Sparse Reconstruction Of Equivalence Classes Of Moving Targets Using Single-Channel Synthetic Aperture Radar

Jake Gunther*, Josh Hunsaker*, Hyrum Anderson[†] and Todd Moon* *Electrical & Computer Engineering, Utah State University, Logan, UT 84322–4120 [†]Sandia National Laboratory, Data Analysis and Data Exploitation

Abstract—Simultaneously estimating position and velocity of moving targets using only phase information from single-channel SAR data is impossible. This paper defines classes of equivalent target motion and solves the GMTI problem up to membership in an equivalence class using single-channel SAR phase data. We present a definitions for endo- and exo-clutter that is consistent with the equivalence classes, and show that most target motion can be detected, i.e. the set of endo-clutter targets is very small. We exploit the sparsity of moving targets in the scene to develop an algorithm to resolve target motion up to membership in an equivalence class, and demonstrate the effectiveness of the proposed technique using simulated data.

Index Terms—Radar signal processing, synthetic aperture radar, motion detection, sparse signal recovery

I. INTRODUCTION

This paper studies the problem of recovering both the positions and velocity vectors of moving targets in a distant ground scene, a problem referred to as ground moving target indication (GMTI). A variety of radar-based approaches to GMTI have been proposed including multi-channel phased arrays [1]–[4], displaced phase center antennas [5], and space-time adaptive processing [6]–[8]. This paper focuses on the exploitation of single-channel synthetic aperture radar (SC-SAR) data for the GMTI problem. Single-channel SAR systems are of interest because they are more available and more readily deployable on unmanned aerial vehicles and satellites where size, weight and power are critically important.

Synthetic aperture radar (SAR) systems move a single antenna through space to mimic the effect of sampling echo returns using a large distributed array of antennas. Processing algorithms applied to collected SAR data can produce nearphotographic quality, highly-focused imagery of a distant scene [9]–[11]. These imaging algorithms assume that objects in the scene are stationary over the collection interval. If the scene contains moving objects, then these objects can appear in the image to be defocused and displaced to an extent that depends on the speed and direction of the motion [12]–[14].

A. Relation to Prior Work

Techniques for single-channel SAR GMTI have been proposed by a number of authors. Fineup [15] used image "sharpness" as a metric to estimate target motion. Ouchi [16] studied how target motion reveals itself in sequences of multi-look SAR images. Kirscht [17] estimated target velocity by watching target position and signal amplitude across a sequence of SAR images. Bioucas-Dias *et al* [18] constructed matched filters that account for both amplitude and phase of echoes returned from moving targets. Werness *et al* [19] assumed the presence of three prominent point scatters on each target to estimate its motion. Barbarossa [20] applied optimal maximum likelihood schemes for detecting and focusing moving targets. Samczynski *et al* [21] applied spectral estimation to estimate the range component of velocity and applied a variation on map-drift auto-focus to obtain the along-track component. Marques *et al* [22] applied digital spotlighting to separate multiple moving targets before applying motion estimation.

A result of Chapman *et al* [13] establishes ultimate limits on moving target detection and estimation using only phase measurements from a single-aperture SAR. SAR inherently measures the two-way range to a target, and for every stationary target there exist many moving targets having the same range history, and therefore none of these movers can be distinguished from the stationary target using the phase history alone [13]. Thus there are equivalence classes of moving and stationary objects that are indistinguishable. Winkler [14, Chapter 5] provides nice graphical illustrations of these equivalence classes and proposes a single-channel SAR GMTI algorithm that uses contextual information such as roadways to remove the class ambiguity. Marques [23] also uses prior knowledge of the road network.

B. Contributions of This Paper

The present paper shows that most target motion is actually distinguishable from the static background. However, there is ambiguity in determining both location and velocity simultaneously. A technique is proposed to recover moving targets up to membership in an equivalence class. The proposed technique simultaneously estimates an image of the stationary background of the scene as well as the moving target information. The sparsity of moving targets in the scene is exploited in the recovery. The effectiveness of the method is evaluated using simulated data.

II. DATA MODEL

This section develops a model for the echo return from a moving point target. Though this discussion follows a spotlight mode SAR collection scheme [24], the results apply equally well to strip-map mode SAR [14].

The transmitted and measured echo return signals are given by

$$u_t(t) = \operatorname{rect}\left(\frac{t}{T_p}\right) \cos(2\pi [ft + \alpha t^2]) \tag{1}$$

$$u_r(t) = A \operatorname{rect}\left(\frac{t-\tau}{T_p}\right) \cos(2\pi [f(t-\tau) + \alpha(t-\tau)^2]) \quad (2)$$

where T_p is the pulse duration, α is the LFM chirp rate, and rect(t) = 1 for $|t| \le 1/2$ and is zero otherwise. The variable A in (2) models any change in amplitude due to scattering and path loss. The two-way time delay τ of the echo return is related to the SAR-to-target range R by

$$\tau = 2R/c,\tag{3}$$

where c is the speed of light.

In spot-light SAR, an echo return from an imaginary target in the scene center is used as a reference for de-chirping the received signal [24]. If τ_c is the round trip delay to scenecenter, then the received signal is mixed with

$$u_c(t) = \exp(-j2\pi [f(t-\tau_c) + \alpha(t-\tau_c)^2]).$$
 (4)

After low pass filtering and removing the residual video phase [24], we obtain

$$u(t) = A \operatorname{rect}\left(\frac{t}{T_p}\right) \exp(j2\pi\varphi(t))$$
 (5)

$$\varphi(t) = (f + 2\alpha t)(\tau_c - \tau). \tag{6}$$

Finally, the radar receiver samples u(t). Due to relative motion between the SAR and points in the scene, the delays τ_c and τ vary with time as discussed below.

For simplicity assume that the SAR platform motion is straight and level with a constant velocity vector \mathbf{v}_{SAR} = $(0, v_{SAR}, 0)$. Then the SAR trajectory is given by

$$\mathbf{x}_{\mathsf{SAR}}(t) = (0, tv_{\mathsf{SAR}}, h),\tag{7}$$

and the nadir track of the SAR is the y-axis. At time t = 0, the SAR passes through the point (0, 0, h) lying directly above the coordinate origin.

Assume a point target moves along the ground with constant velocity vector $\mathbf{v}_{target} = (v_x, v_y, 0)$. The time-varying position of the target is given by

$$\mathbf{x}_{\text{target}}(t) = (x_0 + tv_x, y_0 + tv_y, 0),$$
 (8)

where $(x_0, y_0, 0)$ is the target position at time t = 0.

Define the range vector

$$\mathbf{r}(t) = \mathbf{x}_{\text{target}}(t) - \mathbf{x}_{\text{SAR}}(t), \tag{9}$$

$$= (x_0 + tv_x, y_0 + t(v_y - v_{SAR}), h),$$
(10)

then the SAR-to-target range history is given by the positive root of

$$R(t)^{2} = \|\mathbf{r}(t)\|^{2} = At^{2} + 2Bt + C,$$
(11)

$$A = v_x^2 + (v_y - v_{\text{SAR}})^2, \tag{12}$$

$$B = v_x x_0 + (v_y - v_{\text{SAR}}) y_0 \tag{13}$$

$$C = x_0^2 + y_0^2 + h^2. (14)$$

The scene center is an imaginary stationary point at $(x_c, y_c, 0)$. The SAR-to-scene center range history $R_c(t)$ is given by (11) with $(x_c, y_c, 0, 0)$ substituted for (x_0, y_0, v_x, v_y) in the calculation of (A, B, C).



Fig. 1. Left: Position circles having radii $\sqrt{C-h^2}$. Right: Velocity circles having radii \sqrt{A} .

Combining the range histories with (3), the phase history (6) can be written as

$$\varphi(t) = (f + 2\alpha t)(2/c)[R_c(t) - R(t)].$$
(15)

(16)

Given the coordinates of the scene center (x_c, y_c) , the moving target parameters (x_0, y_0, v_x, v_y) , and the SAR altitude h and speed v_{SAR} , (15) and (5) can be evaluated at sample times to simulate measured single-channel SAR data.

III. EQUIVALENCE MOTION CLASSES FOR SINGLE-CHANNEL SAR GMTI

Equations (12-14) define a many-to-one map

$$g: (x_0, y_0, v_x, v_y) \mapsto (A, B, C).$$
 (16)
The inverse image of a point (A, B, C) is defined to be the
set $g^{-1}(A, B, C) = \{(x_0, y_0, v_x, v_y) : g(x_0, y_0, v_x, v_y) = (A, B, C)\}$. We call this set an equivalence motion class
because it describes different combinations of starting posi-
tions (x_0, y_0) and velocities (v_x, v_y) that map to the same
 (A, B, C) . Because the range history $R(t)$ in (11) depends
upon position and velocity through (A, B, C) , all points in an
equivalence class produce the same range history $R(t)$ and the
same measurement $u(t)$ in (5) also.

The ambiguity in the single-channel SAR GMTI problem arises because the inverse image of (A, B, C) contains many points (x_0, y_0, v_x, v_y) . To see this note that for fixed (A, B, C), (14) defines a circle in (x_0, y_0) -space with radius $\sqrt{C - h^2}$ centered at the origin. Call this the "position circle". Equation (12) describes a circle in (v_x, v_y) -space with radius \sqrt{A} centered at the point $(0, v_{SAR})$. Call this the "velocity circle". Figure 1 illustrates these circles. The 4-tuple (x_0, y_0, v_x, v_y) characterizes target motion. Equivalent targets (belonging to the same equivalence class) have starting positions on the same position-circle and velocities on the same velocity circle.

Equation (13) is an inner product between the starting position (x_0, y_0) and the shifted velocity vector $(v_x, v_y - v_{SAR})$ and may be expressed as

$$B = \sqrt{A}\sqrt{C - h^2}\cos\theta,\tag{17}$$

where θ is the angle between (x_0, y_0) and $(v_x, v_y - v_{SAR})$. To be in the same equivalence class, this angle is a constant. Because cosine is an even function of θ , two velocity vectors pair with each position: one making an angle θ with (x_0, y_0) and another making an angle $-\theta$. As the point (x_0, y_0) moves around the position-circle, the corresponding velocity vectors (v_x, v_y) move around the velocity circle.

An equivalence class may be constructed by choosing one



Fig. 2. Illustration of three equivalence motion classes. The solid black arcs are position circles. The dashed box is the ground scene of interest. The vectors show starting positions (x_0, y_0) and velocity vectors (v_x, v_y) that produce the same (A, B, C).

of the four variables x_0, y_0, v_x, v_y and computing the other three for a fixed A, B, and C in equations (12-14), assuming that h and v_{SAR} are also known and fixed. For example, let (A, B, C) be fixed and choose $y_0 \in [-\sqrt{C - h^2}, \sqrt{C - h^2}]$. Then compute $x_0 = \sqrt{C - h^2 - y_0^2}$. The positive square root may be taken because the SAR antenna pattern illuminates only the positive-x half plane. Next assume $y_0 \neq 0$ and solve (13) for $v_y - v_{\text{SAR}}$ yielding

$$v_y - v_{\text{SAR}} = (B - v_x x_0)/y_0.$$
 (18)

Substitute this into (12) and solve the quadratic equation $(y_0^2 + x_0^2)v_x^2 - 2Bx_0v_x + (B^2 - Ay_0^2) = 0$ (19)

 $(y_0^2 + x_0^2)v_x^2 - 2Bx_0v_x + (B^2 - Ay_0^2) = 0$ (19) for v_x , yielding two solutions that can be substituted back into (18) to compute corresponding values for v_y . Repeating this procedure for every y_0 in the interval $-\sqrt{C - h^2} \le y_0 \le \sqrt{C - h^2}$ yields the equivalence motion class for the point (A, B, C). Figure 2 illustrates several example equivalence classes.

Though not necessary, physical constraints may be introduced to restrict the size of the motion classes. For example, the coordinates (x_0, y_0) may be limited to the ground region that is illuminated by the antenna beam pattern. We may also choose a maximum speed of interest and constrain the velocities of interest to be contained in the disk $\{(v_x, v_y) :$ $\sqrt{v_x^2 + v_y^2} \le v_{\text{max}}\}$. An example rectangular ground region and a velocity disk are illustrated in Fig. 3.

GMTI is interested in moving targets. Therefore, echo returns from the stationary background scene are considered to be clutter. Because of the ambiguity inherent in single-channel SAR GMTI, some moving targets are indistinguishable from stationary targets. Therefore, we adopt the following definitions.

• **Static-clutter:** the stationary (zero velocity) background scene.

• Endo-clutter: all moving targets that lie in an equivalence



Fig. 3. The domain (top) and image (bottom) of the mapping g. Top: Portions of several position and velocity circles are overlaid on top of the ground region of interest (shaded square) and the velocities of interest (shaded disk). Bottom: The inset surface is the set of static/endo-clutter equivalence classes.

class containing a stationary target.

• **Exo-clutter:** all moving targets that do not have stationary targets in their equivalence classes.

Note that static-clutter and endo-clutter moving targets are members of the same equivalence classes. One naturally wonders about the respective sizes of the sets of static/endo-clutter and exo-clutter targets. To answer this question, consider the image under q of a rectangular scene of interest and (Cartesian product) the velocity disk illustrated in Fig. 3. The image of g is the volume bounded below by the blue surface shown and by the top and back faces which are flat and not shown. The top and back of the image of g have been left open in order to see the set of static/endo-clutter equivalence classes, which fall on the two-dimensional embedded surface (black). We can see that the volume of the static/endo-clutter is zero. Except for a set of measure zero, moving targets can be detected using a single-channel SAR! However, the exact position and velocities of the movers can only be determined up to membership in an equivalence motion class. The next section describes an algorithm for estimating motion class membership.

IV. SC-SAR GMTI ON EQUIVALENCE CLASSES

Let the vector d be all the measured data samples obtained over a collection interval using a single-channel SAR system. Often such samples are arranged in a two-dimensional array with one axis representing (fast time) sample index and the other representing (slow time) pulse index. Arrange all these samples into the one-dimensional vector d in some suitable order. Let $\mathbf{u}(x_0, y_0, v_x, v_y)$ be a vector constructed by sampling the point target response u(t) in (5), and assume that the same set of sample times are used to construct $\mathbf{u}(x_0, y_0, v_x, v_y)$ as were used in the measured vector **d**. The response u(t)depends on the position and velocity of the target. Therefore, the additional notation is included. Let $a(x_0, y_0, v_x, v_y)$ be the strength of the echo return from a target at (x_0, y_0) moving with velocity (v_x, v_y) . Assuming the principle of superposition holds, the measurement **d** is a linear combination of point target responses

$$\mathbf{d} = \sum_{x_0, y_0, v_x, v_y} \mathbf{u}(x_0, y_0, v_x, v_y) a(x_0, y_0, v_x, v_y) + \mathbf{n}, \quad (20)$$

where a Gaussian noise vector **n** has been included. The sum is over all ground positions and possible velocities. Because of equivalence classes, $\mathbf{u}(x_0, y_0, v_x, v_y) = \mathbf{u}(x'_0, y'_0, v'_x, v'_y)$ whenever the target positions and velocities belong to the same equivalence class, i.e. $g(x_0, y_0, v_x, v_y) = g(x'_0, y'_0, v'_x, v'_y) =$ (A, B, C). Therefore, there are repeated terms in (20). A more parsimonious parameterization for the target response uses the coordinates of the equivalence class (A, B, C). With this simplification, (20) becomes

$$\mathbf{d} = \sum_{A,B,C} \mathbf{u}(A,B,C) a(A,B,C) + \mathbf{n} = [\mathbf{S} \mid \mathbf{M}] \begin{bmatrix} \mathbf{s} \\ \mathbf{m} \end{bmatrix} + \mathbf{n},$$
(21)

where the columns of matrix **S** consist of all those responses $\mathbf{u}(A, B, C)$ corresponding to the static-clutter and their equivalence classes which contain all endo-clutter movers. The columns of matrix **M** consist of all $\mathbf{u}(A, B, C)$ for equivalence classes of exo-clutter movers.

The background scene may be described as being dense in the sense that the radar receives echo returns from most locations in the scene. The number of moving targets in the scene is relatively small. These working assumptions are used to estimate the parameters s and m in the model (21) using the following regularized least-squares problem:

$$\min_{\mathbf{s},\mathbf{m}} \|\mathbf{d} - \mathbf{S}\mathbf{s} - \mathbf{M}\mathbf{m}\|_2 + \lambda \|\mathbf{m}\|_0,$$
(22)

where $||\mathbf{m}||_0$ is the number of non-zero elements in \mathbf{m} . The problem in (22) seeks \mathbf{s} and \mathbf{m} that provide a good fit to the measurements \mathbf{d} such that \mathbf{m} is a sparse vector, which agrees with the assumption that there are a small number of moving targets in the scene. The user-selectable parameter λ emphasizes the importance of sparsity in the solution: large (small) values for λ make \mathbf{m} more (less) sparse.

It is well known from the theory of compressive sensing and sparse reconstruction [25] that the zero-norm regularized problem (22) is NP hard, and it is common to substitute a one-norm leading to the relaxed problem

$$\min_{\mathbf{s},\mathbf{m}} \|\mathbf{d} - \mathbf{Ss} - \mathbf{Mm}\|_2 + \lambda \|\mathbf{m}\|_1, \quad (23)$$

yielding only approximations to the true solution to (22). Solving this problem first for s yields $s = S^{\dagger}(d - Mm)$. Substituting this back into (24) yields the reduced problem

$$\min_{\mathbf{m}} \|\mathbf{P}_{\mathbf{S}}^{\perp}(\mathbf{d} - \mathbf{M}\mathbf{m})\|_{2} + \lambda \|\mathbf{m}\|_{1}, \quad (24)$$

where $\mathbf{P}_{\mathbf{S}}^{\perp}$ projects orthogonally onto the left null space of \mathbf{S} . We assume that \mathbf{S} is a tall matrix so that $\mathbf{P}_{\mathbf{S}}^{\perp} \neq \mathbf{0}$.



Fig. 4. Simulation result showing five reconstructed static point targets and the estimated equivalence class (red) corresponding to the true moving target (blue).

V. SIMULATION RESULTS

Data for a small scene consisting of five stationary point targets and one moving point target was simulated. The L1-regularized least-squares problem in (24) was solved for m and then s was computed. The background grayscale image in Fig. 4 shows the estimated static scene; the five point targets are clearly reconstructed. The true initial position and velocity vector of the moving target is shown using the blue vector. Only one equivalence class was active in the solution for m. This equivalence class is depicted in Fig. 4 as the family of red vectors. The red line is a small piece of the position circle for this equivalence class. It appears as a straight line because the scene is very small compared to the radius of the position circle. Notice that the velocity vector of the mover and its range are very well determined, but its azimuth (y) position is not well resolved.

VI. CONCLUSION

This paper showed that the set of endo-clutter moving targets is quite small (i.e., it has measure zero), and most moving targets are exo-clutter (i.e., they can theoretically be discriminated from the static scene using single-channel SAR phase history data). However, moving targets can not always be differentiated from one another. This paper defined equivalence classes of moving targets and gave a mathematical problem formulation leads to simultaneous estimation of the static scene and moving target information. The moving targets were identified up to their membership in an equivalence class. The stability of this estimation problem is related to the sampling in the space of equivalence classes of moving targets. Part of the future work aims to understand the width of the main lobe of moving target point responses in this space.

REFERENCES

- L. Lidicky, "Fourier approach to moving target indication and detection in multichannel SAR data," in *Computational Intelligence in Image and Signal Processing*, 2007. CIISP 2007. IEEE Symposium on, 2007, pp. 397–402.
- [2] B. Guo, D. Vu, L. Xu, M. Xue, and J. Li, "Ground moving target indication via multichannel airborne SAR," *Geoscience and Remote Sensing, IEEE Transactions on*, vol. 49, no. 10, pp. 3753–3764, 2011.
- [3] J. Qian, X. Lv, M. Xing, L. Li, and Z. Bao, "Motion parameter estimation of multiple ground fast-moving targets with a three-channel synthetic aperture radar," *Radar, Sonar Navigation, IET*, vol. 5, no. 5, pp. 582–592, 2011.
- [4] S. Zhu, G. Liao, Y. Qu, Z. Zhou, and X. Liu, "Ground moving targets imaging algorithm for synthetic aperture radar," *Geoscience and Remote Sensing, IEEE Transactions on*, vol. 49, no. 1, pp. 462–477, 2011.
- [5] H. S. C. Wang, "Mainlobe clutter cancellation by DPCA for spacebased radars," in *Aerospace Applications Conference*, 1991. Digest., 1991 IEEE, 1991, pp. 1/1–128.
- [6] R. Klemm, "Introduction to space-time adaptive processing," *Electronics Communication Engineering Journal*, vol. 11, no. 1, pp. 5–12, 1999.
- [7] C. D. Peckham, A. M. Haimovich, T. F. Ayoub, J. S. Goldstein, and I. S. Reid, "Reduced-rank stap performance analysis," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 36, no. 2, pp. 664–676, 2000.
- [8] W. L. Melvin, "A STAP overview," Aerospace and Electronic Systems Magazine, IEEE, vol. 19, no. 1, pp. 19–35, 2004.
- [9] Jr. C. V. Jakowatz, D. E. Wahl, P. H. Eichel, D. C. Ghiglia, and P. A. Thompson, *Spotlight-Mode Synthetic Aperture Radar: A Signal Processing Approach*, Springer, 1996.
- [10] M. Soumekh, Synthetic Aperture Radar Signal Processing, Wiley, 1999.
- [11] I. G. Cumming and F. H. Wong, *Digital Processing of Synthetic Aperture Radar Data*, Artech House, 2005.
- [12] Jen King Jao, "Theory of synthetic aperture radar imaging of a moving target," *Geoscience and Remote Sensing, IEEE Transactions on*, vol. 39, no. 9, pp. 1984–1992, 2001.
- [13] R. D. Chapman, C. M. Hawes, and M. E. Nord, "Target motion ambiguities in single-aperture synthetic aperture radar," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 46, no. 1, pp. 459–468, 2010.
- [14] J. W. Winkler, "An investigation into ground moving target indication (gmti) using a single-channel synthetic aperture radar (sar)," M.S. thesis, Brigham Young University, Provo, UT, Mar. 2013.
- [15] J. R. Fienup, "Detecting moving targets in SAR imagery by focusing," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 37, no. 3, pp. 794–809, 2001.
- [16] K. Ouchi, "On the multilook images of moving targets by synthetic aperture radars," *Antennas and Propagation, IEEE Transactions on*, vol. 33, no. 8, pp. 823–827, 1985.
- [17] M. Kirscht, "Detection, velocity estimation and imaging of moving targets with single-channel SAR," in *Proc. EUSAR*, 1998, pp. 587–590.
- [18] J. M. Bioucas Dias and P. A. C. Marques, "Multiple moving target detection and trajectory estimation using a single SAR sensor," *Aerospace* and Electronic Systems, IEEE Transactions on, vol. 39, no. 2, pp. 604– 624, 2003.
- [19] S. A. S. Werness, W. G. Carrara, L. S. Joyce, and B. D. Franczak, "Moving target imaging algorithm for SAR data," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 26, no. 1, pp. 57–67, 1990.
- [20] S. Barbarossa, "Detection and imaging of moving objects with synthetic aperture radar. 1. Optimal detection and parameter estimation theory," *Radar and Signal Processing, IEE Proceedings F*, vol. 139, no. 1, pp. 79–88, 1992.
- [21] P. Samczynski and K. Kulpa, "Fast technique for along-track motion estimation of moving targets in SAR images," in *Radar Conference -Surveillance for a Safer World, 2009. RADAR. International*, 2009, pp. 1–6.
- [22] P. A. C. Marques and J. M. Bioucas Dias, "Velocity estimation of fast moving targets using a single SAR sensor," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 41, no. 1, pp. 75–89, 2005.
- [23] P. Marques, "SAR-MTI improvement using a-priori knowledge of the road network," in *Radar Conference (EuRAD)*, 2010 European, 2010, pp. 244–247.

- [24] A. W. Doerry, "Basics of polar-format algorithm for processing synthetic aperture radar images," Tech. Rep. SAND2012-3369, Sandia National Laboratories, May 2012.
- [25] M. Elad, Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing, Springer, 2010.