))$, with the dependency on $\boldsymbol{\theta}$ for the correlation element ρ_1^x being emphasized. Data, $N = 1000$ samples, are generated for $a_0 = 2$ and $h = 0.1$, and a_0 is estimated as $\hat{a}$. The data generation and the estimation is repeated 1000 times in a Monte Carlo study and the resulting empirical mean and variance are shown in Table 1. In the same table, the theoretical variance (4) of $\hat{a}$ is also given together with the Cramér-Rao bound (CRB), computed using Slepian-Bangs formula [19, 20], for two different situations. The first bound, denoted $\mathrm{CRB}_{\mathrm{zc}}$, is for the estimation of a_0 based on the number of zero crossings and the second bound, denoted CRB_x, used for comparison purposes, is for the estimation of a_0 based on the N samples $\{x_k\}_{k=0}^{N-1}$. It is seen that the theoretical variance can describe the empirical variance, that the estimator reaches the bound $\mathrm{CRB}_{\mathrm{zc}}$, and that $\mathrm{CRB}_{\mathrm{zc}}$ is about a factor three larger than CRB_x.

In the second example, the noisy case is considered, and based on (2), $\boldsymbol{\theta}_0 = [a_0\ \lambda_0^2]^T$ is estimated from $\hat{\mathbf{s}} = [\hat{s}_1\ \hat{s}_2]^T$ according to (3), without applying any weighting in the criterion function $V(\boldsymbol{\theta})$. Here, $\hat{s}_1$ and $\hat{s}_2$ are the registered number of zero crossings for the processes y_k and the differenced process $d_k = y_k - y_{k-1}$ with correlation functions ρ_L^y and ρ_L^d, respectively. Note that ρ_L^y and ρ_L^d are expressed in terms of ρ_L^x and λ_0^2. Furthermore, $\mathbf{f}(\boldsymbol{\theta}) = [f_1(\boldsymbol{\theta})\ f_2(\boldsymbol{\theta})]^T$, where $f_1(\boldsymbol{\theta}) = ((N-1)/\pi)\arccos(\rho_1^y(\boldsymbol{\theta}))$ and $f_2(\boldsymbol{\theta}) = ((N-2)/\pi)\arccos(\rho_1^d(\boldsymbol{\theta}))$, with the dependency on $\boldsymbol{\theta}$ for the correlation elements ρ_1^y and ρ_1^d being emphasized. Note that the factor $N-2$ appears in the expression for $f_2(\boldsymbol{\theta})$ as there is only $N-1$ samples of d_k available. Data, $N = 10000$ samples, are generated for $a_0 = 2$ and $h = 0.1$ for different signal-to-noise ratios (SNRs) obtained by varying λ_0^2, and $\boldsymbol{\theta}_0$ is estimated as $\hat{\boldsymbol{\theta}}$. The data generation and the estimation is repeated 1000 times in a Monte Carlo study and the resulting empirical variances are shown together with the theoretical variances given by (4) and the CRBs

Table 1. The emprical mean and variance for $\hat{a}$, the theoretical variance for $\hat{a}$, and the bounds $\mathrm{CRB}_{\mathrm{zc}}$ and CRB_x for the estimation of $a_0 = 2$ when there is no noise. Here, $N = 1000$ and $h = 0.1$.

empirical		theoretical		
mean($\hat{a}$)	var($\hat{a}$)	var($\hat{a}$)	$\mathrm{CRB}_{\mathrm{zc}}$	CRB_x
2.0134	0.1266	0.1259	0.1259	0.0403

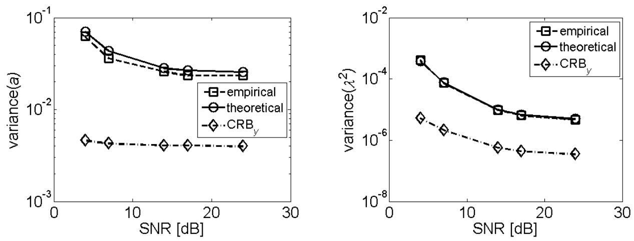

Fig. 1. The empirical and theoretical variances, and the bound CRB_y for the estimation of a_0 (left) and λ_0^2 (right) as functions of the SNR. Here, $a_0 = 2$, $N = 10000$, and $h = 0.1$.

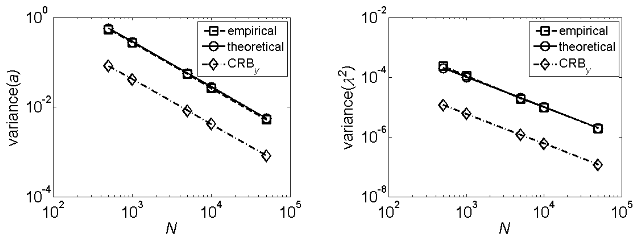

Fig. 2. The empirical and theoretical variances, and the bound CRB_y for the estimation of a_0 (left) and λ_0^2 (right) as functions of N. Here, $a_0 = 2$, $\lambda_0^2 = 0.01$, and $h = 0.1$.

computed by Slepian-Bangs formula as functions of the SNR for the estimation of a_0 and λ_0^2 in Figure 1. In (4), the approximation $\mathbb{E}\{\varepsilon_1\varepsilon_2\} = \mathbb{E}\{\varepsilon_2\varepsilon_1\} \approx 0$ is made as it is known from empirical data that it is much smaller than $\mathbb{E}\{\varepsilon_1^2\}$ and $\mathbb{E}\{\varepsilon_2^2\}$. The bound, denoted CRB_y, used for comparison purposes, is for the estimation of $\boldsymbol{\theta}_0$ based on the N samples $\{y_k\}_{k=0}^{N-1}$. From the figures, it is clear that the theoretical variances can describe the empirical variances. It is also illustrated how much the estimation accuracy, could be improved in case the N samples $\{y_k\}_{k=0}^{N-1}$ were used instead of just two registered number of zero crossings. However, it is important to keep in mind that such a gain in estimation accuracy would come at the cost of a largely increased bandwidth requirement.

In the final, third example, the setup and all details are the same as in the second example, with the difference that λ_0^2 is kept constant ($\lambda_0^2 = 0.01$) whereas N is varied. The empirical variances are shown together with the theoretical variances given by (4) and the CRBs computed by Slepian-Bangs formula as functions of the SNR for the estimation of a_0 and λ_0^2 in Figure 2. Yet again, it is illustrated that the theoretical variances can describe the empirical variances.

7. DISCUSSIONS

The quality of the estimation results for the noisy case in Section 6 can be improved. First of all, information of the number of zero crossings for several other filtered versions of the process of interest can be included. As a consequence, the number of equations becomes greater than the number of unknowns in the loss function $V(\boldsymbol{\theta})$. Furthermore, use can be made of the weighting matrix $\mathbf{Q}$. It is well known that in the Gaussian case, an optimal choice of $\mathbf{Q}$ is the inverse of the covariance matrix of $\hat{\mathbf{s}}$. The covariance matrix can be computed using the results in Section 5. However, as there is a dependency on the true and unknown process parameters, it is suggested to first apply the estimator without weighting and then use the estimated parameters in the computation of the covariance matrix, thus obtaining an approximately optimal weighting. Note that the asymptotic normalized covariance matrix of the estimated parameter vector given in Section 4 can be used for theoretical investigations of what can be gained in terms of accuracy by increasing the number of equations and using an approximately optimal weighting.

8. CONCLUSIONS

The parameters in a general Gaussian process, including the parameters in an additive Gaussian noise process, have been estimated based on zero crossing data for the total process and arbitrarily filtered versions thereof. A nonlinear weighted least squares estimate has been considered and an analysis of the asymptotic covariance matrix of the estimated parameter vector has been made. The proposed estimator and the use of zero crossing data are suitable when information of a process is sent from wireless sensors to a node center for further processing. This is due to an efficient use of available bandwidth as only the number of zero crossings for a process is sent from a sensor to a node center. Numerical examples have illustrated the proposed method and the asymptotic covariance matrix. This includes a comparison with the Cramér-Rao bound for the case when all samples, and not only zero crossing data, are available. The gain in estimation accuracy in that case would come at the cost of a largely increased bandwidth requirement.

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