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THE FAREY-DICTIONARY FOR SPARSE REPRESENTATION OF PERIODIC SIGNALS

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ABSTRACT

A finite duration sequence exhibiting periodicities does not in general admit a sparse representation in terms of the DFT basis unless the period is a divisor of the duration. This paper develops a dictionary called the Farey dictionary for the efficient representation of such sequences. It is shown herein that this representation is especially useful for identifying hidden periodicities in a finite data record. The properties of the Farey dictionary are studied, and the dictionary is shown to be superior to the conventional DFT based uniform dictionary, from the view point of identifying hidden periods.

Index Terms— Farey dictionary, uniform DFT dictionary, sparse reconstruction, hidden periodicities.

1. INTRODUCTION

Consider a finite duration signal $x(n), 0 \leq n \leq q-1$ with DFT $X[k]$. Suppose $x(n)$ has the property

$$x(n) = x(n+N) \tag{1}$$

for some integer $N < q$, and for all n such that the arguments n and $n+N$ are in $[0, q-1]$. Then we say that N is the period of $x(n)$ assuming N is the smallest such integer. Suppose we represent $x(n)$ using the DFT as a basis:

$$x(n) = \sum_{k=1}^{q} \alpha_k W_q^{nk} \tag{2}$$

Here $W_q = e^{-j2\pi/q}$ and $\alpha_k = X[-k]/q$ (with k interpreted modulo q). Then, depending on the period N, the coefficients $\{\alpha_k\}$ may or may not exhibit sparsity. For example suppose N is a divisor of q, say $q = 12$ and $N = 4$, as in Fig. 1(a) (the first period is highlighted for clarity). Then $\alpha_k = 0$ unless k is a multiple of q/N. But if N is not a divisor of q, e.g., $q = 12$, and $N = 5$ as in Fig. 1(b), then α_k can be nonzero for all k because we do not have an integer number of periods in the duration q. Thus the DFT is in general not an economic representation in this case. For applications such as period identification, or extraction of a periodic signal from noise, it is desirable to have a more economic representation.

If a signal $x(n)$ has the form $x(n) = x_N(n) + x_M(n)$ where $x_N(n)$ and $x_M(n)$ have period N and M (both $< q$) respectively, then its period is the lcm of N and M (or possibly a divisor). If the data length q is less than this lcm (although $M, N < q$), then the signal does not "look" periodic

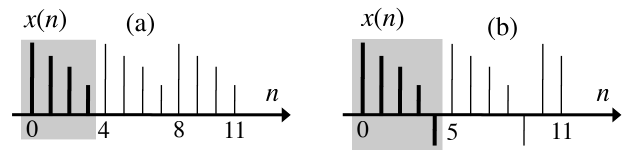

Fig. 1. (a) A 12-point sequence with hidden periodicity 4, and (b) A 12-point sequence with hidden periodicity 5. In the latter case $x(n)$ does not have an integer number of periods.

(see Fig 3 later). In this case we say that N and M are the **hidden periods**. Identifying two or more hidden periods is more challenging than identifying a single explicit period $N < q$, which is rather straightforward.[1]

In this paper we address the problem of representing periodic signals (hidden or otherwise) in terms of a dictionary which we call the Farey dictionary. This is motivated by the well known Farey series in the theory of numbers [9]. We can use sparse reconstruction techniques such as basis pursuit, Lasso, etc. [2]–[7], [14], [16] for the identification of hidden periods, and compare the results with the use of a conventional uniform-grid (DFT style) dictionary. The Farey series is reviewed briefly in Sec. 2. The Farey dictionary is introduced in Sec. 3, and its properties are studied in Sec. 4, including its Kruskal rank (which is crucial for its suitability in sparse reconstruction). In Secs. 4, 5 we explain how the Farey dictionary can be used to identify the hidden periodicities, and compare this with the uniform dictionary. Examples of this application are given in the end, which demonstrate that the Farey dictionary is indeed well suited to identify hidden periods.

Notations: (a) (k, m) stands for the gcd of the integers k and m. So $(k, m) = 1$ means that they are coprime. (b) $N|q$ means that N is a divisor of q. (c) $\phi(m)$ is the *Euler totient* function (number of integers in $1 \leq i \leq m$ coprime to m) [9]. (d) Finally $W_q = e^{-j2\pi/q}$.

2. THE FAREY SEQUENCE

Given an integer q, consider all irreducible rational numbers $x = k/m$ (i.e., with $(k, m) = 1$) in the range $0 \leq x \leq 1$, with denominator $m \leq q$. There is clearly a finite number of them, namely $1+\sum_{m=1}^{q} \phi(m)$. If these numbers are arranged

[1] If $x(n)$ has a period $N < q$, then in absence of noise we can compute $\Delta(n) = x(n) - x(n-K)$ for $K = 1, 2, \ldots$, to identify N. The smallest K for which $\Delta(n) = 0$ for $K \leq n \leq q-1$ is $K = N$.

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in increasing order we get a sequence of rationals which we denote as F_q. For example if $q = 6$ we get F_6:

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \tag{3}$$

This sequence F_q of rationals is called the Farey sequence of order q (see [9]).[2] Figure 2 shows the step by step construction of F_6 from its definition. The Farey sequences F_8, F_{10} and F_{14} are also shown.

Notice that if k/m is an element of F_q then so is $(m - k)/m$ because if $(k, m) = 1$ then so is $(m - k, m) = 1$. Thus the elements in the array are *symmetric* with respect to $1/2$ as seen from the examples in Fig. 2. The term *Farey array* is applicable if we are building a linear array of sensors with these sensor locations. In this paper our goal is to construct a dictionary which discretizes the frequency (or parameter) space using a Farey grid, and show that such a dictionary is useful in the representation of signals with hidden periodicities.

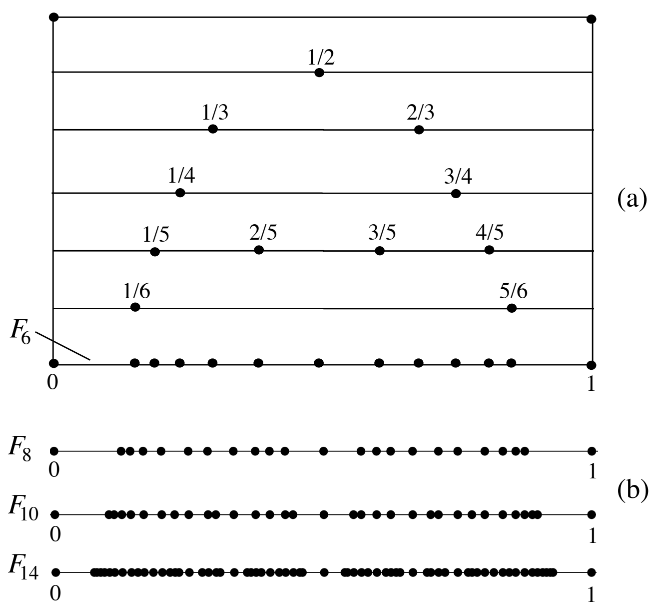

Fig. 2. (a) Step by step construction of the Farey sequence or array F_6. First, $F_1 = \{0, 1\}$ is shown. The individual arrays k/m ($(k, m) = 1$) for $2 \leq m \leq 6$ are then shown, and their union F_6 is shown in the bottom. (b) Farey arrays for $q = 8, 10$, and 14.

3. FAREY DICTIONARY AS A FREQUENCY-GRID

The m-point DFT represents a signal in terms of a set $\mathcal{X}_m$ of frequencies $\omega_m(i) = 2\pi i/m, \quad 1 \leq i \leq m$. Consider the *union* of all these frequencies, as m varies in the range $1 \leq m \leq q$. The set $\mathcal{X}_m$ has m frequencies, but the union has less than $\sum_{m=1}^{q} m$ frequencies because of overlap among the sets $\mathcal{X}_m$. To avoid this overlap, define the set $\mathcal{Y}_m$ of frequencies

$$\omega_m(i) = \frac{2\pi i}{m}, \quad (i, m) = 1, \text{ for } 1 \leq i \leq m. \tag{4}$$

[2] It is actually called the Farey *series* in the literature but since no summations are involved, it is appropriate to call it a sequence.

Then the sets $\mathcal{Y}_m$ are disjoint for different m. Each set has $\phi(m)$ elements. So the total number of distinct elements in the set

$$\mathcal{F}_q = \bigcup_{m=1}^{q} \mathcal{Y}_m \tag{5}$$

is

$$\Phi(q) \triangleq \sum_{m=1}^{q} \phi(m) \tag{6}$$

For example if $q = 6$, then

$$\begin{aligned} \Phi(6) &= \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5) + \phi(6) \\ &= 1 + 1 + 2 + 2 + 4 + 2 = 12 \end{aligned} \tag{7}$$

So $\mathcal{F}_6$ has 12 elements. Now define a column vector of size q:

$$\mathbf{v}(m, k) = \begin{bmatrix} 1 \\ W_m^k \\ W_m^{2k} \\ \vdots \\ W_m^{(q-1)k} \end{bmatrix}, \tag{8}$$

and the $q \times \phi(m)$ matrix

$$\mathbf{V}_m = \begin{bmatrix} \mathbf{v}(m, k_1) & \mathbf{v}(m, k_2) & \ldots & \mathbf{v}(m, k_{\phi(m)}) \end{bmatrix} \tag{9}$$

where $(m, k_i) = 1$ and $1 \leq k_i \leq m$. Based on this we define the dictionary

$$\mathbf{A}_q^{(f)} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \ldots & \mathbf{V}_q \end{bmatrix} \tag{10}$$

which is a $q \times \Phi(q)$ matrix. For example if $q = 6$ this dictionary has the form shown at the top of the next page. (Even though $W_6^{10} = W_6^4$, etc., we have kept the raw numbers in the exponents for clarity.) This dictionary can be used to give a representation for $\mathbf{x}$ in the form

$$\mathbf{x} = \mathbf{A}_q^{(f)} \mathbf{d} \tag{11}$$

Even though $\mathbf{d}$ is not unique for a given $\mathbf{x}$, we can impose a sparsity constraint on it to obtain efficient representations for signals dominated by hidden periodicities.

Returning now to the dictionary (10) we see that the frequencies covered by the columns of $\mathbf{A}_q^{(f)}$ have the form $2\pi f$ where $f \in F_q$. So we refer to $\mathbf{A}_q^{(f)}$ as the Farey dictionary. Since $2\pi f$ represents the same frequency for $f = 0$ and $f = 1$, we can eliminate one of them. So the Farey dictionary has $\Phi(q) = \sum_{m=1}^{q} \phi(m)$ atoms.

4. APPLICABILITY OF FAREY DICTIONARY IN SPARSE RECOVERY

Recall that the Kruskal rank ρ of a matrix is the largest integer such that any set of ρ columns is linearly independent. It is well known [7] that if the kruskal rank is ρ then we can solve

$$\begin{array}{c} m \rightarrow \quad 1 \quad 2 \quad 3 \quad 3 \quad 4 \quad 4 \quad 5 \quad 5 \quad 5 \quad 5 \quad 6 \quad 6 \end{array}$$

$$\mathbf{A}_6^{(f)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_2 & W_3 & W_3^2 & W_4 & W_4^3 & W_5 & W_5^2 & W_5^3 & W_5^4 & W_6 & W_6^5 \\ 1 & W_2^2 & W_3^2 & W_3^4 & W_4^2 & W_4^6 & W_5^2 & W_5^4 & W_5^6 & W_5^8 & W_6^2 & W_6^{10} \\ 1 & W_2^3 & W_3^3 & W_3^6 & W_4^3 & W_4^9 & W_5^3 & W_5^6 & W_5^9 & W_5^{12} & W_6^3 & W_6^{15} \\ 1 & W_2^4 & W_3^4 & W_3^8 & W_4^4 & W_4^{12} & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} & W_6^4 & W_6^{20} \\ 1 & W_2^5 & W_3^5 & W_3^{10} & W_4^5 & W_4^{15} & W_5^5 & W_5^{10} & W_5^{15} & W_5^{20} & W_6^5 & W_6^{25} \end{pmatrix}$$

uniquely for $\mathbf{d}$ from (11) as long as $\mathbf{d}$ has sparsity $s \leq \rho/2$. (i.e., the number of nonzero elements in $\mathbf{d}$ is $s \leq \rho/2$). We now prove that $\mathbf{A}_q^{(f)}$ has full *Kruskal rank*.

Lemma 1. The $q \times \Phi(q)$ dictionary $\mathbf{A}_q^{(f)}$ has Kruskal rank q, i.e., it has full Kruskal rank. $\diamondsuit$

Proof. The column vector in Eq. (8) is a Vandermonde vector, so the matrix $\mathbf{A}_q^{(f)}$ is a Vandermonde matrix. Furthermore the first element W_m^k in Eq. (8) is different for the different columns in $\mathbf{V}_m$ because $1 \leq k \leq m$. We now claim that this first element is different for different columns in $\mathbf{A}_q^{(f)}$ also. To see this assume

$$W_{m_1}^{k_1} = W_{m_2}^{k_2} \tag{12}$$

where $m_1 < m_2$, and k_i are such that $(k_i, m_i) = 1$ and $1 \leq k_i \leq m_i$. Then $W_{m_1 m_2}^{m_2 k_1} = W_{m_1 m_2}^{m_1 k_2}$, or

$$m_1 k_2 - m_2 k_1 = l m_1 m_2 \tag{13}$$

for integer l. That is, $m_1 k_2 = m_2 l_1$ for some integer l_1, which implies $k_2/m_2 = l_1/m_1$. But since $(k_2, m_2) = 1$ and $m_1 < m_2$, this is impossible. This shows that the Vandermonde matrix $\mathbf{A}_q^{(f)}$ is such that the first element W_m^k is distinct for any two columns. Since $\mathbf{A}_q^{(f)}$ has q rows, this proves that any set of q columns in $\mathbf{A}_f$ is linearly independent. That is, $\mathbf{A}_q^{(f)}$ has Kruskal rank q. $\bigtriangledown \bigtriangledown \bigtriangledown$

Procedurally, how do we identify hidden periods with the Farey dictionary? First assume the period is not hidden, that is, $x(n) = x(n + N)$, with $N < q$. Then it can be represented as $x(n) = \sum_{k=0}^{N-1} \alpha_k W_N^{nk}$ for $0 \leq n \leq q - 1$. Assuming $N \leq q/2$ (q being the Kruskal rank of $\mathbf{A}_q^{(f)}$), this means that $\mathbf{x}$ can be represented as a linear combination of $N \leq q/2$ atoms in the dictionary. Since the Kruskal rank is large enough we can identify these atoms using sparse recovery techniques. These atoms are Vandermonde vectors generated by W_N^k (where $0 \leq k \leq N - 1$). In the dictionary W_N^k appears in the reduced form $W_{N_i}^{k_i}$ where $N_i | N$ and $(N_i, k_i) = 1$. Thus, the largest of these N_i's represents the period N.

More interestingly, suppose the signal has two **hidden periods**, that is, $x(n) = x_N(n) + x_M(n)$ with $M < N < q$ but $\text{lcm}(M, N)$ is larger than the data length q. This signal needs two sets of Vandermonde vectors from the dictionary atoms: (a) $W_{N_i}^{k_i}$ where $N_i | N$ and $(N_i, k_i) = 1$, and (b) $W_{M_i}^{l_i}$ where $M_i | M$ and $(M_i, l_i) = 1$. As long as $M + N \leq q/2$, we can identify the atoms. The hidden periods M and N are the two unique integers in the set $\mathcal{S} = \{N_1, N_2, \ldots, N, M_1, M_2, \ldots, M\}$, which are *not* divisors of any other integer in $\mathcal{S}$. So we can readily identify the hidden periods. Since some of the W_{N_i} might overlap with some of the W_{M_i}, we may not be able to separate out $x_M(n)$ and $x_N(n)$ in general. For the special case where M and N are coprime, we can separate them out except for a constant ("DC") term. If there are more than two periods, the idea works similarly.

5. THE FAREY DICTIONARY VERSUS THE UNIFORM DICTIONARY

The Farey dictionary $\mathbf{A}_q^{(f)}$ has size $q \times \Phi(q)$. Notice that $\Phi(q)$ is usually much larger than q. For example:

$$\Phi(8) = 22, \Phi(10) = 32, \Phi(14) = 64, \Phi(32) = 324, \ldots \tag{14}$$

It can be shown [9] that

$$\Phi(q) = \frac{3q^2}{\pi^2} + O(q \log q) \tag{15}$$

So for large q, we have $\Phi(q) \approx 3q^2/\pi^2$. The dictionary $\mathbf{A}_q^{(f)}$ therefore has $O(q^2)$ columns where q is the number of rows. It gets fatter as q grows.

Now consider a dictionary of the same size $q \times \Phi(q)$ constructed as follows: with $\mathbf{W}_{\Phi(q)}$ denoting the $\Phi(q) \times \Phi(q)$ DFT matrix, simply retain the first q rows. For example, with $q = 6$, this matrix is of size 6×12 and is:

$$\mathbf{A}_6^{(u)} = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & W_{12} & W_{12}^2 & W_{12}^3 & \ldots & W_{12}^{10} & W_{12}^{11} \\ 1 & W_{12}^2 & W_{12}^4 & W_{12}^6 & \ldots & W_{12}^{20} & W_{12}^{22} \\ \vdots & \vdots & \vdots & \ldots & \vdots & & \\ 1 & W_{12}^5 & W_{12}^{10} & W_{12}^{15} & \ldots & W_{12}^{50} & W_{12}^{55} \end{bmatrix} \tag{16}$$

We shall call this the **uniform (grid) dictionary** because each row contains *successive* powers of $W^{nl}_{\Phi(q)}$. So each row is a Vandermonde vector, and so is each column. Clearly any set of q columns is linearly independent and the uniform dictionary also has Kruskal rank q. Notice by the way that the Farey dictionary has Vandermonde columns but not Vandermonde rows.

Given a q point signal $\mathbf{x}$ with hidden periodicity, suppose we represent it with the dictionary $\mathbf{A}^{(u)}_q$, that is

$$\mathbf{x} = \mathbf{A}^{(u)}_q \mathbf{a} \tag{17}$$

How does this compare with the representation based on the Farey dictionary, i.e.,

$$\mathbf{x} = \mathbf{A}^{(f)}_q \mathbf{d} \tag{18}$$

The lth column of the uniform dictionary represents a length q signal of the form

$$W^{ln}_{\Phi(q)}, \quad 0 \le n \le q-1 \tag{19}$$

Even though l ranges from 0 to $\Phi(q)-1$ (equivalently 1 to $\Phi(q)$), a sparse representation results from the uniform dictionary only when the periodicities are the divisors of $\Phi(q)$. But the Farey dictionary can represent any hidden periodicity efficiently (when the sum of hidden periods $\le q/2$) because such a signal can be fully represented by the union of the columns of appropriate DFT matrices which are included in the Farey dictionary. The following examples for signals with a single, non-hidden period, already demonstrate the power of the Farey dictionary:

Example 1. Let $q = 42$. Then $\Phi(q) = 542 = 2 \cdot 271$. The factors of $\Phi(q)$ which are $\le q/2$ are 1 and 2. So the uniform dictionary yields a sparse representation only for these two periods. For other periods the representation is in general not sparse. But the Farey dictionary is guaranteed to give a sparse representation for 21 periods, namely $1 \le N \le 21$.

Example 2. Let $q = 32$. Then $\Phi(q) = 324 = 2^2 \cdot 3^4$. The factors of $\Phi(q)$ which are $\le q/2$ are $1, 2, 3, 4, 6, 9$, and 12. So the uniform dictionary yields a sparse representation for these 7 periods. The Farey dictionary is guaranteed to give a sparse representation for all the 16 periods, namely $1 \le N \le 16$.

The procedure to identify periods $N|q$ with the uniform dictionary is as follows. Starting from the measurement $\mathbf{x}$ we identify the sparse solution $\mathbf{a}$ for $\mathbf{x} = \mathbf{A}^{(u)}_q \mathbf{a}$. So we know the $q/2$ (or fewer) atoms of the dictionary that represent $\mathbf{x}$. These are Vandermonde atoms generated by $W^k_N = W^{(\Phi(q)/N)k}_{\Phi(q)}$. So, in the dictionary these are present in the column positions which are multiples of $\Phi(q)/N$. From this pattern, N can be identified. If there are two hidden periods M and N (both divisors of q), the procedure is similar.

Computer experiment: identifying hidden periods. We consider a signal with hidden periods $N = 12$ and $M = 15$, and total duration $q = 59$. Figs. 3 (a)–(c) show the signal $x(n)$ (vector $\mathbf{x}$ in Eqs. (17) and (18)), and the hidden periodic components $x_M(n)$ and $x_N(n)$. The hidden periodicities cannot be seen from the plot of $x(n)$. Since $\Phi(q) = 1086$, the dictionaries have size 59×1086. Figs. 3(d)-(e) show a plot of the components of $\mathbf{d}$ in the Farey representation (18)), and those of $\mathbf{a}$ in the uniform representation (17). The vectors $\mathbf{a}$ and $\mathbf{d}$ were identified using l_1 minimization [4]. In presence of noise, an algorithm like Lasso [16] will be more appropriate. We see that the Farey representation is much sparser as expected (it uses 24 atoms whereas the uniform dictionary uses 48). The first 74 coefficients of the Farey representation (which includes all the 24 nonzero coefficients) are shown separately in Fig. 3(f) for clarity. From the location of the dictionary atoms which contribute to the nonzero components of the Farey representation $\mathbf{d}$, we can identify all the integers m such that powers of W_m participate in this representation. The set of all such m is: $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 12, 15\}$. The only integers in this set which are *not* divisors of any other integer in the set are 12 and 15. So the hidden periods $N = 12$ and $M = 15$ have been identified.

6. CONCLUDING REMARKS

In this paper we introduced the Farey dictionary, and showed that it can be used to obtain a sparse representation for signals with hidden periodicities. By using standard sparse recovery techniques, we can identify the hidden periods. Notice that if there are K hidden periods, then the Farey dictionary can identify the integer K: it is equal to the number of elements in $\mathcal{S}$ which are not divisors of other integers in $\mathcal{S}$. For low noise situations, the above method continues to work well, when Lasso is used. For high noise scenarios, the representation is not sparse, and optimal thresholding techniques would have to be introduced to extract the periodicity information. This will be a topic for future research. The connection to Ramanujan-sum expansions (see [15] and references therein) and integer dictionaries will be explored in future. The relation to modern super resolution methods [5], sparse spectrum sensing [8], and algorithms such as MUSIC [11], [13] also remain to be explored.

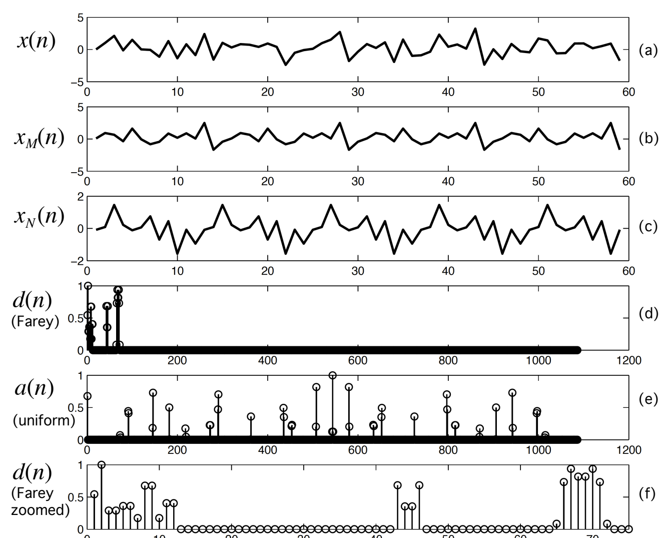

Fig. 3. (a) A signal $x(n) = x_M(n) + x_N(n)$, with hidden periodic components $x_M(n)$ and $x_N(n)$. (b), (c) The hidden periodic components. (d) The sparse representation $d(n)$ using a Farey dictionary. (e) The representation $a(n)$ using a uniform dictionary. (f) Zoomed-in plot of the sparse representation $d(n)$ using a Farey dictionary.

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