

NOISE SQUARED NORM IN OFDM SYSTEMS INTERFERED BY IMPULSE NOISE

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ABSTRACT

In this paper we derive exact analytical expressions for the noise squared norm in the frequency domain for Orthogonal Frequency-Division Multiplex (OFDM) considering the two generic scenarios: i) OFDM signal is affected by M randomly positioned Gaussian impulses within the L -sized frame and ii) OFDM signal is affected by the Gaussian white noise (background noise) and M randomly positioned Gaussian impulses within the L -sized frame. Analysis of the noise squared norm in (OFDM) is a completely novel approach, and we exploit it in an accurate evaluation of the performance including the distribution of errors per OFDM frame and the frame error rate. Numerical results obtained for the noise squared distribution are compared with Monte Carlo simulations. The comparison shows a full agreement even when the number of impulses and the frame size are low.

Index Terms— Impulse noise, noise squared norm, OFDM systems.

1. INTRODUCTION

Impulsive noise is an unwanted escort of communication signals in wireless channels which can significantly impair performance of OFDM systems [1, 2, 3]. Performance evaluation of OFDM systems affected by impulsive noise commonly relies on assumption that the number of carriers per frame is large [4, 1]. Such approximate approach works well in evaluating the *average* bit error rate (BER) and the symbol error rate (SER) [3, 2]. *Average* BER and SER might be the most interesting performance parameters. However, in many applications, the distribution of the number of errors per frame and the frame error rate (FER) are interesting as well in ARQ and frequency selective systems [5, 6]. When OFDM signal is affected only by Gaussian white i.i.d. noise, the noise in the frequency domain is also Gaussian i.i.d. and the SER, BER and FER performance can be easily calculated [3, 7]. Owing to i.i.d. noise property, number of errors per frame is a binomial random variable (RV). In this case the noise squared norms per real and imaginary parts in the frequency domain

are also i.i.d. chi-square RVs. However, when OFDM signal is affected by M Gaussian i.i.d. randomly positioned impulses within the L -sized frame, the noise in the frequency domain is not i.i.d. but is correlated among samples. Correlation affects evaluation of distribution of the number of errors per frame (which in this case is not a binomial RV), although does not affect evaluation of *average* SER and BER. Distribution of the number of errors per frame, and consequently the FER, can not be accurately evaluated by only knowing the noise density in the frequency domain. In this case, knowledge of a sum of squares of noise samples (the squared norm) and its distribution are important. This issue is analysed and discussed in [8, 9]. In this paper the exact closed-form formulas for the noise squared norm are derived for the two generic noise scenarios mentioned in the abstract. Specifically, for Bernoulli-Gaussian (BG) impulse noise scenario [3, 2] it is easy to extend the formulas obtained for generic cases. For more realistic scenarios including Middleton Class A, B impulse noise [10], particularly an alpha-stable impulse noise [11, 12] which is important due to its generality in representation the non-Gaussian heavy-tailed distributions, closed-form expressions for the noise squared norms can be easily derived when Class A or alpha-stable RVs are considered as the finite Gaussian mixture [13].

The remainder of the paper is organized as follows. In Section II we first show that the noise squared norm in the frequency domain is the scaled mixture RV consisting of $M+1$ components, and in Section III we develop the distribution of the scaled factors. Section IV shows numerical results and their comparison with the results obtained by Monte Carlo simulations. Section V gives some concluding remarks.

2. NOISE SQUARED NORM IN THE FREQUENCY DOMAIN

Let M randomly positioned noise impulses interfere an L -sized OFDM signal in the time domain. Let L be even, as it is a realistic case in OFDM systems. If noise impulse amplitude is Gaussian $n \sim N(0, \sigma_n^2)$ and i.i.d., the instantaneous noise amplitude in the frequency domain is (due to the central limit theorem) Gaussian as well [3].

Let $\mathbf{N} = [N_0 \cdots N_{L-1}]^T$ be a complex valued noise sequence in the frequency domain, and $\mathbf{n} = [n_0 \cdots n_{L-1}]^T$ (T denotes transposition) a complex-valued noise sequence

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in the time domain with M non-zero values randomly positioned within the L -sized frame.

The L -sized noise vector element in the frequency domain is given by

$$N_m = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} n_l W_L^{-lm}, \quad \forall m = \{0, 1, \dots, L-1\} \quad (1)$$

where $W_L = e^{j\frac{2\pi}{L}}$, $j = \sqrt{-1}$, and letters l, m denote discrete time and frequency indices, respectively.

Due to the symmetry of the Vandermonde matrix W_L , noise sample N_m depends on the impulse position symmetry scenario in the time domain. Let s denote the impulse position in the first half of the L -sized frame i.e. $s = \{0, \dots, \frac{L}{2}\}$. Let p denote the impulse position in the second half of the frame, i.e. $p = \{\frac{L}{2} + 1, \dots, L-1\}$. Then the impulse positions s and p form a symmetry pair when the condition $s + p = L$ is satisfied.

Lemma 1: A symmetric impulse noise pair in the time domain produces noise squared norms $Y_r^{(f)}$ and $Y_i^{(f)}$ which are central chi-squared RVs with two degrees of freedom each, in the frequency domain, i.e. holds

$$Y_r^{(f)} = n_{r,1}^2 + n_{r,2}^2; \quad Y_i^{(f)} = n_{i,1}^2 + n_{i,2}^2 \quad (2)$$

where $n_{r,1}, n_{r,2}, n_{i,1}, n_{i,2}$ are independent real/imaginary Gaussian impulses in the time domain at positions s and $p = L - s$.

In a special symmetry case $s = 0$ and $p = L \equiv p = 0$, or $s = L/2$ and $p = L/2$, holds $Y_r^{(f)} = 2n_r^2, Y_i^{(f)} = 0$.

Proof: The squared norm of the real noise part in the frequency domain is

$$Y_r^{(f)} = \|N_r\|^2 = \sum_{m=0}^{L-1} N_{r,m}^2 \quad (3)$$

With $N = \text{DFT}\{n\}$ and $N = [N_0 \dots N_{L-1}]^T$ we write $N_m = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} n_l \exp(-j2\pi lm/L)$, and $N_{r,m} = \text{Re}\{N_m\}, N_{i,m} = \text{Im}\{N_m\}$. Hence

$$\begin{aligned} N_{r,m} &= \frac{1}{\sqrt{L}} \times \\ &\times \text{Re} \left\{ \sum_{l=0}^{L-1} (n_{r,l} + jn_{i,l}) [\cos(2\pi lm/L) + j \sin(2\pi lm/L)] \right\} \\ &= \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} [n_{r,l} \cos(2\pi lm/L) - n_{i,l} \sin(2\pi lm/L)] \end{aligned} \quad (4)$$

Let a symmetric impulse pair occupy positions s and $p=L-s$. Then

$$N_{r,m} = \frac{1}{\sqrt{L}} [n_{r,s} \cos(2\pi sm/L) - n_{i,s} \sin(2\pi sm/L)] + \frac{1}{\sqrt{L}} [n_{r,p} \cos(2\pi pm/L) - n_{i,p} \sin(2\pi pm/L)]$$

Since $\cos[2\pi(L-s)m/L] = \cos(2\pi sm/L)$, $\sin[2\pi(L-s)m/L] = -\sin(2\pi sm/L)$, follows

$$N_{r,m} = \frac{1}{\sqrt{L}} \times [(n_{r,s} + n_{r,p}) \cos(2\pi sm/L) - (n_{i,s} - n_{i,p}) \sin(2\pi sm/L)] \quad (5)$$

Given $N_{r,m}$ by (5), the squared norm (3) is

$$Y_r^{(f)} = \sum_{m=0}^{L-1} N_{r,m}^2 = \frac{1}{L} \sum_{m=0}^{L-1} [(n_{r,s} + n_{r,p}) \cos(2\pi sm/L) - (n_{i,s} - n_{i,p}) \sin(2\pi sm/L)]^2$$

Through elementary calculation, follows

$$Y_r^{(f)} = \frac{1}{2} (n_{r,s} + n_{r,p})^2 + \frac{1}{2} (n_{i,s} - n_{i,p})^2$$

Due to the impulse noise independence, we can write

$$Y_r^{(f)} = n_{r,1}^2 + n_{r,2}^2; \text{ and similarly } Y_i^{(f)} = n_{i,1}^2 + n_{i,2}^2.$$

In a special symmetry case when $s = 0$, or $s = L/2$, from (5) follows $Y_r^{(f)} = n_r^2$, and $Y_i^{(f)} = 0$.

This concludes the proof of Lemma 1.

Lemma 2: Noise impulse at position l ($l \neq 0, l \neq L/2$) in the time domain which has not its symmetric pair produces the noise squared norms $Y_r^{(f)}$ and $Y_i^{(f)}$ which are the central chi-squared RVs each with two degrees of freedom, in the frequency domain, and holds

$$Y_r^{(f)} = \frac{1}{2} (n_{r,l}^2 + n_{i,l}^2); \quad Y_i^{(f)} = \frac{1}{2} (n_{r,l}^2 + n_{i,l}^2) \quad (6)$$

Proof: Let the noise impulse appear at position l ($l \neq 0, l \neq L/2$). Then from (4) follows

$$N_{r,m} = [n_{r,l} \cos(2\pi lm/L) - n_{i,l} \sin(2\pi lm/L)]. \quad (7)$$

Given $N_{r,m}$ in (7), the squared norm (6) is

$$\begin{aligned} Y_r^{(f)} &= \sum_{m=0}^{L-1} N_{r,m}^2 = \\ &= \frac{1}{L} \sum_{m=0}^{L-1} [n_{r,l} \cos(2\pi lm/L) - n_{i,l} \sin(2\pi lm/L)]^2 \end{aligned} \quad (8)$$

Through elementary calculation we obtain $Y_i^{(f)} = (n_{r,l}^2 + n_{i,l}^2)/2$.

This concludes the proof of Lemma 2.

Theorem 1: When M randomly positioned noise impulses interfere L -sized OFDM signal, the impulse noise squared norm density in the frequency domain $p_{Y_r^{(f)}}(y)$ is defined by the scaled mixture as

$$p_{Y_r^{(f)}}(y) = \sum_{k=0}^M p_K(k) p_{Y_{r,k}^{(f)}}(y) \quad (9)$$

where

$$Y_r^{(f)} = \sum_{l=0}^{M-k} n_{r,l}^2 + \frac{1}{2} \sum_{m=0}^{2k} n_{i,m}^2 \quad (10)$$

and k denotes the impulse noise symmetry scenario.

Proof: Let k denote the number of impulses without corresponding symmetric impulse pairs. As it is shown, when M randomly positioned impulses appear within the L -sized frame, $M+1$ impulse symmetry scenarios are possible. The impulse squared norm related to the each scenario is defined as follows:

$k=0$: there are no unpaired impulses (all impulses are paired). Given (2), the noise squared norm is $Y_{r,0}^{(f)} = \sum_{l=1}^M n_{r,l}^2$,

$k=1$: one impulse is not paired. Given (2) and (6), the noise squared norm is $Y_{r,1}^{(f)} = \sum_{j=1}^{M-1} n_{r,j}^2 + \frac{1}{2} \sum_{l=1}^2 n_{r,l}^2$,

$k=2$: two impulses are not paired. Given (2) and (6), the noise squared norm is $Y_{r,2}^{(f)} = \sum_{j=1}^{M-2} n_{r,j}^2 + \frac{1}{2} \sum_{l=1}^4 n_{r,l}^2$, and so on. Finally

$k=M-1$: only one impulse is paired. Given (2) and (6), the noise squared norm is $Y_{r,M-1}^{(f)} = n_{r,1}^2 + \frac{1}{2} \sum_{l=1}^{2M-2} n_{r,l}^2$,

$k=M$: no paired impulses. Given (6), the noise squared norm is $Y_{r,M}^{(f)} = \frac{1}{2} \sum_{l=1}^{2M} n_{r,l}^2$,

Since impulses are independent, the total squared norm is superposition of the each. Therefore, holds

$$Y_{r,k}^{(f)} = \sum_{j=1}^{M-k} n_{r,j}^2 + \frac{1}{2} \sum_{l=1}^{2k} n_{i,l}^2 \quad (11)$$

Let the impulse noise symmetry scenario k appear with a probability $p_K(k)$, $k = 0, 1, \dots, M$. Then the total impulse noise squared norm density is

$$p_{Y_r^{(f)}}(y) = \sum_{k=0}^M p_K(k) p_{Y_{r,k}^{(f)}}(y) \quad (12)$$

which is identical to (9). Expression for $p_K(k)$ is derived in Section III.

This concludes the proof of Theorem 1.

Sumands in (11) are independent chi-square RVs with $M-k$ and $2k$ degrees of freedom (DOD), respectively. Let $p_{Z_1}(z)$ and $p_{Z_2}(z)$ denote their densities, then the PDF of the k th noise squared norm $p_{Y_k^{(f)}}(y)$ can be expressed by

$$p_{Y_k^{(f)}}(y) = p_{Z_1}(z) * p_{Z_2}(z) \quad (13)$$

where $*$ denotes convolution.

In a general scenario when an independent background white noise $w \sim CN(0, \sigma_w^2)$ is present, the noise can be modelled as a combination of two impulse noises where one is related to M randomly positioned impulses with variance $\sigma_n^2 + \sigma_w^2$, and the another one is related to “impulse” noise which interferes the rest of $L-M$ samples with variance σ_w^2 .

Let $u_l = n_l + w_l$ denote the impulse noise which affects M positions, then similarly to derivation of (11), we can write

$$Y_{r,k}^{(f)} = \sum_{j=1}^{M-k} u_{r,j}^2 + \sum_{\substack{l=1 \\ l \neq j}}^{L-M-k} w_{r,l}^2 + \frac{1}{2} \sum_{\substack{m=1 \\ m \neq j,l}}^{2k} (u+w)_{i,m}^2 \quad (14)$$

As discussed in relation to (11), summands in (14) are independent chi-square RVs with $M-k$, $L-M-k$ and $2k$ DOD, respectively. Let $p_{Z_1}(z)$, $p_{Z_2}(z)$ and $p_{Z_3}(z)$ denote their densities, then the PDF of the k th noise squared norm $p_{Y_k^{(f)}}(y)$ can be expressed by

$$p_{Y_k^{(f)}}(y) = p_{Z_1}(z) * p_{Z_2}(z) * p_{Z_3}(z). \quad (15)$$

3. DISTRIBUTION OF THE SCENARIO K

The total number of M -sized complexions within the L -sized frame is

$$L_0 = C_L^{(M)} = \frac{L!}{M!(L-M)!} = \frac{1}{M!} \prod_{j=0}^{M-1} (L-j) \quad (16)$$

The number of complexions that generate $Y_{r,k}^{(f)}$ and $Y_{i,k}^{(f)}$ at $k = 0, 1, \dots, M$ depends on the triplet $\{L, M, k\}$. Let $w_k(L, M)$ denote a number of complexions at k . We can write

$$w_k(L, M) = L_k(L, M) f_k(L, M) \quad (17)$$

where $L_k(L, M)$ denotes the number of complexions which occur at the maximum number of impulses s_k that occupy the first $N/2$ positions within the L -sized OFDM frame, and $f_k(L, M)$ is the related multiplication factor.

The maximum number of impulses s_k that occupy the first $N/2$ positions is

$$s_k = \frac{M+k}{2}, \text{ for } M \text{ odd, } k \text{ odd or } M \text{ even, } k \text{ even}$$

$$s_k = \frac{M+k+1}{2}, \text{ for } M \text{ odd, } k \text{ even or } M \text{ even, } k \text{ odd.} \quad (18)$$

Within each complexion, one among s_k impulses occupies one (fixed) position within the first $L/2$ positions. Therefore, a number of complexions $L_k(L, M)$ that occur with $s_k - 1$ impulses within $L/2 - 1$ positions is

$$\begin{aligned} L_k(L, M) &= C_{L/2-1}^{(s_k-1)} = \frac{\left(\frac{L}{2}-1\right)!}{(s_k-1)! \left(\frac{L}{2}-s_k\right)!} = \\ &= \frac{\left(\frac{L}{2}-1\right)!}{\left(\frac{M+k-2}{2}\right)! \left(\frac{L-M-k}{2}\right)!}, \quad \begin{array}{c} \text{for } M \text{ odd, } k \text{ odd} \\ \text{OR} \\ M \text{ even, } k \text{ even} \end{array} \\ L_k(L, M) &= C_{L/2-1}^{(s_k-1)} = \frac{\left(\frac{L}{2}-1\right)!}{(s_k-1)! \left(\frac{L}{2}-s_k\right)!} = \\ &= \frac{\left(\frac{L}{2}-1\right)!}{\left(\frac{M+k-1}{2}\right)! \left(\frac{L-M-k-1}{2}\right)!}, \quad \begin{array}{c} \text{for } M \text{ odd, } k \text{ even} \\ \text{OR} \\ M \text{ even, } k \text{ odd} \end{array} \end{aligned} \quad (19)$$

Factor $f_k(L, M)$ is clearly determined by the binomial coefficients $C_a^{(b)}$ (odd and even diagonals of the Pascal triangle) as

$$f_k(L, M) = \frac{2^k}{k} \left(\frac{L}{2} - k\right) C_{(M+k-2)/2}^{(L-k)/2} = \frac{2^k \left(\frac{L}{2} - k\right) \left(\frac{M+k-2}{2}\right)!}{k! \left(\frac{M-k}{2}\right)!}, \quad \text{for } M \text{ odd, } k \text{ odd} \quad \text{OR} \quad M \text{ even, } k \text{ even}$$

$$f_k(L, M) = 2^{k+1} C_{(M+k-1)/2}^{(M-k-1)/2} = \frac{2^{k+1} \left(\frac{M+k-1}{2}\right)!}{k! \left(\frac{M-k-1}{2}\right)!}, \quad \text{for } M \text{ odd, } k \text{ even} \quad \text{OR} \quad M \text{ even, } k \text{ odd} \quad (20)$$

Given (19) and (20) we obtain (17). Then the total number of complexions is

$$L_0 = \sum_{k=0}^M w_k(L, M) = \frac{L!}{M!(L-M)!}$$

which, as expected, equals to (16).

Let us define the PDF of the index random variable K as

$$p_K(k) = w_k(L, M)/L_0. \quad (21)$$

Substituting (16) and (??) into (21) it follows

$$p_K(k) = \frac{2^k M!(L-M)! \left(\frac{L}{2} - 1\right)! \left(\frac{L}{2} - k\right)}{k! L! \left(\frac{M-k}{2}\right)! \left(\frac{L-M-k}{2}\right)!}, \quad \text{for } M \text{ odd, } k \text{ odd} \quad \text{OR} \quad M \text{ even, } k \text{ even}$$

$$p_K(k) = \frac{2^{k+1} M!(L-M)! \left(\frac{L}{2} - 1\right)!}{k! L! \left(\frac{M-k-1}{2}\right)! \left(\frac{L-M-k-1}{2}\right)!}, \quad \text{for } M \text{ odd, } k \text{ even} \quad \text{OR} \quad M \text{ even, } k \text{ odd} \quad (22)$$

Expression (22) represents the exact closed-form formula for calculation the PDF of the index RV k which describes probability of the k th position scenario for M randomly positioned impulses in the L -sized frame.

Note that the discrete distribution $p_K(k)$ given by (22) is the same for M and $L-M$. Thus, if $M > L/2$, it is enough to calculate $p_K(k)$ for $k = 0, 1, \dots, M$. This result also applies in a general scenario “impulse noise plus background noise” (see Eq. (14)).

4. NUMERICAL AND SIMULATION RESULTS

The i.i.d. Gaussian impulse noise has been generated by a standard i.i.d. Gaussian noise source with variance $\sigma_n^2 = 1$ masked by an L -sized vector of randomly positioned M ones and the rest of $L - M$ zeros. In a scenario “impulse noise plus background white noise” the white noise Gaussian samples are drawn from an additional i.i.d. Gaussian source with variance $\sigma_w^2 = \gamma \sigma_n^2$.

Fig. 1 shows the distribution of the noise squared norm calculated from Eq. (12) for $L = 16$, $M = 5$ and $\gamma = 0, 0.2, 0.5, 1$. For $\gamma = 0$ the mixture component distributions for $k = 0, 1, 2, 3, 4, 5$ calculated from Eq. (13) are also

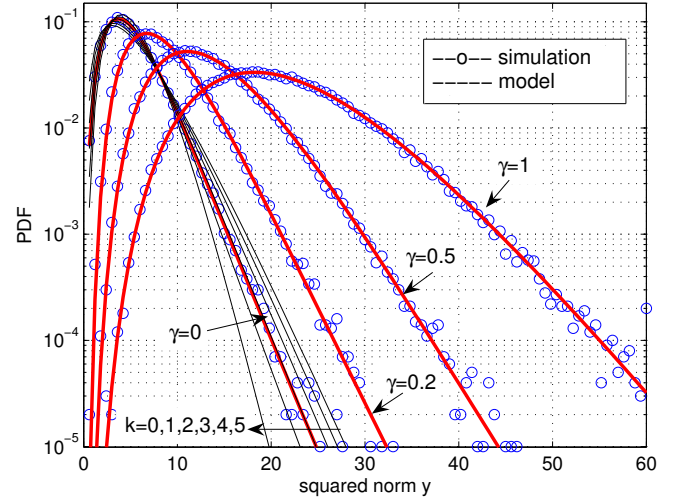


Fig. 1. Distribution of the noise squared norm for $L = 16$, $M = 5$ and $\gamma = 0, 0.2, 0.5, 1$ (Eq. (12)). For $\gamma = 0$ the mixture component distributions for $k = 0, 1, 2, 3, 4, 5$ (Eq. (13)) are also shown.

shown. The presented simulation results are obtained from 10^5 tests. It is obvious that the total noise squared norm densities given by Eq. (12) (thick red lines) accurately describe the results obtained by Monte Carlo simulations.

Let us consider the distribution of the mixture scaling factor K (22) for $L = 8, 16, 32, 64, 128, 256$, $M = 5$. Clearly, the largest scale factor depends on the frame size L . So for $L = 8$ the largest scale factor amounts to about 0.42 at $k = 2$, for $L = 16$ the largest scale factor amounts to about 0.32 at $k = 3$, and for $L = 32$ it amounts to about 0.47 at $k = M = 5$. As L increases to 64, 128 and 256 the largest scale factor appears always at $k = M$ and increases to about 0.72, 0.85 and 0.93, respectively. As frame size increases to infinity the largest scale factor approaches 1. Hence, for the large frame size ($L > 512$), the noise squared norm PDF can be approximated by a chi-square PDF with $2M$ DOD at half the noise variance.

5. CONCLUSION

In this paper we have developed exact analytical expressions for the noise squared norm density in the frequency domain for OFDM affected by M randomly positioned Gaussian impulses within the L -sized frame. The derived formulas can be exploited for accurate evaluation of the OFDM performance related to distribution of errors per frame and the frame error rate even when the number of noise impulses and frame size are low. By using the noise squared norm density we overcome the problem of the error correlation within the frame in cases when the standard approach based on the assumption on the Binomial error distribution does not hold.

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