BLOCK-RECURSIVE IAA-BASED SPECTRAL ESTIMATES WITH MISSING SAMPLES USING DATA INTERPOLATION

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ABSTRACT

In this work, we examine a computationally efficient block-updating scheme for estimating the spectral content of signals with missing samples. The work is an extension of our recent single-sample data interpolation updating of the Iterative Adaptive Approach (IAA), being reformulated to incorporate blocks of samples. The proposed implementation offers a substantial complexity reduction as compared to earlier presented updating schemes, without sacrificing the quality of the resulting spectral estimates more than marginally (if at all).

Index Terms— Spectrum estimation theory and methods, Iterative Adaptive Approach (IAA), fast algorithms

1. INTRODUCTION

Missing samples occur for a variety of reasons, for instance due to samples being lost or unmeasurable, sensor failure, or various forms of disturbances, and will, if not treated properly, corrupt the identification or estimation procedure for such data sequences (see, e.g. [1–3], and the references therein). In this work, we examine high-resolution spectral estimation of data sets containing missing samples, a problem that has attracted notable attention in the recent literature, with solutions ranging from classical estimation methods such as the Lomb-Scargle periodogram [4, 5], to data-adaptive algorithms such as MAPES [2] and MIAA [3], where both the latter are formed under the assumption that the missing samples share the same spectral content as the given samples. Of the latter, the MIAA algorithm, which is based on the iterative adaptive approach (IAA) [6], has been shown to yield preferable performance as compared to other conventional data-adaptive methods (see also [7] and the references therein). We have in earlier work examined how to form computationally efficient time-recursive (TR) formulations of IAA-based spectral estimators [8–10], exploiting the inherently low displacement rank of the updating, together with the development of suitable Gohberg-Semencul (GS) representations. Most recently, in [10], we examined the possibility to form an approximative data interpolation updating, such that the TR-MIAA was updated using the TR-IAA updating presented in [8] if the updating sample was available, and using a data interpolation scheme, reconstructing the sample, if missing. The resulting interpolated TR-MIAA algorithm, termed TRI-MIAA, was found to be dramatically cheaper computationally as compared to the TR-MIAA updating. In this work, we generalize the fast TRI-MIAA (FTRI-MIAA) implementation presented in [10] to allow also for block-recursive updating, such that blocks of data, possibly containing missing samples, are processed at each updating step.

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2. TIME-VARYING SPECTRAL ESTIMATION USING THE MIAA APPROACH

Let

$$\mathbf{y}_N(n) = \left[\begin{array}{ccc} y(n-N+1) & \ldots & y(n) \end{array} \right]^T \qquad (1)$$

denote a snapshot of the discrete time signal $y(n)$ within a sliding window of size N, with $(\cdot)^T$ denoting the transpose. When all samples are available, the corresponding spectrum at time instant n can be computed in various ways [11], and a range of computationally affordable time-updating schemes are also available that allows for a sliding window update as an additional sample becomes available [8, 12–15]. However, when the data set may also contain missing samples located at random, but known positions, the derivation of computationally efficient spectral estimation methods becomes a more challenging task [1, 10]. Suppose that at time instant n, G_n samples among those in (1) are available, whereas the remaining $M_n = N - G_n$ data are missing. The so-called MIAA algorithm presented in [3] is then formed by iterating (2)-(4), until practical convergence,

$$\alpha_n(\omega_k) = \frac{\mathbf{f}_{G_n}^H(\omega_k)\mathbf{R}_{G_n}^{-1}(n)\mathbf{y}_{G_n}(n)}{\mathbf{f}_{G_n}^H(\omega_k)\mathbf{R}_{G_n}^{-1}(n)\mathbf{f}_{G_n}(\omega_k)} \qquad (2)$$

$$\Phi_n(\omega_k) = |\alpha_n(\omega_k)|^2 \qquad (3)$$

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$$\mathbf{R}_{G_n}(n) = \sum_{k=0}^{K-1} \Phi_n(\omega_k)\mathbf{f}_{G_n}(\omega_k)\mathbf{f}_{G_n}^H(\omega_k) \quad (4)$$

with $(\cdot)^H$ denoting the conjugate transpose, and

$$\mathbf{y}_{G_n}(n) = \mathbf{S}_{G_n}\mathbf{y}_N(n) \quad (5)$$
$$\mathbf{f}_{G_n}(\omega_k) = \mathbf{S}_{G_n}\mathbf{f}_N(\omega_k) \quad (6)$$

denoting the vector of the given data and the corresponding frequency vector respectively, where

$$\mathbf{f}_N(\omega_k) \triangleq \left[\begin{array}{cccc} 1 & e^{\jmath\omega_k} & \ldots & e^{\jmath\omega_k(N-1)} \end{array}\right]^T$$

The matrix $\mathbf{S}_{G_n}$ is a $G_n \times N$ time-varying selection matrix, with zeros and ones indicating the presence of a sample in (1). As it was shown in [7, 16], the MIAA algorithm allows for a fast implementation, requiring approximately $m[G_n^3 + 1.5K\log_2(K)]$ operations, where m denotes the number of iterations applied in (2)-(4), with usually 10 -15 iterations being adequate for practical convergence. When $M_n < G_n$, an even faster implementation is also available [17]. When block processing is considered, the MIAA may be applied directly on each block of data in (1), providing a new spectral estimate after each L time instances, where $L < N$ is the size of the block step. The resulting scheme, here termed the block MIAA (B-MIAA) will thus produce spectral estimates every L time instances, processing consecutive and overlapped data blocks one at a time, as being in batch mode. However, this form of updating will result in an unnecessarily heavy work load, equal to that required when each individual data set is processed in batch mode. Fortunately, a TR block-updating is feasible. The consecutive data sets $\mathbf{y}_{G_{n-L}}(n-L)$ and $\mathbf{y}_{G_n}(n)$ share $\bar{G} - L$ common elements, where $\bar{G}$ denotes the mean number of missing samples within (1). Thus, it is reasonable to use the previously estimated spectrum $\Phi_{n-L}(\omega_k)$ for the computation of $\mathbf{R}_{G_n}(n)$ in (2). The spectral estimate at time instant n is subsequently computed using (2) and (3), resulting in this way in a considerable savings in the required computation, without sacrificing much (if any) of the quality in the estimated spectra, yielding the updating

$$\mathbf{R}_{G_n}(n) = \sum_{k=0}^{K-1} \Phi_{n-L}(\omega_k)\mathbf{f}_{G_n}(\omega_k)\mathbf{f}_{G_n}^H(\omega_k) \quad (7)$$
$$\alpha_n(\omega_k) = \frac{\mathbf{f}_{G_n}^H(\omega_k)\mathbf{R}_{G_n}^{-1}(n)\mathbf{y}_{G_n}(n)}{\mathbf{f}_{G_n}^H(\omega_k)\mathbf{R}_{G_n}^{-1}(n)\mathbf{f}_{G_n}(\omega_k)} \quad (8)$$
$$\Phi_n(\omega_k) = |\alpha_n(\omega_k)|^2 \quad (9)$$

We here term this form of TR block-updating the block TR-MIAA (BTR-MIAA). The computational complexity of the BTR-MIAA is about $[G_n^3 + 1.5K\log_2(K)]$ operations per processed block, although, as noted, when $L = 1$, corresponding to single sample updates, efficient implementations are already available [9, 10]. Building upon and extending these works, we here focus on the more general case when $1 \leq L < N$.

3. DATA INTERPOLATION BTR-MIAA

By supposing that all but the most $L < N$ recent samples

$$\mathbf{y}_L(n) = \left[\begin{array}{ccc} y(n-L+1) & \ldots & y(n) \end{array}\right]^T \quad (10)$$

are either given or have been replaced by some form of reconstruction estimates, the thus reconstructed data vector, $\hat{\mathbf{y}}_{\hat{G}_n}(n)$, may be expressed as

$$\hat{\mathbf{y}}_{\hat{G}_n}(n) = \left[\begin{array}{cc} \hat{\mathbf{y}}_{N-L}^T(n-L) & \mathbf{y}_{g_n}^T(n) \end{array}\right]^T \quad (11)$$

where $\mathbf{y}_{g_n}(n)$ denotes the most recent given samples, being related to (10) by

$$\mathbf{y}_{g_n}(n) = \mathbf{S}_{g_n}\mathbf{y}_L(n) \quad (12)$$

whereas $\mathbf{S}_{g_n}$ is a $g_n \times L$ selection matrix, with zeros and ones indicating the presence of a sample, with g_n denoting the number of available samples in $\mathbf{y}_L(n)$. Clearly,

$$\mathbf{S}_{\hat{G}_n} = \left[\begin{array}{cc} \mathbf{I}_{\tilde{N}} & \mathbf{0}_{\tilde{N},L} \\ \mathbf{0}_{\tilde{N},g_n}^T & \mathbf{S}_{g_n} \end{array}\right] \quad (13)$$

where $\tilde{N} = N - L$ and $\hat{G}_n = N - L + g_n$. The BTR-MIAA is then formed as

$$\mathbf{R}_{\hat{G}_n}(n) = \sum_{k=0}^{K-1} \Phi_{n-L}(\omega_k)\mathbf{f}_{\hat{G}_n}(\omega_k)\mathbf{f}_{\hat{G}_n}^H(\omega_k) \quad (14)$$
$$\alpha_n(\omega_k) = \frac{\mathbf{f}_{\hat{G}_n}^H(\omega_k)\mathbf{R}_{\hat{G}_n}^{-1}(n)\hat{\mathbf{y}}_{\hat{G}_n}(n)}{\mathbf{f}_{\hat{G}_n}^H(\omega_k)\mathbf{R}_{\hat{G}_n}^{-1}(n)\mathbf{f}_{\hat{G}_n}(\omega_k)} \quad (15)$$
$$\Phi_n(\omega_k) = |\alpha_n(\omega_k)|^2 \quad (16)$$

It remains to show how the $m_n \triangleq L - g_n$ missing data $\hat{\mathbf{y}}_{m_n}(n)$ within the most recent data block $\mathbf{y}_L(n)$ are reconstructed using the previous estimates. This task is performed using the time-domain MIAA-t missing data interpolation method proposed in [3], such that

$$\hat{\mathbf{y}}_{m_n}(n) = \sum_{k=0}^{K-1} \Phi_n(\omega_k)\mathbf{f}_{\hat{G}_n}^H(\omega_k)\tilde{\mathbf{R}}_{\hat{G}_n}^{-1}(n)\hat{\mathbf{y}}_{\hat{G}_n}(n)\mathbf{f}_{m_n}(\omega_k) \quad (17)$$

where

$$\mathbf{f}_{m_n}(\omega_k) \triangleq \mathbf{S}_{m_n}\left[\begin{array}{ccc} e^{\jmath\omega_k(\tilde{N}-1)} & \ldots & e^{\jmath\omega_k(N-1)} \end{array}\right]^T \quad (18)$$
$$\tilde{\mathbf{R}}_{\hat{G}_n}(n) = \sum_{k=0}^{K-1} \Phi_n(\omega_k)\mathbf{f}_{\hat{G}_n}(\omega_k)\mathbf{f}_{\hat{G}_n}^H(\omega_k) \quad (19)$$

The resulting combined algorithm, here termed the block TR interpolation MIAA (BTRI-MIAA) is constructed by the updating of (14)-(15), noting that the initialization should be performed by a complete MIAA-t step being applied for

spectral estimation and missing data interpolation. Unfortunately, neither $\mathbf{R}_{\hat{G}_n}(n)$ or $\tilde{\mathbf{R}}_{\hat{G}_n}(n)$, given by (14) and (15), are structured matrices, and thus the efficient implementation schemes developed in the past [7, 8, 18] cannot be applied directly. However, due to the special form of (13), $\mathbf{R}_{\hat{G}_n}(n)$ and $\tilde{\mathbf{R}}_{\hat{G}_n}(n)$ may be partitioned in such a way that their upper left part is Toeplitz. Thus, the derivation of fast implementation schemes is still feasible, although at some extra effort. With this observation, we proceed to detail the resulting efficient implementation.

4. EFFICIENT IMPLEMENTATION OF BTRI-MIAA

As noted, the elements of $\mathbf{R}_{\hat{G}_n}(n)$ may be extracted from a circulant matrix of increased dimensions defined as

$$\sum_{k=0}^{K-1} \Phi_{n-L}(\omega_k)\mathbf{f}_K(\omega_k)\mathbf{f}_K^H(\omega_k) \tag{20}$$

where the first column may be computed efficiently using the FFT [7, 8, 18]. Using (13), (14) is partitioned as

$$\mathbf{R}_{\hat{G}_n}(n) = \begin{bmatrix} \mathbf{R}_{\tilde{N}}(n) & \mathbf{R}_{\tilde{N},g_n}(n) \\ \mathbf{R}_{\tilde{N},g_n}^H(n) & \mathbf{R}_{g_n,g_n}(n) \end{bmatrix} \tag{21}$$

noting that the upper left part $\mathbf{R}_{\tilde{N}}(n)$ is a $\tilde{N} \times \tilde{N}$ Toeplitz matrix. Applying the matrix inversion lemma for partitioned matrices yields

$$\mathbf{R}_{\hat{G}_n}^{-1}(n) = \begin{bmatrix} \mathbf{R}_{\tilde{N}}^{-1}(n) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \mathcal{B}_n \tag{22}$$

where $\mathcal{B}_n = \mathbf{B}_{\hat{G}_n,g_n}(n)\mathbf{B}_{\hat{G}_n,g_n}^H(n)$,

$$\mathbf{B}_{\hat{G}_n,g_n}(n) \triangleq \begin{bmatrix} -\mathbf{R}_{\tilde{N}}^{-1}(n)\mathbf{R}_{\tilde{N},g_n}(n) \\ \mathbf{I}_{g_n} \end{bmatrix} \mathbf{A}_{g_n,g_n}^{-1/2}(n) \tag{23}$$

and

$$\mathbf{A}_{g_n,g_n}(n) = \mathbf{R}_{g_n,g_n}(n) - \mathbf{R}_{\tilde{N},g_n}^H(n)\mathbf{R}_{\tilde{N}}^{-1}(n)\mathbf{R}_{\tilde{N},g_n}(n) \tag{24}$$

As $\mathbf{R}_{\tilde{N}}(n)$ is a Toeplitz matrix, its inverse may be represented using the GS factorization as a sum of products of triangular Toeplitz matrices, whose leading column may be computed using the Levinson-Durbin algorithm [11]. Furthermore, $\mathbf{B}_{\hat{G}_n}(n)$ may be computed column-wise from (23), using the GS representation of $\mathbf{R}_{\tilde{N}}^{-1}(n)$ and fast Toeplitz vector multiplication via the FFT (see also [19–21]). Thus, all parameters in (21)-(24) may be computed at a cost of about $\tilde{N}^2 + 6g_n\tilde{N}\log_2(2\tilde{N}) + g_n^2\tilde{N} + g_n^3$ operations. Further, the sought spectral coefficients in (22) may be expressed in terms of trigonometric polynomials, defined as

$$\alpha_n(\omega_k) = \frac{\psi_n(\omega_k)}{\varphi_n(\omega_k)} \tag{25}$$

$$\psi_n(\omega_k) \triangleq \mathbf{f}_{\hat{G}_n}^H(\omega_k)\mathbf{d}_{\hat{G}_n}(n) \tag{26}$$

$$\varphi_n(\omega_k) \triangleq \mathbf{f}_{\hat{G}_n}^H(\omega_k)\mathbf{R}_{\hat{G}_n}^{-1}(n)\mathbf{f}_{\hat{G}_n}(\omega_k) \tag{27}$$

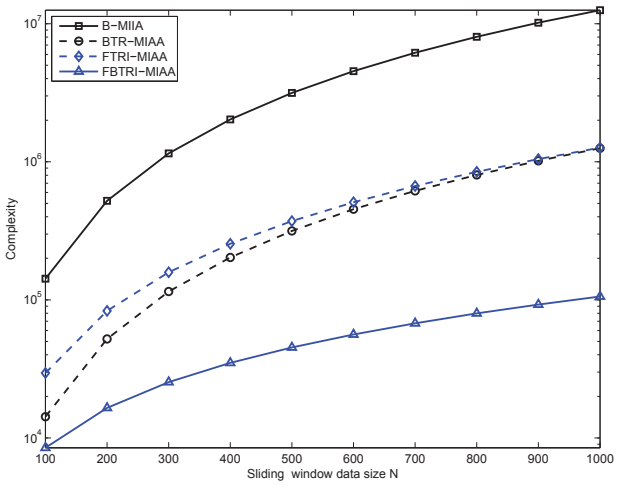

Fig. 1. Computational complexity of the B-MIAA, the BTR-BIAA, the FTRI-MIAA, and the proposed FBTRI-MIAA spectral estimation algorithms, versus the sliding window data size N.

where

$$\mathbf{d}_{\hat{G}_n}(n) \triangleq \mathbf{R}_{\hat{G}_n}^{-1}(n)\hat{\mathbf{y}}_{\hat{G}_n}(n) \tag{28}$$

Using (22),

$$\mathbf{d}_{\hat{G}_n}(n) = \begin{bmatrix} \mathbf{R}_{\tilde{N}}^{-1}(n)\hat{\mathbf{y}}_{\tilde{N}}(n-L) \\ 0 \end{bmatrix} + \mathcal{B}_n\hat{\mathbf{y}}_{\hat{G}_n}(n) \tag{29}$$

Once again, the GS factorization of $\mathbf{R}_{\tilde{N}}^{-1}(n)$ is used for the fast computation of the inverse matrix vector product that appears in (29), at a cost of about $6\tilde{N}\log_2(2\tilde{N}) + 2g_n\hat{G}_n$ operations. Inserting (22) into (27) yields

$$\varphi_n(\omega_k) = \phi_n(\omega) + \xi_n(\omega_k) \tag{30}$$

where

$$\phi_k(\omega_k) = \mathbf{f}_{\tilde{N}}^H(\omega_k)\mathbf{R}_{\tilde{N}}^{-1}(n)\mathbf{f}_{\tilde{N}}(\omega_k) \tag{31}$$

$$\xi_n(\omega_k) \triangleq \sum_{\ell=1}^{g_n} \chi_n^\ell(\omega_k)\chi_n^{\ell\,*}(\omega_k) \tag{32}$$

$$\chi_n^\ell(\omega_k) \triangleq \mathbf{f}_{\hat{G}_n}^H(\omega_k)\mathbf{b}_{\hat{G}_n}^\ell(n) \tag{33}$$

for $\ell = 1, \ldots, L_g(n)$, where $\mathbf{b}_{\hat{G}_n}^\ell(n)$ denotes the ℓ-th column of the matrix $\mathbf{B}_{\hat{G}_n,g_n}(n)$, and with $(\cdot)^*$ denoting the conjugate. The coefficients of the trigonometric polynomial

$$\phi_n(\omega_k) = \sum_{\ell=-\tilde{N}+1}^{\tilde{N}-1} c_n^\ell e^{\jmath\ell\omega_k} \tag{34}$$

are computed using the GS representation of $\mathbf{R}_{\tilde{N}}^{-1}(n)$, as proposed in [7, 18], without the need of forming the inverse matrix explicitly, at a cost of about $3\tilde{N}\log_2(2\tilde{N})$ operations.

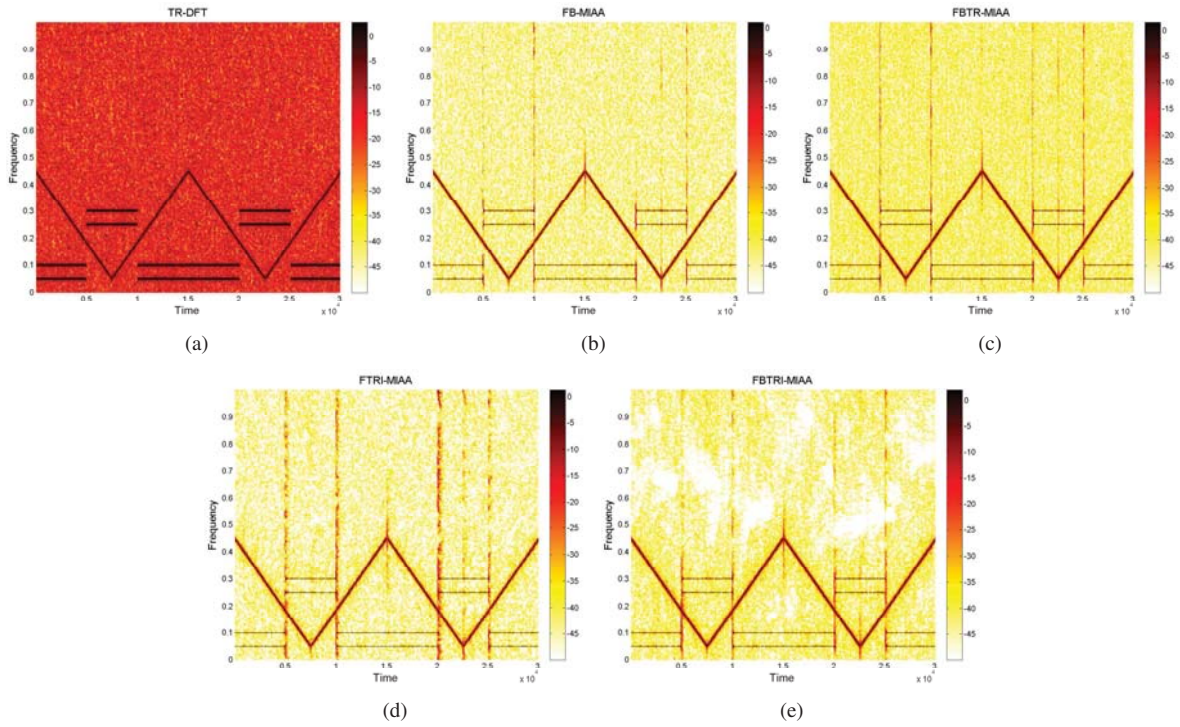

Fig. 2. (a) DFT, (b) B-MIAA, (c) BTR-MIAA, (d)FTRI-MIAA and (e) the proposed FBTRI-MIAA

Furthermore, the coefficients of the trigonometric polynomial

$$\xi_n(\omega_k) = \sum_{\ell=-N+g_n+1}^{N-g_n-1} h_n^{\ell} e^{\jmath \ell \omega_k} \tag{35}$$

are computed from (31) and (32) using fast polynomial multiplication and the FFT [22], at a cost of about $g_n \hat{G}_n \log_2(2\hat{G}_n)$ operations. Finally, (26) and (30) are evaluated using the FFT, at a cost of $K \log_2(K)$ operations, where K denotes the number of frequencies equally spaced on the unit circle. We proceed to examine the data reconstruction steps in (15) and (17), which require the computation $\hat{\mathbf{d}}_{\hat{G}_n}(n) \triangleq \tilde{\mathbf{R}}^{-1}_{\hat{G}_n}(n)\hat{\mathbf{y}}_{\hat{G}_n}(n)$. This task is similar to the one in (28); note that $\tilde{\mathbf{R}}^{-1}_{\hat{G}_n}(n)$, given by (15), allows for an left upper partition along the lines of (21), and thus all the involved computations can be performed using the GS representation of $\tilde{\mathbf{R}}^{-1}_{\tilde{N}}(n)$. Noting that $\mathbf{R}_{\tilde{N}}(n) = \tilde{\mathbf{R}}_{\tilde{N}}(n-L)$, a single GS representation is propagated through time, avoiding in this way the application of the Levinson-Durbin algorithm twice, i.e., once for the computation of the GS representation $\mathbf{R}_{\tilde{N}}(n)$ and once for that of $\tilde{\mathbf{R}}_{\tilde{N}}(n)$. As a result, the computational complexity of the resulting fast BTRI-MIAA (FBTRI-MIAA) algorithm is approximately given by $C \approx N^2 + \mathcal{O}(g_n N \log_2(N)) + K \log_2(K)$ per spectral updating. The exact value of the $\mathcal{O}(\cdot)$ term depends on the implementation details of the adopted fast Toeplitz vector multiplication method [19–21].

5. NUMERICAL EVALUATION

The computational efficiency offered by the proposed fast implementation is illustrated in Fig. 1, where the computational complexity of B-MIAA, BTR-BIAA, FTRI-MIAA, and the proposed FBTRI-MIAA spectral estimation algorithms are shown as a function of the sliding window data size N. Here, the block processing step L has been set equal to $L = 0.1N$, corresponding to 10% overlapping between successive data windows. The level of the missing data is set equal to 50%, implying that, on average, $g_n = 0.5L$. It should further be noted that B-MIAA, BTR-BIAA, and FBTRI-MIAA yields an updated spectral estimate only every L time instances, whereas the FTRI-MIAA will produce an updated spectral estimate at each time instant, i.e., using $L = 1$. The number of B-MIAA iterations that are performed on each individual data window was set equal to $m = 10$. In all cases, the number of frequency points was set equal to $K = 10N$. To illustrate the performance of the proposed algorithm, Figure 2 shows the estimated spectral estimates of a time-varying signal consisting of a mixtures of two cisoids with abruptly changing frequencies and a complex-valued linear chirp with descending/ascending linear frequency variations, being corrupted by an additive zero-mean complex Gaussian noise (see [8] for a detailed description of the signal). Here, $N = 120$, $L = 10$, and 50% of the data are missing. The DFT spectrogram is computed from the given data every other L time instances.

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