ON THE COMPLEXITY OF SINR OUTAGE CONSTRAINED MAX-MIN-FAIRNESS MULTICELL COORDINATED BEAMFORMING PROBLEM

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ABSTRACT

Max-min-fairness (MMF), which concerns optimizing the worst signal-to-interference-plus-noise ratio (SINR) performance of receivers, is a popular transmitter design criterion in multiuser communications. In the single-input single-output (SISO), multiple-input single-output (MISO), and single-input multiple-output (SIMO) interference channels with perfect channel state information at the transmitters, it has been shown that the MMF power allocation and beamforming design problems are polynomial-time solvable, and efficient optimization algorithms exist. In this paper, we assume that the transmitters have channel distribution information only, and study the MMF coordinated beamforming design problem under probabilistic SINR outage constraints. While such a problem is nonconvex, it was not clear if it is polynomial-time solvable. We propose a complexity analysis, showing that the SINR outage constrained MMF problem is polynomial-time solvable in the SISO scenario whereas it is NP-hard in the MISO scenario. The NP-hardness is established by showing that the MISO MMF problem is at least as difficult as the 3-satisfiability problem which is NP-complete.

Index Terms— Interference channel, max-min fairness, outage probability constraint, transmit beamforming, complexity analysis

1. INTRODUCTION

Inter-cell interference is one of the main bottlenecks in wireless cellular networks, and various interference management techniques have been proposed to enhance the system performance [1]. In this paper, we consider the *coordinated beamforming* (CoBF) design problem in a multiple-input single-output (MISO) interference channel (IFC) where K pairs of transmitters and receivers communicate simultaneously over a common spectrum [2]. Our interest lies in the max-min-fairness (MMF) CoBF design, which aims to optimize the worst signal-to-interference-plus-noise ratio (SINR) performance of the K receivers. The MMF CoBF design problem has been extensively studied, under the assumption that the transmitters have perfect knowledge of the channel state information (CSI). In particular, it has been shown in [3, 4] that the MMF CoBF design problem is polynomial-time solvable, in both MISO and single-input multiple-output (SIMO) scenarios, and efficient algorithms were proposed. However, for the multiple-input multiple-output (MIMO) IFC, the MMF joint transmit and receive beamforming design problem is NP-hard, as shown in [5].

In view of that obtaining instantaneous CSI is not always feasible, especially in fast fading scenarios, we assume that the transmitters have only channel distribution information (CDI). Under such circumstances, SINR outage probability has been often used as a measure of receiver performance. In [6], considering the single-input single-output (SISO) IFC, the power control problem for minimizing the worst SINR outage probability was studied. Specifically, by the concave Perron-Frobenius theory [7], it was shown that the problem can be globally solved efficiently by the algorithms developed in [6, 8]. The same design criterion was further studied in [9] and [10] for the MISO and MIMO scenarios, respectively. However, the associated design problems are not convex and it was not clear if the problems can be solved in polynomial-time or not.

In this paper, we consider the SINR outage constrained MMF CoBF design problem, which maximizes the worst SINR performance of the receivers under probabilistic SINR outage constraints. We are interested in the complexity analysis. Specifically, we show that, in the SISO scenario, the outage constrained MMF problem is polynomial-time solvable whereas it is NP-hard in the MISO scenario. The former is obtained by the close relation between the considered MMF problem and the outage probability minimization problem studied in [6] which is polynomial-time solvable. The latter is obtained by showing that, in the MISO scenario, solving the outage constrained MMF problem is at least as difficult as solving a 3-satisfiability (SAT) problem which is known NP-complete [11]. Since the MMF CoBF problem is polynomial-time solvable in the perfect CSI case [3, 4], the presented analysis result indicates that the outage constrained MMF CoBF problem is indeed more challenging, and is in fact computationally intractable. Besides, our analysis can also be used to show that the MISO and MIMO outage probability minimization problems studied in [9, 10] are NP-hard.

2. SIGNAL MODEL AND PROBLEM STATEMENT

The MISO IFC where K pairs of multiple-antenna transmitters and single-antenna receivers share a common spectrum is considered. Each transmitter is equipped with N_t antennae, and communicates with its intended receiver by transmit beamforming. Let $\boldsymbol{h}_{ki}$ denote the Rayleigh faded channel vector between transmitter k and receiver i, i.e., $\boldsymbol{h}_{ki} \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q}_{ki})$ where $\mathbf{Q}_{ki} \succeq \mathbf{0}$ is the channel

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covariance matrix. The received signal at receiver i is given by

$$\boldsymbol{x}_i = \boldsymbol{h}_{ii}^H \boldsymbol{w}_i s_i + \sum_{k\neq i} \boldsymbol{h}_{ki}^H \boldsymbol{w}_k s_k + n_i,$$

where $s_k \sim \mathcal{CN}(0,1)$ is the Gaussian-encoded information signal transmitted from the kth transmitter, $\boldsymbol{w}_k \in \mathbb{C}^{N_t}$ is the associated beamforming vector for $k = 1,\ldots,K$, and $n_i \sim \mathcal{CN}(0,\sigma_i^2)$ is the additive noise at receiver i with variance $\sigma_i^2 > 0$. Assuming single user detection at receivers, the performance of each receiver depends on the SINR of the received signal, which is given by

$$\mathsf{SINR}_i = \frac{|\boldsymbol{h}_{ii}^H \boldsymbol{w}_i|^2}{\sum_{k\neq i} |\boldsymbol{h}_{ki}^H \boldsymbol{w}_k|^2 + \sigma_i^2},\ i = 1,\ldots,K.$$

In this paper, we investigate the MMF CoBF design problem for maximizing the worst SINR performance of the receivers. Mathematically, the problem can be formulated as:

$$\max_{\substack{\boldsymbol{w}_i\in\mathbb{C}^{N_t},\\ i=1,\ldots,K}} \ \min_{i\in\{1,\ldots,K\}} \ \mathsf{SINR}_i \tag{1a}$$

$$\text{s.t.}\ \ \|\boldsymbol{w}_i\|_2^2 \leq P_i,\ i=1,\ldots,K, \tag{1b}$$

where P_i is the power budget of transmitter i. It has been shown that problem (1) can be optimally solved in polynomial time via solving a finite number of second-order cone programs (SOCPs) [3].

The design formulation in (1) assumes that the transmitters have perfect knowledge of the instantaneous CSI. However, this may not be a feasible assumption for practical wireless systems, especially when the channels are fast faded. In view of this, we assume that the transmitters only know the CDI, i.e., $\mathbf{Q}_{ki}$, $i,k = 1,\ldots,K$. In this scenario, given any SINR requirement $\gamma_i > 0$, there is a non-zero probability of SINR outage, i.e., $\Pr\{\mathsf{SINR}_i < \gamma_i\} > 0$ for $i = 1,\ldots,K$. As a result, we consider the *outage-constrained* MMF CoBF problem, where the beamformers are designed to maximize the minimal SINR under a tolerable probability of outage. Mathematically, this outage-constrained MMF CoBF problem can be formulated as:

$$\max_{\substack{\boldsymbol{w}_i\in\mathbb{C}^{N_t},\gamma_i\geq 0,\\ i=1,\ldots,K}} \ \min_{i\in\{1,\ldots,K\}} \ \gamma_i \tag{2a}$$

$$\text{s.t.}\ \ \text{Prob}\left\{\mathsf{SINR}_i < \gamma_i\right\} \leq \epsilon_i, \tag{2b}$$

$$\|\boldsymbol{w}_i\|_2^2 \leq P_i,\ i=1,\ldots,K, \tag{2c}$$

where $\epsilon_i \in (0,1)$ is the tolerable outage probability for receiver i.

According to [12], the SINR outage constraints (2b) can be explicitly expressed as:

$$\rho_i e^{\frac{\gamma_i\sigma_i^2}{\boldsymbol{w}_i^H \mathbf{Q}_{ii}\boldsymbol{w}_i}} \prod_{k\neq i}\left(1 + \frac{\gamma_i \boldsymbol{w}_k^H \mathbf{Q}_{ki}\boldsymbol{w}_k}{\boldsymbol{w}_i^H \mathbf{Q}_{ii}\boldsymbol{w}_i}\right) \leq 1, \tag{3}$$

where $\rho_i \triangleq 1-\epsilon_i$, for $i = 1,\ldots,K$. Due to the min function in (2a) and the fact that the left-hand side of (3) is strictly increasing with γ_i, we can impose $\gamma_1 = \cdots = \gamma_K = \gamma$ without loss of optimality, and write problem (2) as

$$\max_{\substack{\boldsymbol{w}_i\in\mathbb{C}^{N_t},\gamma\geq 0,\\ i=1,\ldots,K}} \ \gamma \tag{4a}$$

$$\text{s.t.}\ \ \rho_i e^{\frac{\gamma\sigma_i^2}{\boldsymbol{w}_i^H \mathbf{Q}_{ii}\boldsymbol{w}_i}} \prod_{k\neq i}\left(1+\frac{\gamma \boldsymbol{w}_k^H \mathbf{Q}_{ki}\boldsymbol{w}_k}{\boldsymbol{w}_i^H \mathbf{Q}_{ii}\boldsymbol{w}_i}\right) \leq 1, \tag{4b}$$

$$\|\boldsymbol{w}_i\|_2^2 \leq P_i,\ i=1,\ldots,K. \tag{4c}$$

3. COMPUTATIONAL COMPLEXITY ANALYSIS

In this section, we investigate the computational complexity of problem (4). To this end, let us consider the following feasibility problem: Given a target SINR $\bar{\gamma} \geq 0$,

$$\text{find}\ \ \boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \tag{5a}$$

$$\text{s.t.}\ \ \rho_i e^{\frac{\bar{\gamma}\sigma_i^2}{\boldsymbol{w}_i^H \mathbf{Q}_{ii}\boldsymbol{w}_i}} \prod_{k\neq i}\left(1+\frac{\bar{\gamma} \boldsymbol{w}_k^H \mathbf{Q}_{ki}\boldsymbol{w}_k}{\boldsymbol{w}_i^H \mathbf{Q}_{ii}\boldsymbol{w}_i}\right) \leq 1, \tag{5b}$$

$$\|\boldsymbol{w}_i\|_2^2 \leq P_i,\ i=1,\ldots,K. \tag{5c}$$

The feasibility problem (5) is the core problem for handling problem (4). Their relation is stated in the following lemma.

Lemma 1 *Let $\gamma^\star$ denote the optimal value of problem* (4)*. It holds true that, for any $\bar{\gamma} \geq 0$, $\gamma^\star \geq \bar{\gamma}$ if and only if problem* (5) *is feasible. Moreover, the set of optimal beamformers to problem* (4) *is a subset of the feasible set of problem* (5) *when $\gamma^\star > \bar{\gamma}$, and these two sets coincide when $\gamma^\star = \bar{\gamma}$.*

Lemma 1 can be easily proved by the monotonicity of the outage constraint function in (5b), and the details of the proof is omitted here. Lemma 1 infers that problem (4) is polynomial-time solvable if and only if problem (5) is polynomial-time solvable. In particular, if problem (5) can be efficiently solved, then problem (4) can be efficiently solved by a bisection methodology which involves solving a series of problem (5) [13, §4.2.5]. On the other hand, if one can solve problem (4), then one can correctly determine whether problem (5) is feasible, and, if yes, obtain a set of feasible beamformers. Therefore, these two problems belong to the same complexity class. In the next two subsections, we respectively show that problem (5) can be solved in polynomial time when $N_t = 1$ (the SISO scenario) and is NP-hard when $N_t > 1$ (the MISO scenario).

3.1. Single Transmit Antenna Case

We assume that each of the transmitters is equipped with a single antenna. In that case, the channel covariance matrices $\mathbf{Q}_{ki} \succeq \mathbf{0}$, $i,k = 1,\ldots,K$, degenerate to the channel variances $Q_{ki} \geq 0$, $\forall i,k$. Moreover, by (4b) and (4c), the effective optimization variables are the transmit powers, i.e., $p_i \triangleq \|\boldsymbol{w}_i\|_2^2$, $i = 1,\ldots,K$. In this case, problem (4) degenerates to the following power control problem.

$$\max_{\substack{p_i\geq 0,\gamma\geq 0,\\ i=1,\ldots,K}} \ \gamma \tag{6a}$$

$$\text{s.t.}\ \ \rho_i e^{\frac{\gamma\sigma_i^2}{p_i Q_{ii}}} \prod_{k\neq i}\left(1 + \frac{\gamma p_k Q_{ki}}{p_i Q_{ii}}\right) \leq 1, \tag{6b}$$

$$p_i \leq P_i,\ i=1,\ldots,K. \tag{6c}$$

Accordingly, problem (5) reduces to

$$\text{find}\ \ p_1,\ldots,p_K \tag{7a}$$

$$\text{s.t.}\ \ \rho_i e^{\frac{\bar{\gamma}\sigma_i^2}{p_i Q_{ii}}} \prod_{k\neq i}\left(1 + \frac{\bar{\gamma} p_k Q_{ki}}{p_i Q_{ii}}\right) \leq 1, \tag{7b}$$

$$0 \leq p_i \leq P_i,\ i=1,\ldots,K. \tag{7c}$$

By considering the logarithmic change of variables $\tilde{p}_i = \ln p_i$ for $i = 1,\ldots,K$, one can further write (7) as

$$\min_{\tilde{p}_i, \tilde{\alpha}, i=1,\ldots,K} \quad \tilde{\alpha} \tag{8a}$$

$$\text{s.t.} \;\; \ln \rho_i + \frac{\bar{\gamma}}{Q_{ii}} e^{-\tilde{p}_i} + \sum_{k \neq i} \ln\left(1 + \frac{\bar{\gamma} Q_{ki}}{Q_{ii}} e^{\tilde{p}_k - \tilde{p}_i}\right) \leq \tilde{\alpha}, \tag{8b}$$

$$\tilde{p}_i \leq \ln P_i, \; i = 1, \ldots, K. \tag{8c}$$

Specifically, one can see that problem (7) is feasible if and only if the optimal $\tilde{\alpha}$ of problem (8) is less than or equal to zero. Problem (8) is in fact equivalent to the outage probability minimization problem studied in [6] which can be efficiently solved by a nonlinear Perron-Frobenius theory-based algorithm with overall complexity $\mathcal{O}(K \ln(K/\epsilon))$ (see [6, Algorithm 1]), where ϵ specifies the solution accuracy. Thus, problem (6) can be solved by bisection as described in Algorithm 1. The overall complexity of Algorithm 1 is $\kappa \cdot \mathcal{O}(K \ln(K/\epsilon))$, where

$$\kappa \triangleq \left\lceil \log_2 \left(\delta^{-1} \cdot \min_i \left(\frac{P_i Q_{ii} \ln(1/\rho_i)}{\sigma_i^2} \right) \right) \right\rceil \tag{9}$$

is the number of bisection iterations required for the convergence of Algorithm 1.[1]

Algorithm 1 Bisection method for solving problem (6)

Set $\gamma_\ell := 0$, $\gamma_u := \min_i \frac{P_i Q_{ii} \ln(1/\rho_i)}{\sigma_i^2}$, and set the solution accuracy to $\delta > 0$;
repeat
  Set $\bar{\gamma} := (\gamma_\ell + \gamma_u)/2$;
  Solve problem (8), and denote the solution as $(\{\tilde{p}_i^\star\}_{i=1}^K, \tilde{\alpha}^\star)$;
  Set $\gamma_\ell := \bar{\gamma}$ if $\tilde{\alpha}^\star \leq 0$; otherwise, set $\gamma_u := \bar{\gamma}$;
until $\gamma_u - \gamma_\ell < \delta$;
Output $p_i^\star = e^{\tilde{p}_i^\star}$, $\gamma^\star = \bar{\gamma}$, $i = 1, \ldots, K$, as a solution to problem (6).

As a result, Algorithm 1 has a polynomial-time complexity under the mild assumption that κ is finite, which is always the case in practical situations. We thus conclude this subsection with the following theorem.

Theorem 1 *When $N_t = 1$, the outage constrained MMF CoBF problem* (4) *(i.e., problem* (6)*) is polynomial-time solvable.*

3.2. Multiple Transmit Antenna Case

In this subsection, we show that problem (4), in contrast to the SISO case, is NP-hard when each of the transmitters is equipped with multiple antennas. The complexity analysis result is described in the following theorem.

Theorem 2 *When $N_t \geq 2$, the outage-constrained MMF CoBF problem* (4) *is NP-hard in the number of users K.*

Proof: As inferred from Lemma 1, it suffices to show that solving the feasibility problem (5) is NP-hard when $N_t \geq 2$. Our idea is to show that the 3-satisfiability (3-SAT) problem, which is known to be NP-complete [11], is reducible to problem (5), i.e., solving the 3-SAT problems cannot be harder than solving problem (5). The 3-SAT problem is defined below.

[1] From constraints (6b), it is observed that the maximal SINR of the ith transmitter-receiver pair is attained when $p_i = P_i$ and $p_k = 0$ $\forall k \neq i$. Thus, we have $\gamma \leq P_i Q_{ii} \ln(1/\rho_i)/\sigma_i^2$ for all i. Consequently, $\min_i P_i Q_{ii} \ln(1/\rho_i)/\sigma_i^2$ is an upper bound to the optimal value of problem (6). On the other hand, $\gamma = 0$ is obviously a lower bound to the optimal value of (6). Therefore, the number of bisection iterations for achieving the solution accuracy $\delta > 0$ is given by (9).

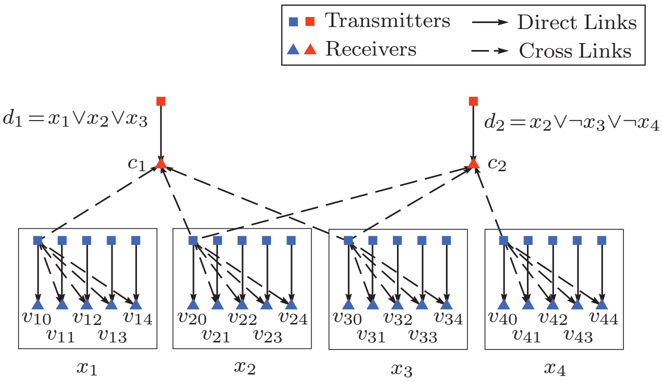

Fig. 1. Illustration of the correspondence between a 3-SAT problem instance $(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4)$, which consists of $N = 4$ Boolean variables and $M = 2$ clauses, and a MISO IFC with $K = 5 \times 4 + 2$ pairs of transmitters and receivers.

Definition 1 *Given N Boolean variables and M clauses each containing exactly three literals of different Boolean variables, the 3-SAT problem is to determine whether there exists a truth assignment of the Boolean variables such that the conjunction of the M clauses is true.*

For ease of exposition, we use "$\wedge$", "$\vee$", and "$\neg$" to denote the logical conjunction (AND), disjunction (OR), and negation afterwards. Given a problem instance of 3-SAT, i.e., given N Boolean variables $x_1, \ldots, x_N$ and M clauses $d_m = y_{i_m} \vee y_{j_m} \vee y_{k_m}$, $m = 1, \ldots, M$, where y_{i_m} is either x_{i_m} or its negation $\neg x_{i_m}$ and so are y_{j_m}, y_{k_m}, we construct the following problem instance of (5). Let $K = 5N + M$ and denote the set of transmitter-receiver pairs as

$$\mathcal{U} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4 \cup \mathcal{C},$$

where $\mathcal{V}_\ell \triangleq \{v_{1\ell}, v_{2\ell}, \ldots, v_{N\ell}\}$ denotes the ℓth user set associated with $x_1, \ldots, x_N$, for $\ell = 0, \ldots, 4$, and $\mathcal{C} \triangleq \{c_1, \ldots, c_M\}$ denotes the users associated with $d_1, \ldots, d_M$. The other parameters for this problem instance of (5) are chosen as:

$$N_t = 2, \; \bar{\gamma} = 1, \; P_u = 1, \; \rho_u = \rho = 0.9, \; \forall u \in \mathcal{U}, \tag{10a}$$

$$\sigma_{v_{n0}}^2 = \ln\left(\frac{1}{\rho}\right), \; \sigma_{v_{n\ell}}^2 = \ln\left(\frac{10}{11\rho}\right), \; \sigma_{c_m}^2 = 0.01, \; \forall n, m, \forall \ell \neq 0, \tag{10b}$$

$$\mathbf{Q}_{uu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \forall u \in \mathcal{U}, \; \mathbf{Q}_{v_{n0}, v_{n\ell}} = \frac{\mathbf{A}_\ell}{10}, \; \forall n, \forall \ell \neq 0 \tag{10c}$$

$$\mathbf{Q}_{v_{n0}, c_m} = \begin{cases} \frac{1}{25} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } x_n \in \{y_{i_m}, y_{j_m}, y_{k_m}\}, \\ \frac{1}{25} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } \neg x_n \in \{y_{i_m}, y_{j_m}, y_{k_m}\}, \end{cases} \quad \forall n, \tag{10d}$$

$$\mathbf{Q}_{u_1, u_2} = \mathbf{0}, \text{ for all } u_1, u_2 \text{ not specified above}, \tag{10e}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; \mathbf{A}_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; \mathbf{A}_3 = \begin{bmatrix} 1 & \iota \\ -\iota & 1 \end{bmatrix}, \; \mathbf{A}_4 = \begin{bmatrix} 1 & -\iota \\ \iota & 1 \end{bmatrix},$$

and $\iota^2 = -1$. An illustrative example for the correspondence between a 3-SAT problem and a problem instance of (5) is provided in Fig. 1. In this example, a problem instance of 3-SAT consisting of four Boolean variables, x_1, x_2, x_3, x_4, and two clauses, $d_1 =$

$(x_1 \vee x_2 \vee x_3)$, $d_2 = (x_2 \vee \neg x_3 \vee \neg x_4)$ is considered. Each Boolean variable x_n corresponds to five pairs of transmitter and receivers, i.e., $v_{n0}, v_{n1}, \ldots, v_{n4}$, for $n = 1, \ldots, 4$. For these transmitter-receiver pairs, only the transmitter of v_{n0} interferes with the receivers of $v_{n1}, \ldots, v_{n4}$ for $n = 1, \ldots, 4$, and the channel covariance matrices are given in (10c). On the other hand, each clause d_m corresponds to one transmitter-receiver pair c_m for $m = 1, 2$, and the receiver of c_m is interfered by the transmitter of v_{n0} if the Boolean variable x_n or its negation $\neg x_n$ exists in clause d_m. Moreover, the covariance matrix of the channel between the receiver of c_m and the transmitter of v_{n0} depends on whether x_n or the negation $\neg x_n$ appears in d_m, as indicated in (10d). In Fig. 1, the transmitters and receivers are not connected if the associated cross-link channel covariance matrix is zero as indicated in (10e).

According to the construction specified in (10), the corresponding problem instance of (5) can be expressed as

$$\text{find } \{\boldsymbol{w}_{n0}^v\}_{n=1}^N, \{\boldsymbol{w}_{n1}^v\}_{n=1}^N, \ldots, \{\boldsymbol{w}_{n4}^v\}_{n=1}^N, \{\boldsymbol{w}_m^c\}_{m=1}^M \quad (11a)$$

$$\text{s.t. } \rho e^{\frac{\sigma_{v_{n0}}^2}{\|\boldsymbol{w}_{n0}^v\|_2^2}} \leq 1, \forall n, \quad (11b)$$

$$\rho e^{\frac{\sigma_{v_{n\ell}}^2}{\|\boldsymbol{w}_{n\ell}^v\|_2^2}} \left(1 + \frac{(\boldsymbol{w}_{n0}^v)^H \mathbf{A}_\ell \boldsymbol{w}_{n0}^v}{10\|\boldsymbol{w}_{n\ell}^v\|_2^2}\right) \leq 1, \forall n, \forall \ell \neq 0, \quad (11c)$$

$$\rho e^{\frac{\sigma_{c_m}^2}{\|\boldsymbol{w}_m^c\|_2^2}} \prod_{\tau = i_m, j_m, k_m} \left(1 + \frac{(\boldsymbol{w}_{\tau 0}^v)^H \mathbf{Q}_{v_{\tau 0}, c_m} \boldsymbol{w}_{\tau 0}^v}{\|\boldsymbol{w}_m^c\|_2^2}\right) \leq 1, \forall m, \quad (11d)$$

$$\|\boldsymbol{w}_{n\ell}^v\|_2^2 \leq 1, \|\boldsymbol{w}_m^c\|_2^2 \leq 1, \forall n, \forall m, \forall \ell, \quad (11e)$$

where $\boldsymbol{w}_{n\ell}^v$ and $\boldsymbol{w}_m^c$ denote the beamformers of transmitter $v_{n\ell}$ and c_m, respectively. Next, we will show that the given 3-SAT problem instance is satisfiable if and only if (11) is feasible.

We first show that any feasible point of problem (11) corresponds to a truth assignment satisfying the given instance of 3-SAT. Note that we can rewrite constraint (11b) as

$$\|\boldsymbol{w}_{n0}^v\|_2^2 \geq \sigma_{v_{n0}}^2 (\ln \rho^{-1})^{-1} = 1, \; n = 1, \ldots, N, \quad (12)$$

where the equality comes from (10b). Hence, constraints (11b) and (11e) imply $\|\boldsymbol{w}_{n0}^v\|_2^2 = 1$ for $n = 1, \ldots, N$. Similarly, we can rewrite (11c) as

$$(\boldsymbol{w}_{n0}^v)^H \mathbf{A}_\ell \boldsymbol{w}_{n0}^v \leq 10\|\boldsymbol{w}_{n\ell}^v\|_2^2 \left(\rho^{-1} \exp\left(\frac{-\sigma_{v_{n\ell}}^2}{\|\boldsymbol{w}_{n\ell}^v\|_2^2}\right) - 1\right)$$
$$\overset{(*)}{\leq} \max_{\|\boldsymbol{w}_{n\ell}^v\|_2^2 \leq 1} 10\|\boldsymbol{w}_{n\ell}^v\|_2^2 \left(\rho^{-1} \exp\left(\frac{-\sigma_{v_{n\ell}}^2}{\|\boldsymbol{w}_{n\ell}^v\|_2^2}\right) - 1\right) \overset{(**)}{=} 1, \quad (13)$$

for $n = 1, \ldots, N$, $\ell = 1, 2, 3, 4$, where $(*)$ holds with equality if and only if $\|\boldsymbol{w}_{n\ell}^v\|_2^2 = 1$, and $(**)$ comes from (10b). Furthermore,

$$(\boldsymbol{w}_{n0}^v)^H \mathbf{A}_\ell \boldsymbol{w}_{n0}^v + (\boldsymbol{w}_{n0}^v)^H \mathbf{A}_{\ell+1} \boldsymbol{w}_{n0}^v = 2\|\boldsymbol{w}_{n0}^v\|_2^2 = 2, \; \ell = 1, 3, \quad (14)$$

for all $n = 1, \ldots, N$. Combining (13) and (14) yields

$$(\boldsymbol{w}_{n0}^v)^H \mathbf{A}_\ell \boldsymbol{w}_{n0}^v = 1, \text{ and } \|\boldsymbol{w}_{n\ell}^v\|_2^2 = 1, \quad (15)$$

for all $n = 1, \ldots, N$ and $\ell = 1, 2, 3, 4$. Then, we have

$$(\boldsymbol{w}_{n0}^v)^H (\mathbf{A}_1 - \mathbf{A}_2) \boldsymbol{w}_{n0}^v = 4\text{Re}\{([\boldsymbol{w}_{n0}^v]_2)^H [\boldsymbol{w}_{n0}^v]_1\} = 0, \quad (16a)$$
$$(\boldsymbol{w}_{n0}^v)^H (\mathbf{A}_3 - \mathbf{A}_4) \boldsymbol{w}_{n0}^v = 4\text{Im}\{([\boldsymbol{w}_{n0}^v]_2)^H [\boldsymbol{w}_{n0}^v]_1\} = 0, \quad (16b)$$

for all $n = 1, \ldots, N$, where $[\boldsymbol{w}_{n0}^v]_1$ and $[\boldsymbol{w}_{n0}^v]_2$ denote the first and second elements of $\boldsymbol{w}_{n0}^v$, respectively. Thus, the feasible $\boldsymbol{w}_{n0}^v$'s must satisfy either $[\boldsymbol{w}_{n0}^v]_1 = 0$ or $[\boldsymbol{w}_{n0}^v]_2 = 0$. Besides, by observing constraints (11d) and (11e), one can let $\|\boldsymbol{w}_m^c\|_2^2 = 1$ for all $m = 1, \ldots, M$ without loss of generality. By (15), (16), and $\|\boldsymbol{w}_m^c\|_2^2 = 1$ for all $m = 1, \ldots, M$, we can equivalently reformulate problem (11) as

$$\text{find } \{\boldsymbol{w}_{n0}^v\}_{n=1}^N, \{\boldsymbol{w}_{n1}^v\}_{n=1}^N, \ldots, \{\boldsymbol{w}_{n4}^v\}_{n=1}^N, \{\boldsymbol{w}_m^c\}_{m=1}^M \quad (17a)$$

$$\text{s.t. } \boldsymbol{w}_{n0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [e^{j\theta} \; 0]^T \Big\} \cup \Big\{ \bigcup_{\theta \in [0, 2\pi]} [0 \; e^{j\theta}]^T \Big\}, \forall n, \quad (17b)$$

$$\rho e^{\sigma_{c_m}^2} \prod_{\tau = i_m, j_m, k_m} \left(1 + (\boldsymbol{w}_{\tau 0}^v)^H \mathbf{Q}_{v_{\tau 0}, c_m} \boldsymbol{w}_{\tau 0}^v\right) \leq 1, \forall m, \quad (17c)$$

$$\|\boldsymbol{w}_{n\ell}^v\|_2^2 = 1, \|\boldsymbol{w}_m^c\|_2^2 = 1, \forall n, \forall m, \forall \ell. \quad (17d)$$

By (10d) and constraint (17b), one can see that

$$(\boldsymbol{w}_{\tau_m 0}^v)^H \mathbf{Q}_{v_{\tau_m 0}, c_m} \boldsymbol{w}_{\tau_m 0}^v$$
$$= \begin{cases} 0, & \text{if } y_{\tau_m} = x_{\tau_m}, \boldsymbol{w}_{\tau_m 0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [e^{j\theta} \; 0]^T \Big\}, \\ 0, & \text{if } y_{\tau_m} = \neg x_{\tau_m}, \boldsymbol{w}_{\tau_m 0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [0 \; e^{j\theta}]^T \Big\}, \\ 1/25, & \text{if } y_{\tau_m} = x_{\tau_m}, \boldsymbol{w}_{\tau_m 0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [0 \; e^{j\theta}]^T \Big\}, \\ 1/25, & \text{if } y_{\tau_m} = \neg x_{\tau_m}, \boldsymbol{w}_{\tau_m 0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [e^{j\theta} \; 0]^T \Big\}. \end{cases} \quad (18)$$

Furthermore, constraint (17c) is violated if and only if

$$(\boldsymbol{w}_{\tau 0}^v)^H \mathbf{Q}_{v_{\tau 0}, c_m} \boldsymbol{w}_{\tau 0}^v = 1/25, \; \forall \tau \in \{i_m, j_m, k_m\}.$$

Therefore, supposing that $\hat{\boldsymbol{w}}_{n\ell}^v, \hat{\boldsymbol{w}}_m^c, n = 1, \ldots, N, m = 1, \ldots, M, \ell = 0, \ldots, 4$, are feasible to (17), it must be true that

$$(\hat{\boldsymbol{w}}_{\tau 0}^v)^H \mathbf{Q}_{v_{\tau 0}, c_m} \hat{\boldsymbol{w}}_{\tau 0}^v = 0, \text{ for some } \tau \in \{i_m, j_m, k_m\}, \forall m. \quad (19)$$

According to (18) and (19), the conjunction of the M clauses is satisfied (i.e., is true) by the following truth assignment:

$$x_n = \begin{cases} 1, & \text{if } \hat{\boldsymbol{w}}_{n0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [e^{j\theta} \; 0]^T \Big\}, \\ 0, & \text{if } \hat{\boldsymbol{w}}_{n0}^v \in \Big\{ \bigcup_{\theta \in [0, 2\pi]} [0 \; e^{j\theta}]^T \Big\}. \end{cases}$$

On the other hand, suppose that the 3-SAT problem instance is satisfiable and $\hat{x}_n \in \{0, 1\}$, $n = 1, \ldots, N$, is a truth assignment such that the conjunction of $d_1, \ldots, d_M$ is true. Then, it is straightforward to verify that

$$\boldsymbol{w}_{n0}^v = \begin{cases} [1 \; 0]^T, & \hat{x}_n = 1, \\ [0 \; 1]^T, & \hat{x}_n = 0, \end{cases} \forall n,$$
$$\|\boldsymbol{w}_{n\ell}^v\|_2^2 = \|\boldsymbol{w}_m^c\|_2^2 = 1, \forall n, \forall m, \forall \ell \neq 0,$$

is feasible to problem (11). Thus, we have proved the equivalence between the 3-SAT problem instance and problem (11), implying that the 3-SAT problem is reducible to problem (5). As a result, determining the feasibility of problem (5) is NP-hard when $N_t \geq 2$, and hence problem (4) is also NP-hard according to Lemma 1. ■

In conclusion, we have proved that the SINR outage constrained MMF CoBF problem (4) is NP-hard in general, and have identified a subclass, i.e., single transmit antenna case, of this problem that is polynomial-time solvable. Finally, we make a remark that, our analysis can also be used to show that the outage probability minimization problem studied in [6, 9, 10] is also NP-hard for the MISO and MIMO IFCs.

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