

OPTIMALITY OF PROPER SIGNALING IN GAUSSIAN MIMO BROADCAST CHANNELS WITH SHAPING CONSTRAINTS

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ABSTRACT

Proper (i.e., circularly symmetric) Gaussian signals are known to be capacity-achieving in Gaussian multiple-input multiple-output (MIMO) broadcast channels with proper noise in the sense that the sum rate capacity under a sum power constraint is achievable with proper Gaussian signaling. In this paper, we generalize this statement by proving that the optimality of proper Gaussian signals also holds under a shaping constraint, i.e., a sum covariance constraint instead of a power constraint. Moreover, we show that not only the sum rate optimal point, but the whole capacity region can be achieved with proper Gaussian signals. Finally, we prove that the worst-case noise in a MIMO broadcast channel with shaping constraints is proper.

Index Terms— Broadcast channels, capacity region, dirty paper coding, multiuser MIMO systems, proper and improper signals, covariance constraints.

1. INTRODUCTION

In the recent literature, it was shown that the use of improper [1] complex transmit signals can be beneficial in various information theoretic models of communication systems. Even though the most famous result of this kind was shown for Gaussian interference channels in [2], performance gains due to improper signals have also been observed for the Gaussian multiple-input multiple-output (MIMO) broadcast channel if a restriction to widely linear transceivers is imposed [3, 4].

On the other hand, it is accepted as common knowledge (e.g., [2–5]) that for Gaussian MIMO broadcast channels without such a restriction, proper (i.e., circularly symmetric) Gaussian signals are the optimal input distribution if the noise is proper (see Section 2 for the details of the system model). Indeed, it was shown in [6] that dirty paper coding (DPC) with proper Gaussian signals achieves the Sato upper bound on the sum rate of the Gaussian MIMO broadcast channel under a sum power constraint. However, from this fact, it cannot be concluded whether the optimality of proper Gaussian input signals also holds under constraints other than a sum power constraint. Moreover, since the result of [6] only concerns the sum rate optimal point, it does not tell us whether all Pareto optimal points of the capacity region are achievable with proper signals. This calls for a rigorous study of the question in what sense the proper Gaussian distribution is optimal in Gaussian MIMO broadcast channels, i.e., for which kind of objective functions and under which constraints.

In [7], it was shown that under any input constraint such that the input covariance matrix belongs to a compact set of positive semidef-

inite matrices, the whole capacity region of a real-valued Gaussian MIMO broadcast channel can be achieved by using dirty paper coding with Gaussian signals. Moreover, it was argued in [7] that by representing a complex Gaussian MIMO broadcast channel by a composite real representation (see Section 3), the results can be directly generalized to complex Gaussian MIMO broadcast channels. However, as explained in detail in Section 3, this only shows that complex Gaussian signals achieve the whole capacity region, but does not tell us whether the optimal Gaussian signals are proper or not.

In our previous work [8], it was observed that a certain mutual information expression that arises when studying the partial decode-and-forward rate in a Gaussian MIMO relay channel is equivalent to a sum rate in a two-user Gaussian MIMO broadcast channel that has to be maximized under a covariance constraint. In order to prove that proper signals are the optimal Gaussian signals for partial decode-and-forward in Gaussian MIMO relay channels, it was thus necessary to first prove optimality of proper Gaussian signals under a sum covariance constraint for the special case of a sum rate maximization in a two-user Gaussian MIMO broadcast channel. However, since the focus of [8] was on the relay channel, this excursus to the broadcast channel was not extended to the general case with more than two users or with an objective function other than the sum rate. Such a generalization is provided in this paper.

Just like in the special case considered in [8], the proof relies on the minimax duality with linear conic constraints from [9, 10], which we briefly review in Section 4. The optimality of proper transmit signals is then proven in Section 5. As an additional result, we show in Section 6 that the worst-case noise for weighted sum rate maximizations with shaping constraints is proper as well.

Notation: We use \mathbf{C}_x and $h(\mathbf{x})$ for the covariance matrix and the differential entropy of a real-valued or complex random vector \mathbf{x} , respectively. The sets $\mathbb{S}^N \subset \mathbb{R}^{N \times N}$ and $\mathbb{H}^N \subset \mathbb{C}^{N \times N}$ are the set of real-valued symmetric matrices and the set of complex Hermitian matrices, respectively. Orthogonal complements of linear subspaces are denoted by \bullet^\perp , and $\bigotimes_{k=1}^K \mathcal{A}_k$ is used to denote the Cartesian product $\mathcal{A}_1 \times \cdots \times \mathcal{A}_K$. The order relations \succ and \succeq have to be understood in the sense of positive (semi-)definiteness. We use the shorthand notation $(\bullet_k)_{\forall k} = (\bullet_1, \dots, \bullet_K)$.

2. SYSTEM MODEL

In a Gaussian MIMO broadcast channel, where a transmitter with M antennas serves a set of K users with N_k receive antennas at user k , data transmission is described by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \boldsymbol{\eta}_k \quad (1)$$

where \mathbf{H}_k is the $N_k \times M$ channel matrix between the base station and user k , and $\boldsymbol{\eta}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\boldsymbol{\eta}_k})$ with $\mathbf{C}_{\boldsymbol{\eta}_k} \succ \mathbf{0}$ is the additive circularly symmetric complex Gaussian noise at user k . If the transmit signal \mathbf{x} is constructed from the input signals $(\mathbf{x}_k)_{\forall k}$ by means of dirty paper coding [6, 7], the data rate

$$r_k = \mu \log \frac{\det(\mathbf{C}_{\boldsymbol{\eta}_k} + \mathbf{H}_k (\sum_{j \in \mathcal{I}_k \cup \{k\}} \mathbf{C}_{\mathbf{x}_j}) \mathbf{H}_k^H)}{\det(\mathbf{C}_{\boldsymbol{\eta}_k} + \mathbf{H}_k (\sum_{j \in \mathcal{I}_k} \mathbf{C}_{\mathbf{x}_j}) \mathbf{H}_k^H)} \quad (2)$$

is achievable for user k , where \mathcal{I}_k is the set of users causing interference to user k , i.e., the set of users encoded after user k . The pre-log factor is $\mu = 1$ in a complex broadcast channel with complex channel matrices, proper complex noise, and proper complex signals. In a real-valued broadcast channel with real-valued channel matrices, real-valued noise, and real-valued signals, we have $\mu = \frac{1}{2}$.

3. PROPER AND IMPROPER GAUSSIAN SIGNALS

To characterize a general complex (proper or improper) zero-mean Gaussian vector, we need the so-called pseudocovariance matrix $\tilde{\mathbf{C}}_{\mathbf{x}} = \mathbb{E}[\mathbf{x}\mathbf{x}^T]$ in addition to the conventional covariance matrix $\mathbf{C}_{\mathbf{x}} = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$ (see [11] and the references therein). A random vector is called *proper* (equivalent to circularly symmetric for zero-mean Gaussian random vectors) if we have $\tilde{\mathbf{C}}_{\mathbf{x}} = \mathbf{0}$ so that $\mathbf{C}_{\mathbf{x}}$ suffices to describe the statistical properties. If $\tilde{\mathbf{C}}_{\mathbf{x}} \neq \mathbf{0}$, \mathbf{x} is called *improper*.

An alternative description can be obtained using the composite real representation

$$\tilde{\mathbf{A}} = \begin{bmatrix} \Re(\mathbf{A}) & -\Im(\mathbf{A}) \\ \Im(\mathbf{A}) & \Re(\mathbf{A}) \end{bmatrix} \text{ and } \tilde{\mathbf{a}} = \begin{bmatrix} \Re(\mathbf{a}) \\ \Im(\mathbf{a}) \end{bmatrix} \quad (3)$$

of matrices \mathbf{A} and vectors \mathbf{a} . The covariance matrix of the composite real representation of $\mathbf{x} \in \mathbb{C}^N$ is given by [12]

$$\mathbf{C}_{\tilde{\mathbf{x}}} = \underbrace{\frac{1}{2} \begin{bmatrix} \Re(\mathbf{C}_{\mathbf{x}}) & -\Im(\mathbf{C}_{\mathbf{x}}) \\ \Im(\mathbf{C}_{\mathbf{x}}) & \Re(\mathbf{C}_{\mathbf{x}}) \end{bmatrix}}_{\mathbf{P}_{\mathbf{x}}} + \underbrace{\frac{1}{2} \begin{bmatrix} \Re(\tilde{\mathbf{C}}_{\mathbf{x}}) & \Im(\tilde{\mathbf{C}}_{\mathbf{x}}) \\ \Im(\tilde{\mathbf{C}}_{\mathbf{x}}) & -\Re(\tilde{\mathbf{C}}_{\mathbf{x}}) \end{bmatrix}}_{\mathbf{N}_{\mathbf{x}}}. \quad (4)$$

The partitioning into a power shaping component $\mathbf{P}_{\mathbf{x}} \in \mathcal{P}^N$ and an impropriety component $\mathbf{N}_{\mathbf{x}} \in \mathcal{N}^N$ with

$$\mathcal{P}^N = \left\{ \mathbf{P} \in \mathbb{S}^{2N} \mid \mathbf{P} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \mathbf{A}, \mathbf{B} \in \mathbb{R}^{N \times N} \right\} \quad (5)$$

$$\mathcal{N}^N = \left\{ \mathbf{N} \in \mathbb{S}^{2N} \mid \mathbf{N} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{D} & -\mathbf{C} \end{bmatrix}, \mathbf{C}, \mathbf{D} \in \mathbb{R}^{N \times N} \right\} \quad (6)$$

was proposed in [8]. As shown in [8, Lemma 1], \mathcal{N}^N is the orthogonal complement of \mathcal{P}^N in \mathbb{S}^{2N} . Therefore, any $\mathbf{C}_{\tilde{\mathbf{x}}}$ can be uniquely decomposed into $\mathbf{P}_{\mathbf{x}}$ and $\mathbf{N}_{\mathbf{x}}$ (which correspond to unique $\mathbf{C}_{\mathbf{x}}$ and $\tilde{\mathbf{C}}_{\mathbf{x}}$).

Only if $\mathbf{N}_{\mathbf{x}} = \mathbf{0}$, the complex random vector \mathbf{x} is proper. Therefore, using the composite real representation without restricting the real-valued covariance matrices to lie in \mathcal{P}^N corresponds to the general (proper or improper) complex case. With this in mind, it is easy to understand why we cannot draw conclusions about the propriety of the optimal Gaussian signals based on the real-valued study in [7]: to do so, we would need to know whether the impropriety components of the optimal real-valued transmit covariance matrices are zero for real-valued channels with the special structure (3). However, since the authors [7] only considered general channel matrices

without a special structure, we cannot easily answer this question based on the results obtained in [7].

Instead, we apply the minimax duality framework from [9, 10], which is reviewed in the next section, to the composite real-representation in order to obtain insights about MIMO broadcast channels with general complex signals.

4. UPLINK DOWNLINK DUALITY

Since the classical uplink downlink duality from [6] only holds under a sum power constraint, we make use of the following recently established duality result.

Consider a MIMO multiple access channel with channel matrices $(\mathbf{H}_k^H)_{\forall k}$, input signals $(\boldsymbol{\xi}_k)_{\forall k}$ and receiver noise $\boldsymbol{\eta}$. Then, the downlink minimax problem

$$\min_{\substack{(\mathbf{C}_{\boldsymbol{\eta}_k} \succeq \mathbf{0})_{\forall k}: (\mathbf{C}_{\boldsymbol{\eta}_k})_{\forall k} \in \mathcal{Y}^\perp \\ \sum_{k=1}^K \text{tr}[\mathbf{B}_k \mathbf{C}_{\boldsymbol{\eta}_k}] = \sigma^2}} \max_{\substack{(\mathbf{C}_{\mathbf{x}_k} \succeq \mathbf{0})_{\forall k}, \mathbf{Z} \in \mathcal{Z} \\ \sum_{k=1}^K \mathbf{C}_{\mathbf{x}_k} \preceq \mathbf{C} + \mathbf{Z}}} \sum_{k=1}^K w_k r_k \quad (7)$$

and the uplink minimax problem

$$\min_{\substack{\mathbf{C}_{\boldsymbol{\eta}} \succeq \mathbf{0}, \mathbf{C}_{\boldsymbol{\eta}} \in \mathcal{Z}^\perp \\ \text{tr}[\mathbf{C} \mathbf{C}_{\boldsymbol{\eta}}] = \sigma^2}} \max_{\substack{(\mathbf{C}_{\boldsymbol{\xi}_k} \succeq \mathbf{0})_{\forall k}, (\mathbf{Y}_k)_{\forall k} \in \mathcal{Y} \\ \tilde{\mathbf{C}}_{\boldsymbol{\xi}_k} \preceq \mathbf{B}_k + \mathbf{Y}_k}} \sum_{k=1}^K w_k r_k^{\text{UL}} \quad (8)$$

have the same optimal value [9, 10].

The linear subspaces $\mathcal{Z} \subseteq \mathbb{H}^M$ and $\mathcal{Y} \subseteq \bigotimes_{k=1}^K \mathbb{H}^{N_k}$ (\mathbb{H} instead of \mathbb{H} in the real-valued case) can be used to model various constraints on the transmit covariance matrices such as sum power constraints or shaping constraints [10]. Their orthogonal complements \mathcal{Y}^\perp and \mathcal{Z}^\perp determine which noise distributions are allowed in the worst-case noise optimizations.

The downlink rate r_k of user k is given by (2), and the uplink rate is

$$r_k^{\text{UL}} = \mu \log \frac{\det(\mathbf{C}_{\boldsymbol{\eta}} + \sum_{j \in \mathcal{I}_k^{\text{UL}} \cup \{k\}} \mathbf{H}_j^H \mathbf{C}_{\boldsymbol{\xi}_j} \mathbf{H}_j)}{\det(\mathbf{C}_{\boldsymbol{\eta}} + \sum_{j \in \mathcal{I}_k^{\text{UL}}} \mathbf{H}_j^H \mathbf{C}_{\boldsymbol{\xi}_j} \mathbf{H}_j)}. \quad (9)$$

The duality established for the proper complex case with $\mu = 1$ in [9] can be easily extended to real-valued broadcast channels with $\mu = \frac{1}{2}$. Just like in [6], the decoding order in the uplink is the reverse downlink encoding order, i.e., the set of interfering users in the uplink is $\mathcal{I}_k^{\text{UL}} = \{1, \dots, K\} \setminus (\mathcal{I}_k \cup \{k\})$. In all optimization problems in this paper, we implicitly assume that the optimal encoding/decoding order is used, i.e., the sets \mathcal{I}_k and $\mathcal{I}_k^{\text{UL}}$ are implicitly optimized.

5. OPTIMALITY OF PROPER SIGNALS

Before stating and proving the main theorem for the Gaussian MIMO broadcast channel, let us consider the uplink scenario (Gaussian MIMO multiple access channel).

Lemma 1. *In a MIMO multiple access channel with proper Gaussian noise, an optimum of any weighted sum rate maximization under constraints on the covariance matrices $(\mathbf{C}_{\mathbf{x}_k})_{\forall k}$ is achieved with proper transmit signals.*

Proof. Let us assume without loss of generality that $w_K \geq w_{K-1} \geq \dots \geq w_1$. Then, by applying [13, Section I] to the composite real representation, we have that the optimal weighted sum rate in the

Gaussian MIMO multiple access channel with general complex Gaussian signals is obtained by maximizing

$$w_1 h(\tilde{\mathbf{z}}_1) - \underbrace{w_K h(\tilde{\boldsymbol{\eta}})}_{\text{const.}} + \underbrace{\sum_{k=2}^K (w_j - w_{j-1}) h(\tilde{\mathbf{z}}_k)}_{\geq 0} \quad (10)$$

with $\mathbf{z}_k = \boldsymbol{\eta} + \sum_{j=k}^K \mathbf{H}_j^H \boldsymbol{\xi}_j$. Introducing a proper Gaussian vector $\mathbf{z}_{k,\text{proper}}$ with the same covariance matrix as \mathbf{z}_k , we have that $h(\tilde{\mathbf{z}}_k) = h(\mathbf{z}_k) \leq h(\mathbf{z}_{k,\text{proper}})$, where the equality is by definition (see, e.g., [14, Section 2.2.3]) and the inequality comes from the fact that the proper Gaussian distribution maximizes the differential entropy [15]. Given $\mathbf{C}_{\boldsymbol{\eta}}$ and arbitrary fixed covariance matrices $(\mathbf{C}_{\boldsymbol{\xi}_k})_{\forall k}$ that comply with the constraints, $h(\mathbf{z}_{k,\text{proper}})$ is constant, and we can achieve equality $h(\mathbf{z}_k) = h(\mathbf{z}_{k,\text{proper}})$ by setting the pseudocovariance matrices $(\tilde{\mathbf{C}}_{\boldsymbol{\xi}_k})_{\forall k}$ to zero since $\tilde{\mathbf{C}}_{\boldsymbol{\eta}} = \mathbf{0}$ by assumption. \square

Our main result for the downlink is stated and proven in the following.

Theorem 1. *In a complex Gaussian MIMO broadcast channel with proper noise with fixed covariance matrices $(\mathbf{C}_{\boldsymbol{\eta}_k} = \mathbf{R}_k \succ \mathbf{0})_{\forall k}$, the complete capacity region under a shaping constraint $\sum_{k=1}^K \mathbf{C}_{\mathbf{x}_k} \preceq \mathbf{C}$ is achievable with proper Gaussian input signals.*

Proof. Due to convexity of the capacity region, any feasible rate point can be achieved by time-sharing (convex combinations) between optimizers of weighted sum rate problems. Therefore, it suffices to prove optimality of proper signals for weighted sum rate maximizations. Moreover, it suffices to consider the case $\mathbf{R}_k = \mathbf{I}_{N_k}$ since any other case could be treated by introducing equivalent channels $\mathbf{H}'_k = \mathbf{R}_k^{-\frac{1}{2}} \mathbf{H}_k$.

We write the weighted sum rate maximization as a minimax problem in (11) in Table 1 using the composite real representation. The constraint set for the worst-case noise optimization contains $(\mathbf{C}_{\tilde{\boldsymbol{\eta}}_k})_{\forall k} = (\frac{1}{2} \mathbf{I}_{2N_k})_{\forall k}$ as the only element. The matrix $\mathbf{Z} \in \mathcal{N}^M$ allows us to choose the impropriety component of the composite real transmit covariance matrix while the power shaping component is constrained to be smaller than or equal to $\mathbf{P} = \frac{1}{2} \mathbf{C} \in \mathcal{P}^M$ in the sense of positive semidefiniteness. The optimal minimax weighted sum rate denoted by R_w and the solution to the uplink minimax problem (12) in Table 1, which we denote by R_w^{UL} , are equal due to Section 4.

Since $(\mathbf{Y}_k)_{\forall k}$ only has to satisfy a trace constraint on the sum $\sum_{k=1}^K \mathbf{Y}_k$, the individual constraints on $\text{tr}[\mathbf{C}_{\tilde{\boldsymbol{\xi}}_k}]$ become equivalent to a sum power constraint $\sum_{k=1}^K \text{tr}[\mathbf{C}_{\tilde{\boldsymbol{\xi}}_k}] \leq \sum_{k=1}^K 2N_k$ (similar as in the proper complex case in [10]), which corresponds to a constraint on the covariance matrices $\mathbf{C}_{\tilde{\boldsymbol{\xi}}_k}$, but not on the pseudocovariance matrices $\tilde{\mathbf{C}}_{\tilde{\boldsymbol{\xi}}_k}$. Moreover, due to the constraint $\mathbf{C}_{\tilde{\boldsymbol{\eta}}} \in \mathcal{P}^M$, we have proper noise in the dual uplink. Therefore, Lemma 1 applies, and we have $R_w^{\text{UL,proper}} = R_w^{\text{UL}} = R_w^{\text{DL}}$, where $R_w^{\text{UL,proper}}$ is the optimizer of (13) in Table 1.

Let $\mathbf{R}_w^{\text{DL,proper}}$ be the optimizer of (14) in Table 1, where the feasible set for the worst-case noise optimization contains only the element $(\mathbf{C}_{\boldsymbol{\eta}_k})_{\forall k} = (\mathbf{I}_{N_k})_{\forall k}$. Again due to the uplink-downlink duality (Section 4), we have $R_w^{\text{DL,proper}} = R_w^{\text{UL,proper}} = R_w^{\text{UL}} = R_w^{\text{DL}}$. \square

Note that the optimal decoding order in the uplink (which we implicitly assume to be chosen) is such that the condition on the weights $(w_k)_{\forall k}$ given in the proof of Lemma 1 would be fulfilled

after reindexing. The optimal downlink encoding order is then given by the reverse ordering.

Since a constraint $\sum_{k=1}^K \mathbf{C}_{\mathbf{x}_k} \in \mathcal{S}^M$ can be related to sum covariance constraints of the form $\sum_{k=1}^K \mathbf{C}_{\mathbf{x}_k} \preceq \mathbf{C}$, $\mathbf{C} \in \mathcal{S}^M$ as in [7, Lemma 1], we obtain the following corollary.

Corollary 1. *In a complex Gaussian MIMO broadcast channel with proper noise with fixed covariance matrices $(\mathbf{C}_{\boldsymbol{\eta}_k} = \mathbf{R}_k \succ \mathbf{0})_{\forall k}$, the complete capacity region under a sum covariance constraint $\sum_{k=1}^K \mathbf{C}_{\mathbf{x}_k} \in \mathcal{S}^M$, where \mathcal{S}^M is an arbitrary compact subset of the set of positive semidefinite $M \times M$ matrices, is achievable with proper Gaussian signals.*

6. PROPRIETY OF THE WORST-CASE NOISE

In the preceding section, we have considered the case of fixed noise covariance matrices in the downlink. However, the duality framework applied to prove Theorem 1 is capable of treating worst-case noise optimizations in the downlink, as well.

If we perform a worst-case noise optimization, but keep the assumption of proper Gaussian noise, the optimality of proper transmit signals still has to hold: since Theorem 1 holds for arbitrary noise covariance matrices in the case of proper noise, it also holds for the optimizers of the worst-case noise optimization.

An interesting question is, however, what we obtain as a result of the worst-case noise optimization if we allow arbitrary (proper or improper) complex Gaussian noise and only fix the total noise power. The answer to this question for the case of a shaping constraint is stated in the following theorem.

Theorem 2. *For the weighted sum rate maximization in a complex Gaussian MIMO broadcast channel with a shaping constraint $\sum_{k=1}^K \mathbf{C}_{\mathbf{x}_k} \preceq \mathbf{C}$, the worst-case noise under a sum noise power constraint is proper.*

Proof. In (11) in Table 1, replace $\tilde{\mathbf{Y}}^\perp$ by $\tilde{\mathbf{Y}}'^\perp = \bigotimes_{k=1}^K \mathbb{S}^{2N_k}$. Then we have $\tilde{\mathbf{Y}}' = \{(\mathbf{0})_{\forall k}\}$ in the uplink, i.e., the uplink shaping constraints become $\mathbf{C}_{\tilde{\boldsymbol{\xi}}_k} \preceq \mathbf{I}_{2N_k}$, $\forall k$. It is easy to verify that the weighted sum rate in the uplink as written in (10) is nondecreasing in all uplink transmit covariance matrices $\mathbf{C}_{\tilde{\boldsymbol{\xi}}_k}$ (assuming that the optimal decoding order is chosen). Therefore, there is an optimizer for which the shaping constraints in the uplink are active, which implies that the signaling is proper since $\mathbf{C}_{\tilde{\boldsymbol{\xi}}_k} = \mathbf{I}_{2N_k} \in \mathcal{P}^{N_k}$.

Let us now consider a relaxed maximization in the uplink with

$$\tilde{\mathbf{Y}}'' = \left\{ (\mathbf{Y}_k)_{\forall k} \in \bigotimes_{k=1}^K \mathbb{S}^{2N_k} \mid \mathbf{Y}_k \in \mathcal{N}^{N_k} \right\}, \quad (16)$$

i.e., the uplink transmit signals may be improper, and the covariance constraint becomes a constraint on the power shaping component only. Moreover, the uplink noise is still restricted to be proper as in the proof of Theorem 1 since $\tilde{\mathbf{Z}}^\perp$ is unchanged. Therefore, due to Lemma 1, proper transmit signals are optimal in the relaxed uplink problem. Consequently, there is an optimizer with $\mathbf{Y}_k = \mathbf{0}$, and the relaxed problem has the same optimal value as the original problem.

Translating the relaxed problem back to the downlink, the new $\tilde{\mathbf{Y}}''^\perp$ restricts the downlink noise to be proper, but the same optimal value as without this restriction is achieved. \square

Combining both theorems of this paper, we obtain that the optimal transmit signals used in combination with the worst-case noise are proper as well.

	composite real ($\hat{=}$ general complex)	proper complex
downlink	$\min_{\substack{(\mathbf{C}_{\tilde{\eta}_k} \succeq \mathbf{0})_{\forall k} : (\mathbf{C}_{\tilde{\eta}_k})_{\forall k} \in \tilde{\mathcal{Y}}^\perp \\ \sum_{k=1}^K \text{tr}[\mathbf{C}_{\tilde{\eta}_k}] = \sum_{k=1}^K N_k}} \max_{\substack{(\mathbf{C}_{\tilde{\mathbf{x}}_k} \succeq \mathbf{0})_{\forall k}, \mathbf{Z} \in \tilde{\mathcal{Z}} \\ \mathbf{C}_{\tilde{\mathbf{x}}_k} \preceq \mathbf{P} + \mathbf{Z}}} \sum_{k=1}^K w_k r_{k,\text{real}} \quad (11)$ $\tilde{\mathcal{Z}} = \mathcal{N}^M$ $\tilde{\mathcal{Y}}^\perp = \left\{ (\mathbf{C}_{\tilde{\eta}_k})_{\forall k} \in \bigotimes_{k=1}^K \mathbb{S}^{2N_k} \mid \mathbf{C}_{\tilde{\eta}_k} = \alpha \mathbf{I}_{2N_k} \forall k, \alpha \in \mathbb{R} \right\}$ $r_{k,\text{real}} = \frac{1}{2} \log \frac{\det(\mathbf{C}_{\tilde{\eta}_k} + \tilde{\mathbf{H}}_k (\sum_{j \in \mathcal{I}_k \cup \{k\}} \mathbf{C}_{\tilde{\mathbf{x}}_j}) \tilde{\mathbf{H}}_k^T)}{\det(\mathbf{C}_{\tilde{\eta}_k} + \tilde{\mathbf{H}}_k (\sum_{j \in \mathcal{I}_k} \mathbf{C}_{\tilde{\mathbf{x}}_j}) \tilde{\mathbf{H}}_k^T)}$	$\min_{\substack{(\mathbf{C}_{\eta_k} \succeq \mathbf{0})_{\forall k} : (\mathbf{C}_{\eta_k})_{\forall k} \in \mathcal{Y}^\perp \\ \sum_{k=1}^K \text{tr}[\mathbf{C}_{\eta_k}] = \sum_{k=1}^K N_k}} \max_{\substack{(\mathbf{C}_{\mathbf{x}_k} \succeq \mathbf{0})_{\forall k} \\ \mathbf{C}_{\mathbf{x}_k} \preceq \mathbf{C}}} \sum_{k=1}^K w_k r_k \quad (14)$ $\mathcal{Z} = (\mathbb{H}^M)^\perp = \{\mathbf{0}\}$ $\mathcal{Y}^\perp = \left\{ (\mathbf{C}_{\eta_k})_{\forall k} \in \bigotimes_{k=1}^K \mathbb{H}^{N_k} \mid \mathbf{C}_{\eta_k} = \alpha \mathbf{I}_{N_k} \forall k, \alpha \in \mathbb{R} \right\}$ $r_k = \log \frac{\det(\mathbf{C}_{\eta_k} + \mathbf{H}_k (\sum_{j \in \mathcal{I}_k \cup \{k\}} \mathbf{C}_{\mathbf{x}_j}) \mathbf{H}_k^H)}{\det(\mathbf{C}_{\eta_k} + \mathbf{H}_k (\sum_{j \in \mathcal{I}_k} \mathbf{C}_{\mathbf{x}_j}) \mathbf{H}_k^H)} \quad (15)$
	duality (Section 4) \Downarrow	duality (Section 4) \Uparrow
uplink	$\min_{\substack{\mathbf{C}_{\tilde{\eta}} \succeq \mathbf{0} : \mathbf{C}_{\tilde{\eta}} \in \tilde{\mathcal{Z}}^\perp \\ \text{tr}[\mathbf{P} \mathbf{C}_{\tilde{\eta}}] = \sum_{k=1}^K N_k}} \max_{\substack{(\mathbf{C}_{\tilde{\xi}_k} \succeq \mathbf{0})_{\forall k}, (\mathbf{Y}_k)_{\forall k} \in \tilde{\mathcal{Y}} \\ \mathbf{C}_{\tilde{\xi}_k} \preceq \mathbf{I}_{2N_k} + \mathbf{Y}_k \forall k}} \sum_{k=1}^K w_k r_{k,\text{real}}^{\text{UL}} \quad (12)$ $\tilde{\mathcal{Z}}^\perp = (\mathcal{N}^M)^\perp = \mathcal{P}^M \text{ (see Section 3)}$ $\tilde{\mathcal{Y}} = \left\{ (\mathbf{Y}_k)_{\forall k} \in \bigotimes_{k=1}^K \mathbb{S}^{2N_k} \mid \sum_{k=1}^K \text{tr}[\mathbf{Y}_k] = 0 \right\}$ $r_{k,\text{real}}^{\text{UL}} = \frac{1}{2} \log \frac{\det(\mathbf{C}_{\tilde{\eta}} + \sum_{j \in \mathcal{I}_k^{\text{UL}} \cup \{k\}} \tilde{\mathbf{H}}_j^T \mathbf{C}_{\tilde{\xi}_j} \tilde{\mathbf{H}}_j)}{\det(\mathbf{C}_{\tilde{\eta}} + \sum_{j \in \mathcal{I}_k^{\text{UL}}} \tilde{\mathbf{H}}_j^T \mathbf{C}_{\tilde{\xi}_j} \tilde{\mathbf{H}}_j)}$	$\min_{\substack{\mathbf{C}_{\eta} \succeq \mathbf{0} \\ \text{tr}[\mathbf{C} \mathbf{C}_{\eta}] = \sum_{k=1}^K N_k}} \max_{\substack{(\mathbf{C}_{\xi_k} \succeq \mathbf{0})_{\forall k}, (\mathbf{Y}_k)_{\forall k} \in \mathcal{Y} \\ \mathbf{C}_{\xi_k} \preceq \mathbf{I}_{N_k} + \mathbf{Y}_k \forall k}} \sum_{k=1}^K w_k r_k^{\text{UL}} \quad (13)$ $\mathcal{Z}^\perp = \mathbb{H}^M \text{ (any } \mathbf{C}_{\eta} \text{ corresponds to a } \mathbf{P}_{\eta} \in \tilde{\mathcal{Z}}^\perp = \mathcal{P}^M)$ $\mathcal{Y} = \left\{ (\mathbf{Y}_k)_{\forall k} \in \bigotimes_{k=1}^K \mathbb{H}^{N_k} \mid \sum_{k=1}^K \text{tr}[\mathbf{Y}_k] = 0 \right\}$ $r_k^{\text{UL}} = \log \frac{\det(\mathbf{C}_{\eta} + \sum_{j \in \mathcal{I}_k^{\text{UL}} \cup \{k\}} \mathbf{H}_j^H \mathbf{C}_{\xi_j} \mathbf{H}_j)}{\det(\mathbf{C}_{\eta} + \sum_{j \in \mathcal{I}_k^{\text{UL}}} \mathbf{H}_j^H \mathbf{C}_{\xi_j} \mathbf{H}_j)}$

Table 1. Steps of the proof of Theorem 1.

7. DISCUSSION

It is intuitively understandable that proper Gaussian signals are optimal in MIMO broadcast channels with proper Gaussian noise: since the user encoded last does not see any interference, the situation for this user is similar as in a point-to-point transmission, where proper transmit signals are optimal. Choosing a proper transmit signal for this user, the noise-plus-interference signal at the second last user is proper so that a proper transmit signal should be chosen for this user as well, and so forth. However, a formal proof under sum covariance constraints is nontrivial and was provided in this paper by exploiting the uplink downlink duality for minimax problems with linear conic constraints from [9, 10].

The result that the worst-case noise under shaping constraints is proper has an intuitive explanation as well: proper Gaussian noise has maximal entropy for given noise power, and it makes sense that this is the worst case unless we require the transmit signals to be improper (e.g., by using a modulation scheme with improper constellation). Again, the formal proof of this result is based on the minimax uplink downlink duality.

8. REFERENCES

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