A MULTIDIMENSIONAL SIGNAL PROCESSING APPROACH TO WAVE DIGITAL FILTERS WITH TOPOLOGY-RELATED DELAY-FREE LOOPS

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ABSTRACT

To avoid the occurrence of noncomputable, delay-free loops, classic Wave Digital Filters (WDFs) usually exhibit a tree-like topology. For the realization of prototype circuits that contain ring-like subnetworks, prior approaches require the decomposition of the structure and thus neglect the notion of modularity of the original Wave Digital concept. In this paper, a new modular approach based on Multidimensional Wave Digital Filters (MDWDFs) is presented. For this, the contractivity property of WDFs is shown. On that basis, the new approach is studied with respect to possible side-effects and an appropriate modification is proposed that counteracts these effects and significantly improves the convergence behaviour.

Index Terms— Wave Digital Filter, Contractivity, Delay-Free Loop, Multidimensional, Bridged-T Model

1. INTRODUCTION

The Wave Digital Filter theory [1] provides an elegant approach to the real-time capable simulation of analog circuits, ranging from virtual analog modeling in audio processing [2] to the numerical solving of different types of partial differential equations [3]. All these applications benefit from the marked similarity of an electrical prototype circuit to the corresponding Wave Digital (WD) structure, where, in particular, numerous favourable properties as passivity, stability and robustness are preserved in the WD model. Additionally, following the basic notion of translating circuit elements and their interconnection topology to corresponding minimal building blocks that may be recombined into a readily computable digital structure, the original WD concept is strictly modular (i.e. with reusable elements) and topology-preserving. This is achieved with two central definitions: First, the discretization of the complex frequency variables ψ_ν, $\nu = 1, ..., m$ is realized by means of the *bilinear transform*,

$$\psi_\nu = \frac{2}{T_\nu} \frac{z_\nu - 1}{z_\nu + 1} \ , \tag{1}$$

where T_ν represents the unit time step in the ν-th time/space variable t_ν. Second, for an arbitrary electrical port with port resistance $R > 0$, current I and voltage U, so-called *wave variables*

$$\begin{aligned} A &= U + RI \\ B &= U - RI \end{aligned} \ , \tag{2}$$

where A and B denote incident and reflected waves, respectively, are introduced. Here, all variables in capital letters indicate steady-state quantities and correspond to instantaneous quantities denoted in lower case. In conjunction with the Kirchhoff laws of conservation, equations (1) and (2) allow for a multitude of WD elements to be derived. This includes lossless blocks that take care of the interconnection (e.g. parallel or serial connections) of WD elements, so-called $adaptors$, which basically are implementations of the local scattering matrices. For a comprehensive review, the reader is referred to [1].

To obtain a realizable model, i.e. a structure without any delay-free directed loops, the bidirectional wave connection between two WD elements has to be terminated reflection-free on at least one side. Unfortunately, for the classic parallel and serial adaptors, typically just one port can be constrained to exhibit no such direct reflection, which limits the realizable structures to such of tree-like form [2]. On the other hand, there are many interesting circuits that, if rewritten to solely consist of serial and parallel connections, contain one or more *ring-like* subnetworks and thus are not representable by the classic WDF approach without introducing *macroscopic* (i.e. non-local) delay-free loops.

In prior work, a derivation procedure of adaptor structures for generalized connection networks is given in [4]. The resulting specialized multi-port adaptors again exhibit just one reflection-free port at most, so the reusability for more complex structures is questionable. Other approaches base on global scattering matrix or state space formulations and thus omit the notion of modularity of the original WD concept entirely [5], [6].

In this paper, a modular approach to address the problem of delay-free macroscopic loops is presented which is derived from MDWDF principles and utilizes a fixed point iteration scheme. For this purpose, first the contractivity properties of general WDFs are studied, which previously has been mentioned only for a specialized local problem in [7]. Then, the new approach is analysed and a modified version with much improved convergence characteristics is presented. Finally, a concrete example is given and analysed.

2. CONTRACTIVITY OF WAVE DIGITAL FILTERS

Given a metric space (M, d) with metric d, a mapping $\varphi : M \to M$ is called *contraction*, if there exists a $\lambda \in [0, 1[$ with the property that for all $x, y \in M$ the *Lipschitz condition*

$$d(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y) \tag{3}$$

holds. It can be shown that such self-mappings converge to a unique fixed point $x^* \in M$ under iteration [8], which solves the implicit

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relation $\varphi(x^*) = x^*$. Similarly, if equation (3) just holds for $\lambda \in [0,1]$, φ is called a *nonexpansive map*, which is weaker in that it does not allow conclusions about the existence of fixed points. For a deeper insight into the topic, the reader is referred to textbooks like [8].

To investigate the contractivity properties of general (nonlinear) WDFs it is feasible to first analyse a strictly linear foundation WDF: Let A_i and B_i, $i = 1, \ldots, n$, be the incident and reflected steady-state waves, respectively, of an n-port MDWDF network with corresponding finite port resistances $R_i > 0$. For practical reasons this circuit is assumed to be passive but not lossless. Let $\boldsymbol{G} = diag\left(\sqrt{G_1}, \ldots, \sqrt{G_n}\right)$, $G_i = 1/R_i$, $\boldsymbol{A} = (A_1 \cdots A_n)^T$, $\boldsymbol{B} = (B_1 \cdots B_n)^T$ and $\varphi(\boldsymbol{A}) = \boldsymbol{B}$ be the linear map that associates incident and reflected waves. Now, for the L^2-norm and the Mahalanobis metric $d_{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(\boldsymbol{x}-\boldsymbol{y})^T \boldsymbol{G}^T \boldsymbol{G} (\boldsymbol{x}-\boldsymbol{y})}$ with positive definite $\boldsymbol{G}^T\boldsymbol{G}$, the steady-state pseudopower P absorbed by that n-port according to [9], [1] can be written as

$$P = \sum_{i=1}^{n} \left(|A_i|^2 - |B_i|^2\right) G_i \overset{!}{>} 0 \tag{4}$$

$$\Leftrightarrow P = \|\boldsymbol{G}\boldsymbol{A}\|_2^2 - \|\boldsymbol{G}\boldsymbol{B}\|_2^2 > 0 \tag{5}$$

$$\Leftrightarrow \|\boldsymbol{G}\boldsymbol{A}\|_2^2 > \|\boldsymbol{G}\boldsymbol{B}\|_2^2 = \|\boldsymbol{G}\varphi(\boldsymbol{A})\|_2^2 \ , \tag{6}$$

and with $\boldsymbol{A} := \boldsymbol{A'} - \boldsymbol{A''}$ and the linearity of φ we have

$$\Leftrightarrow \left\|\boldsymbol{G}\left(\boldsymbol{A'} - \boldsymbol{A''}\right)\right\|_2^2 > \left\|\boldsymbol{G}\left(\varphi\left(\boldsymbol{A'}\right) - \varphi\left(\boldsymbol{A''}\right)\right)\right\|_2^2 \tag{7}$$

$$\Leftrightarrow \quad d_{\boldsymbol{G}}\left(\boldsymbol{A'}, \boldsymbol{A''}\right)^2 > d_{\boldsymbol{G}}\left(\varphi\left(\boldsymbol{A'}\right), \varphi\left(\boldsymbol{A''}\right)\right)^2 \tag{8}$$

$$\Leftrightarrow \quad d_{\boldsymbol{G}}\left(\boldsymbol{A'}, \boldsymbol{A''}\right) > d_{\boldsymbol{G}}\left(\varphi\left(\boldsymbol{A'}\right), \varphi\left(\boldsymbol{A''}\right)\right) \tag{9}$$

$$\Leftrightarrow \quad \begin{array}{l} \exists\, \lambda \in [0,1[\, : \\ \lambda \cdot d_{\boldsymbol{G}}\left(\boldsymbol{A'}, \boldsymbol{A''}\right) \geq d_{\boldsymbol{G}}\left(\varphi\left(\boldsymbol{A'}\right), \varphi\left(\boldsymbol{A''}\right)\right) \ . \end{array} \tag{10}$$

Clearly, an arbitrary lossy, linear WDF is contractive and thus, for constant input, converges towards a unique fixed point under iteration. Note that in order to include lossless circuits, there exists a similar relationship with substitution of "$\geq$" in equation (4), which corresponds to a nonexpansive map in equation (10).

Furthermore, in the analogous description of pseudopower for instantaneous waves a_i and b_i and this time a nonlinear φ, equations (4) and (10) are generally just related by "$\Leftarrow$", so passivity is a result of contractivity in this nonlinear case, but not the other way around. From here, similar to the interconnection considerations with respect to passivity in [9], this contractive foundation n-port can be extended with further contractive or nonexpansive (analogous to passive and lossless, respectively) structures without losing the contractivity property[1], as a composition of contractions and nonexpansive maps remains contractive. So the composition of contractive and nonexpansive WD elements always yields a contractive WD structure as long as there is at least one lossy and thus contractive element present, forcing the Lipschitz constant of the system to drop below 1. This includes general multi-port nonlinear contractions as well. Note though that a direct proof for nonlinear WDFs is not possible by means of the instantaneous and steady-state pseudopowers p and P as in [9], [1], respectively, since there is neither a steady-state description for memoryless nonlinear elements nor an appropriate instantaneous representation for frequency-dependent elements.

[1] with the exception of degenerated cases where a lossless structure is connected formally but the resulting structure is equivalent to at least two mutually independent systems

3. APPLICATION TO PROBLEMATIC WDFS

Consider a lossy $(m\text{-}1)$-dimensional WD network with at least one macroscopic loop, e.g. a WDF with ring-like topology. Such a structure is generally not realizable by means of the original construction principles, where the successive interconnection of the reflection-free port of one adaptor to a port with direct reflection of an adjacent adaptor leads to a tree-like network [2]. In ring-type structures, this connection scheme would result in at least three implicit loops: One for each signal of the bi-directional wave connection, respectively, and a third for the port resistance, which is inherited from stage to stage. Based on that, to break the latter relation, assume an arbitrary fixed common port resistance $R_c > 0$ for two connected wave ports ν and μ involved in this loop. Obviously, this introduces another local delay-free wave loop between both adaptor ports. Generally, in addition to sharing this port resistance R_c, ports ν and μ must fulfill $A_\nu = B_\mu$ and $A_\mu = B_\nu$ to be connected [9]. For vectors $\boldsymbol{A_c} = (A_\nu, A_\mu)^T$ and $\boldsymbol{B_c} = (B_\nu, B_\mu)^T$, this connection may be expressed as a linear map[2]

$$\varphi_c\left(\boldsymbol{A_c}\right) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{A_c} = \boldsymbol{B_c} \ , \tag{11}$$

which is clearly lossless and thus nonexpansive. In order to break both local and macroscopic loops simultaneously, a vectorial unit delay $\varphi_u\left(\boldsymbol{A}\right) = z_m^{-1}\boldsymbol{A}$ with dimensionless stepsize $T_m = 1$ may be introduced with respect to an artificial, additional dimension t_m. The composition

$$\varphi_u\left(\varphi_c\left(\boldsymbol{A_c}\right)\right) = z_m^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{A_c} =: \widehat{\boldsymbol{B_c}} \ , \tag{12}$$

where $\widehat{\boldsymbol{B_c}}$ denotes the new wave variable fed back to the rest of the WD structure, can, again, be shown easily to yield a lossless, nonexpansive relation (cf. [9]). For structures with multiple local or macroscopic loops, this approach may be extended directly to an arbitrary number of wave connections $l = 1, \ldots, N$ by vectorial concatenation $\boldsymbol{X}^N := (\boldsymbol{X_1}^T \cdots \boldsymbol{X_N}^T)^T$, resulting in similar maps $\varphi_c^N(\boldsymbol{A}^N) = \left(\varphi_c(\boldsymbol{A_1})^T \cdots \varphi_c(\boldsymbol{A_N})^T\right)^T = \boldsymbol{B}^N$ and $\varphi_u^N(\boldsymbol{B}^N) = z_m^{-1}\boldsymbol{B}^N$, which again are lossless and nonexpansive. That way, any wave-based implicit relation can be made explicit and thus realizable.

From here, with the findings of section 2, the resulting structure can be interpreted as a composition of a contractive map (the lossy foundation WDF) and two nonexpansive maps (φ_c^N and φ_u^N), which, for constant input, converges towards the correct solution to the non-realizable original WDF under iteration along t_m. Finally, this solution may, after having fallen below a given error threshold, be read out on the hyperplane $t_m = D \cdot T_m > 0$.

Note that the general idea of introducing a delay to break non-computable loops isn't new in itself (cf. [2], [10]), but without assigning it to an artificial extra-dimension, as proposed in this paper, reactance-like behaviour and thus barely controllable errors have to be expected.

3.1. Similarity to Multidimensional Wave Digital Filters

The resulting structure in fact closely resembles a traditional multidimensional WDF, with the exception that the outputs of the original

[2] note that in order to match the conventions of section 2, wave variables $\boldsymbol{A_c}$ denote the waves travelling into the *connection*, not the respective *ports*. Likewise, $\boldsymbol{B_c}$ denote the respective reflected waves.

WD elements with memory, especially delay elements, form inputs to the fixed point iteration and thus have to be held constant along t_m to ensure a clean convergence. But as has been presented for a similar approach in [11], correct convergence is likely to take place even by means of standard MDWDF, i.e. when all elements with memory are allowed to act in their original multidimensional sense. Here, an obvious negative side-effect is introduced by the system's dynamic along all dimensions but t_m, which is applied even to intermediate values of the fixed point iteration, thus distributing this unwanted information to successive time (or space) steps. This means that for a standard MDWDF, convergence is likely to take more iteration steps the longer the simulation is running, thereby depending largely on the decay characteristics of the original system. Apart from that, the similarity to MDWDF offers a straight forward construction approach that is entirely modular and allows the easy to read WDF notation to be used throughout.

3.2. Improved Multidimensional Approach

The approach presented in the following is based upon the multidimensional WDF principles mentioned, but employs modified delay elements to stop the aforementioned system's dynamic on intermediate values, making it independent with respect to the simulation's duration, and to fasten up convergence by utilizing its contractivity property. It is extendable to multidimensional prototype systems in principle, but depends on the processing sequence here and thus is not available in closed-form. Therefore, it is presented for a one-dimensional system that is extended by an artificial dimension $m = 2$ to solve the original computability problem.

First, assume a constant simulation length of D steps with respect to the artificial t_2-direction, whereas the time t_1 remains unbounded. To cut the unwanted dynamic, the values of the delays in t_1-direction are simply held constant along the t_2-axis according to

$$\begin{aligned} b_{T_1}(k_1, k_2) = a_{T_1}(k_1 - 1, D - 1), \\ k_2 = 0, \dots, D - 1, \end{aligned} \tag{13}$$

where a_{T_1} and b_{T_1} denote the delay's in- and output, respectively, which corresponds to a discrete sample and hold element that is refreshed every D steps. Note though that its values are sampled at $k_2 = D - 1$, where the system's state, per construction, already should have approached an equilibrium state. This ensures a clean fixed point iteration scheme. Furthermore, the delays in t_2-direction are modified as well: Here, by re-iterating the WD structure, a fixed point is approached and fed back via

$$b_{T_2}(k_1, k_2) = \begin{cases} a_{T_2}(k_1, k_2 - 1) & , k_2 = 1, \dots, D - 1 \\ a_{T_2}(k_1 - 1, D - 1) & , k_2 = 0 \end{cases} \tag{14}$$

as a starting value to the next t_1-sample's fixed point iteration. Due to the nature of contractive mappings, a reasonably close starting value may speed up convergence a lot. So, for non-erratic signals, even if D is chosen too small and convergence has a significant remaining error, the starting value for the next sample in general is better than some constant boundary value. With this approach, convergence can be balanced between a longer iteration length D or more error along the time axis t_1. Obviously, for constant input and perfect convergence, the starting value is already the solution of the fixed point scheme at any t_1-step. Similarly, for low-frequency inputs it is already close, leading to a fast convergence, which is further helped by anti-aliasing techniques like upsampling.

Note that a sample and hold element in its hold state, replacing the delay element of e.g. a capacitor, corresponds to a WD resistive source, where the input to the element is omitted and a constant source value is fed back. As the system's contractivity is a global property, it is invariant with respect to the addition of a constant and so the Lipschitz condition (3) holds here as well.

Clearly, the proposed modifications are not passive in the conventional sense, but are lossless at least for constant input. And since it has been shown that the circuit settles towards a constant state of equilibrium along the iteration axis t_2, instabilities are unlikely to occur in that case.

4. EXAMPLE: BRIDGED-T NOTCH FILTER

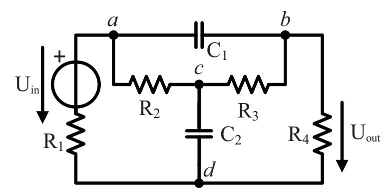


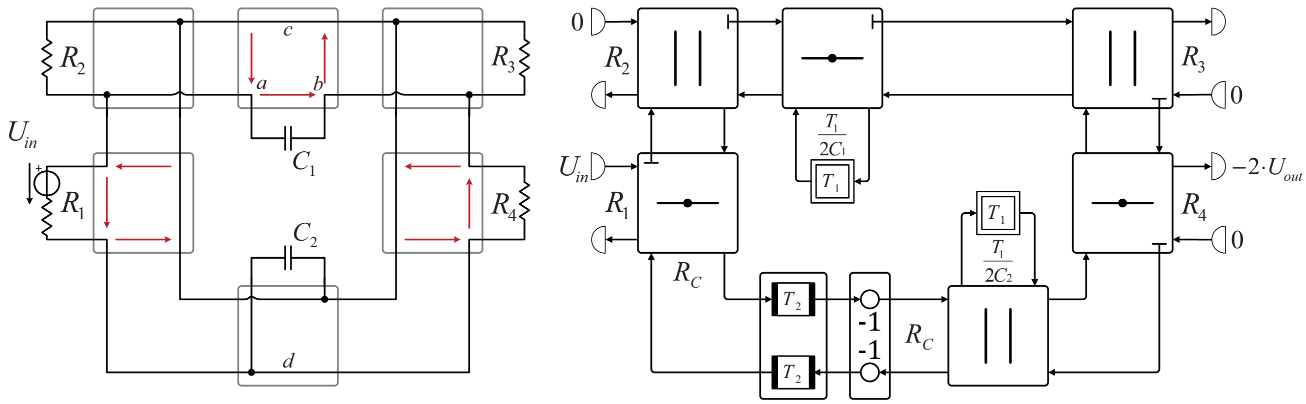
Fig. 1. Prototype circuit with bridged-T topology

To give a classic example to the presented approach, the linear bridged-T filter depicted in Fig. 1 is analysed in the following. For $R_i = R,\ i = 1, \dots, 4$, it yields a notch-type frequency response with regard to input and output voltages U_{in} and U_{out}, respectively. In this example, the values $R = 10k\Omega$, $C_1 = 10nF$ and $C_2 = 1\mu F$ have been chosen.

4.1. Derivation of a Wave Digital Structure

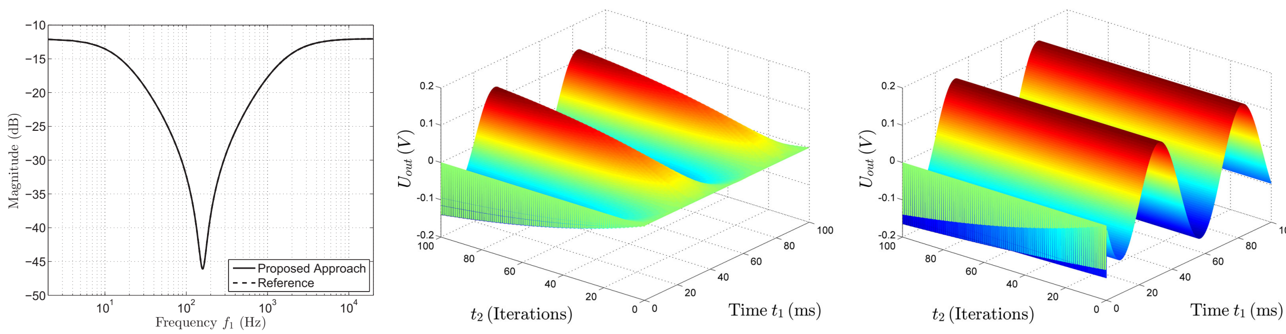
To derive a Wave Digital Structure from a schematic like in Fig. 1, it is advisable to separate the circuit elements from their connection network as depicted in Fig. 2(a) first. While this equivalent structure is still denoted in Kirchhoff domain, the elements' connectivity is dissected into discrete parallel and serial connections, respectively, which already resembles the appearance of the resulting WDF. To achieve and verify such a respresentation, it is sensible to focus on the circuit's nodes, i.e. nodes a, b, c, d in this example, indicated in both figures 1 and 2(a). Clearly, the resulting structure has a ring-like topology, which hinders a direct translation into a corresponding WDF. But, with the findings of sections 2 and 3, a readily computable WDF can be derived, shown in Fig. 2(b). With the introduction of an artificial bidirectional delay to the ring-structure, the occurrence of delay-free loops can be avoided. Here, the modified delay elements as presented in section 3.2 are utilized, denoted by double borders for the T_1 elements and bold borders for the delays T_2, respectively.

Additionally, note that due to the original sign convention of the serial adaptor [1], a direct replacement of the serial connections in Fig. 2(a) with the corresponding adaptor is not possible, as voltages here are measured counterclockwise, denoted by red arrows here. This leads to a sign inversion at every serial adaptor in the loop, which, due to the odd number of serial adaptors, has to be compensated by a voltage inverter, realized by means of simple sign inversion of the respective wave variables (cf. [12]). For the same reason, the output voltage U_{out} appears with inversed sign.



(a) Rewritten prototype circuit (b) WDF realization

Fig. 2. Bridged-T circuit as in Fig. 1 with explicit ring-like topology in Kirchhoff domain (a) and the proposed WDF realization (b).



(a) Frequency response, $D = 3$ (b) Unmodified MDWDF (impedance mismatch) (c) Improved Approach (impedance mismatch)

Fig. 3. Simulation results. (a): Frequency response of the resulting WDF with matched impedances $R_c = R_z$. Reference and WDF simulations produce virtually identical results after $D = 3$ iterations. (b) and (c): Convergence characteristics of MDWDF and improved approach for a deliberate mismatch $R_c = 100 \cdot R_z$ and $U_{in} = -cos(2\pi \cdot 20 \cdot t_1)$ and $D = 100$. The proposed improvements clearly accelerate convergence.

4.2. Specific Convergence Characteristics

One unobvious property of the artificial port resistance R_c introduced in section 3, representing a free parameter up to now, is its strong influence on convergence speed. Here, it is advisable to impedance-match to the rest of the circuit to prevent unnecessary direct reflections in the local loop containing the inserted bidirectional delay from happening. This value R_z can be obtained by means of traditional network analysis, but has to be calculated with the actual port resistance values used in the simulation in place of Kirchhoff impedances. In this example, this leads to $R_z \approx 13.8\,k\Omega$ for the chosen samplerate $F_s = 40kHz$. By matching $R_c = R_z$, the proposed method yields a frequency response as depicted in Fig. 3(a) after a mere $D = 3$ iterations. The correctness of this is easily verified, as the result is virtually identical to the reference simulation generated with Simulink SimPowerSystems, shown superimposed in Fig. 3(a) as well.

For a deliberate mismatch $R_c = 100 \cdot R_z$, convergence speed is much slower, as shown for cosine input in Fig. 3(b) for the unmodified MDWDF approach and in Fig. 3(c) for the proposed improved approach, respectively. Both simulations converge towards the same solution, but the improved approach does so almost instantaneously for slowly changing input. Note that, as expected, the initial jump at $t_1 = 0$ makes both methods converge at a similar rate for this first time step, requiring more than the chosen $D = 100$ iterations.

5. CONCLUSION

In the present paper, a multidimensional approach to overcome topology-induced delay-free loops in Wave Digital structures has been presented. In comparison to prior methods, this approach has the advantage of maintaining the modularity property of the Wave Digital concept. To achieve an appropriate theoretical fundament, the contractivity properties of Wave Digital Filters have been studied. It could be shown that, on a basis of a linear foundation WDF, every lossy WD structure is contractive and can be extended with further contractive (or just nonexpansive) elements, which is similar to the concept of passivity known from the general Wave Digital principles. On that basis, general construction considerations have been made and possible negative side-effects have been brought up. To counteract the latter, a modified multidimensional approach has been introduced, basically representing a structurally modular and easy to implement fixed point iteration scheme. Finally, a concrete example has been given, confirming the former findings.

6. REFERENCES

[1] Fettweis, A., *Wave digital filters: Theory and practice,* Proceedings of the IEEE , vol.74, no.2, pp.270-327, Feb. 1986

[2] De Sanctis, G.; Sarti, A., *Virtual Analog Modeling in the Wave-Digital Domain,* Audio, Speech, and Language Processing, IEEE Transactions on , vol.18, no.4, pp.715,727, May 2010

[3] Bilbao, S., *Wave and Scattering Methods for Numerical Simulation*, John Wiley and Sons, 368 pp., Chichester, UK, 2004

[4] Meerkötter, K.; Fränken, D., *Digital realization of connection networks by voltage-wave two-port adaptors*, Int. J. Electron. Commun, vol. 50, pp. 362367, 1996.

[5] Petrausch, S.; Rabenstein, R., *Wave digital filters with multiple nonlinearities*, in Proc. 12th European Signal Processing Conf. (EUSIPCO-2004), pp.77-80, 2004

[6] Christoffersen, C., *Transient Analysis of Nonlinear Circuits Based on Waves*, Scientific Computing in Electrical Engineering SCEE 2008, Springer Berlin Heidelberg, 2010

[7] Ochs, K., *Passive Integrationsmethoden*, Dissertation, Universität Paderborn, Paderborn, Germany, 2001

[8] Kirk, W.A.; Sims, B. (Eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic, Dordrecht, 2001

[9] Fettweis, A., *Pseudo-passivity, sensitivity, and stability of wave digital filters,* Circuit Theory, IEEE Transactions on , vol.19, no.6, Nov 1972

[10] Pakarinen, J.; Karjalainen, M., *Enhanced Wave Digital Triode Model for Real-Time Tube Amplifier Emulation,* Audio, Speech, and Language Processing, IEEE Transactions on , vol.18, no.4, pp.738-746, May 2010

[11] Schwerdtfeger, T.; Galkowski, K.; Kummert, A., *Stabilization of a Class of Active Ladder Circuits based on a Wave Repetitive Process Approach by means of Multidimensional Wave Digital Filters*, Proceedings of the 8th International Workshop on Multidimensional Systems (nDS), 2013

[12] D'Angelo, S.; Valimaki, V., *Wave-digital polarity and current inverters and their application to virtual analog audio processing,* Acoustics, Speech and Signal Processing (ICASSP), 2012 IEEE International Conference on ,pp.469,472, March 2012