ON THE EPSILON-OPTIMAL DISCRIMINATION OF TWO ONE-DIMENSIONAL SUBSPACES

Lionel Fillatre

Univ. Nice Sophia Antipolis, CNRS, I3S, UMR 7271, 06900 Sophia Antipolis, France

ABSTRACT

This paper addresses the problem of discriminating two different vector lines from a non-zero mean Gaussian noise vector. Under each hypothesis, the Gaussian noise vector is completely characterized by its expected value which belongs to a known vector line. A new criterion of optimality, namely the epsilon most stringent test, is proposed and studied. This criterion consists in minimizing the maximum shortcoming of the test, up to a small loss, subject to a constrained false alarm probability. The maximum shortcoming corresponds to the maximum gap between the power function of the test and the envelope power function which is defined as the supremum of the power over all tests satisfying the prescribed false alarm probability. It is numerically shown that the proposed test outperforms the generalized likelihood ratio test.

Index Terms— Statistical hypothesis, Most stringent test, Subspace classification, Generalized likelihood ratio test.

1. INTRODUCTION

The problem of classifying a vector in noisy measurements appears in many applications including radar and sonar signal processing [1, 2], image processing [3], speech segmentation [4, 5], quantitative non-destructive testing [6], network monitoring [7, 8] and digital communication [9] among others. The problem of discriminating two subspaces given some noisy observations is a common but difficult problem, even in the simplest case where each subspace is a vector line. The main difficulty is due to the fact that each hypothesis is composite since the expected mean of the observation can take an infinite number of values. The common solution is the nearest mean classifier [10, 11], corresponding to the Generalized Likelihood Ratio Test (GLRT), which discriminates two subspaces by affecting the observation vector to the closest subspace but it is not yet established whether the nearest mean classifier is optimal in the case of Gaussian observations.

The observation model has the form $\boldsymbol{y} = x_i \, \boldsymbol{h}_i + \boldsymbol{\xi}$ where $\boldsymbol{y} \in \mathbb{R}^n$ is the measurement vector and $i \in \{1, 2\}$. The unit vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ which define the one-dimensional subspaces are known. The scale factors x_1 and x_2 are unknown and deterministic. The zero-mean Gaussian noise vector $\boldsymbol{\xi}$ has the known covariance matrix $\boldsymbol{\Sigma}$. The main objective is to specify the index i of $\boldsymbol{h}_i$ occurring in $\boldsymbol{y}$ whatever x_i.

To deal with this problem, traditional criterions like most powerful test [2], constrained minimax test [12, 13], Bayesian test [14] and invariant test [15] are not appropriate since the basic assumptions required for their derivation are not satisfied (no one-sided problem, no prior distribution, no indifference zone between hypotheses, no symmetry). A reasonable definition of a general criterion of optimality can be given in terms of the envelope power that corresponds to the supremum of the probability of correct detection over all tests satisfying a prescribed false alarm probability. The Most Stringent (MS) test is the test whose maximum gap between its power of detection and the envelope power is minimum [16, 17] subject to a constrained false alarm probability. Even if the existence of this test is established in [16], its design may be very difficult and the value of the maximum gap, also called the maximum shortcoming, remains unknown.

The first contribution of this paper is the design of an Epsilon Most Stringent (EMS) test solving the detection problem. It is epsilon optimal in the sense that it is optimal with a loss of a small part, say ε, of optimality with respect to the theoretical MS test [18]. This loss of optimality is acceptable since the MS test is untractable due to the difficulties above mentioned. Secondly, the statistical performances of the test, namely its false alarm probability function and its power function, are calculated in a closed-form. Finally, it is numerically shown that the GLRT is suboptimal in the sense of the most stringent criterion.

The paper is organized as follows. Section 2 starts with the problem statement. Section 3 derives the EMS test. Section 4 numerically shows that the EMS test outperforms the GLRT. Finally, Section 5 concludes this paper.

2. PROBLEM STATEMENT

The classification problem is stated as the choice between the two composite hypotheses

$$\begin{aligned} &\mathcal{H}_1 \colon \left\{ \boldsymbol{y} \sim \mathcal{N}(x_1 \, \boldsymbol{h}_1, \boldsymbol{I}_n) \,,\; x_1 \in \mathbb{R} \right\}, \\ &\mathcal{H}_2 \colon \left\{ \boldsymbol{y} \sim \mathcal{N}(x_2 \, \boldsymbol{h}_2, \boldsymbol{I}_n) \,,\; x_2 \in \mathbb{R}^* \right\}, \end{aligned} \tag{1}$$

where n is the size of $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{I}_n$ is the identity matrix of order n and $\mathcal{N}(\boldsymbol{m}, \boldsymbol{\Sigma})$ denotes the multivariate normal distri-

978-1-4799-2893-4/14/$31.00 ©2014 IEEE

bution with the mean $\boldsymbol{m}$ and the positive definite covariance matrix $\boldsymbol{\Sigma}$. Let $\|\cdot\|$ denote the Euclidean norm. The vectors $\boldsymbol{h}_1 \in \mathbb{R}^n$ and $\boldsymbol{h}_2 \in \mathbb{R}^n$ are known, $\boldsymbol{h}_1 \neq \boldsymbol{h}_2$ and it is assumed without any loss of generality that $\|\boldsymbol{h}_1\| = \|\boldsymbol{h}_2\| = 1$ since the scale factors x_1 and x_2 are unknown ($x_2 \neq 0$ ensures that the two hypotheses are distinguishable). Let $\Theta_1 = \mathrm{span}\,(\boldsymbol{h}_1)$ and $\Theta_2 = \mathrm{span}\,(\boldsymbol{h}_2) \setminus \{0\}$ be the one-dimensional spaces spanned by the vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ with and without the null vector, respectively. Testing $\mathcal{H}_1$ against $\mathcal{H}_2$ is equivalent to decide if the mean of $\boldsymbol{y}$ belongs to the space Θ_1 or Θ_2. Without any loss of generality, it is assumed that the covariance matrix of $\boldsymbol{y}$ is reduced to $\boldsymbol{I}_n$. Let η be the angle, defined by

$$\cos(\eta) = \boldsymbol{h}_1^\top \boldsymbol{h}_2, \tag{2}$$

between $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ where $\boldsymbol{h}^\top$ denotes the transpose of $\boldsymbol{h}$. Due to the spherical symmetry of the normal distribution with identity covariance matrix, il is assumed that $0 < \eta \leq \frac{\pi}{2}$.

The detection of sinusoidal signals [1] is a good illustration of this classification problem. Given the received sequence $y(k)$ for $k = 1, \ldots, n$, the goal is to decide between

$$\begin{aligned}
&\mathcal{H}_1 \colon \big\{y(k){=}x_1 \cos(\omega_1 k + \phi_1) + \xi(k),\ x_1 \in \mathbb{R},\ \forall\, k\big\},\\
&\mathcal{H}_2 \colon \big\{y(k){=}x_2 \cos(\omega_2 k + \phi_2) + \xi(k),\ x_2 \in \mathbb{R}^*,\ \forall\, k\big\}
\end{aligned}$$

where the angular frequencies ω_1 and ω_2 and the phases ϕ_1 and ϕ_2 are known. The amplitudes x_1 and x_2 are unknown. The sequence $\xi(k)$ is a (possibly correlated in time) Gaussian noise whose covariance matrix is known.

A statistical test δ is a function of $\mathbb{R}^n$ into $\{\mathcal{H}_1, \mathcal{H}_2\}$ such that hypothesis $\mathcal{H}_i$ is accepted if $\delta(\boldsymbol{y}) = \mathcal{H}_i$ [16, 19]. Let

$$\mathcal{K}_\alpha = \left\{\delta : \sup_{x_1 \in \mathbb{R}} \Pr_{x_1 \boldsymbol{h}_1}(\delta(\boldsymbol{y}) = \mathcal{H}_2) \leq \alpha\right\} \tag{3}$$

be the class of tests of level α with an upper-bounded false alarm probability α, where $\Pr_{\boldsymbol{\theta}}(\cdot)$ stands for $\boldsymbol{y}$ being generated by the distribution $\mathcal{N}(\boldsymbol{\theta}, \boldsymbol{I}_n)$. The statistical performances of the test are characterized by the false alarm function $\alpha_\delta(\boldsymbol{\theta}) = \Pr_{\boldsymbol{\theta}}(\delta(\boldsymbol{y}) = \mathcal{H}_2)$, $\forall \boldsymbol{\theta} \in \Theta_1$, and the power function $\beta_\delta(\boldsymbol{\theta}) = \Pr_{\boldsymbol{\theta}}(\delta(\boldsymbol{y}) = \mathcal{H}_2)$, $\forall \boldsymbol{\theta} \in \Theta_2$. The envelope power function [16] is defined by

$$\beta^*_\alpha(\boldsymbol{\theta}) = \sup_{\delta \in \mathcal{K}_\alpha} \beta_\delta(\boldsymbol{\theta}) \tag{4}$$

for all $\boldsymbol{\theta} \in \Theta_2$. The envelope power function $\beta^*_\alpha(\boldsymbol{\theta})$ is the maximum power that can be attained at level α for testing $\mathcal{H}_1$ against the simple hypothesis $\mathcal{H}_2(\boldsymbol{\theta})$:

$$\mathcal{H}_2(\boldsymbol{\theta}) \colon \big\{\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{\theta}, \boldsymbol{I}_n)\big\}. \tag{5}$$

It is clear that $\mathcal{H}_2 = \cup_{\boldsymbol{\theta} \in \Theta_2} \mathcal{H}_2(\boldsymbol{\theta})$. Let $\gamma_{\delta,\alpha}(\boldsymbol{\theta})$ be the shortcoming of the test δ in $\boldsymbol{\theta} \in \Theta_2$ with respect to the class $\mathcal{K}_\alpha$:

$$\gamma_{\delta,\alpha}(\boldsymbol{\theta}) = \beta^*_\alpha(\boldsymbol{\theta}) - \beta_\delta(\boldsymbol{\theta}) \geq 0. \tag{6}$$

Let $\gamma^{\max}_{\delta,\alpha} = \sup_{\boldsymbol{\theta} \in \Theta_2} \gamma_{\delta,\alpha}(\boldsymbol{\theta})$ be the maximum shortcoming of δ with respect to all possible vectors $\boldsymbol{\theta} \in \Theta_2$.

Definition 1 *A test δ^* is an Epsilon Most Stringent (EMS) test in $\mathcal{K}_\alpha$ between $\mathcal{H}_1$ and $\mathcal{H}_2$ if there exists a (small) positive value ε such that the two following conditions are fulfilled:*

1. *$\delta^* \in \mathcal{K}_\alpha$,*
2. *$\gamma^{\max}_{\delta^*,\alpha} \leq \gamma^{\max}_{\delta,\alpha} + \varepsilon$ for any test $\delta \in \mathcal{K}_\alpha$.*

Obviously, if the constant ε is zero, the EMS test coincides with the Most Stringent (MS) test [16, 20]. Contrary to the MS test, the design of the EMS test tolerates small appropriate approximations of the error probabilities, which can be very useful when the exact calculation of the error probabilities is untractable [13]. Moreover, the MS test may have a very complicated form whereas the EMS test can have a simplified form. Finally, since it is proved that the MS test always exists for testing $\mathcal{H}_1$ against $\mathcal{H}_2$, it guarantees the existence of the EMS test. Choosing the lowest values for ε is of interest but difficult. This paper proposes an EMS test with an acceptable loss of optimality ε as shown in Section 4.

3. EPSILON MOST STRINGENT TEST

This section proposes a new test and studies its statistical performances. It shows that the proposed test is an EMS test by comparing its power function to the envelope power function.

3.1. Envelope Power Function

Let $\boldsymbol{\theta}_2 = x\boldsymbol{h}_2$ for a given value $x \neq 0$. The envelope power function $\beta^*_\alpha(\boldsymbol{\theta}_2)$, given in (4), at point $\boldsymbol{\theta}_2$ can be calculated by looking for the most powerful test for testing

$$\begin{aligned}
&\mathcal{H}_1 \colon \big\{\boldsymbol{y} \sim \mathcal{N}(x_1\, \boldsymbol{h}_1, \boldsymbol{I}_n)\,,\ x_1 \in \mathbb{R}\big\},\\
&\mathcal{H}_2(\boldsymbol{\theta}_2) \colon \big\{\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{\theta}_2, \boldsymbol{I}_n)\big\}.
\end{aligned} \tag{7}$$

Here, the hypothesis $\mathcal{H}_2(\boldsymbol{\theta}_2)$ is simple since $\boldsymbol{\theta}_2$ is fixed. Let $\boldsymbol{\theta}_1 = (\boldsymbol{h}_1^\top \boldsymbol{\theta}_2)\, \boldsymbol{h}_1 = x \cos(\eta) \boldsymbol{h}_1$ be the orthogonal projection of $\boldsymbol{\theta}_2$ onto the vector line $\mathrm{span}\,(\boldsymbol{h}_1)$. From [19, Section 45.4], the test $\delta_{\boldsymbol{\theta}_2}(\boldsymbol{y})$, given by

$$\delta_{\boldsymbol{\theta}_2}(\boldsymbol{y}) = \begin{cases} \mathcal{H}_1 \ \text{ if } \ \Lambda_{\boldsymbol{\theta}_2}(\boldsymbol{y}) = \frac{\boldsymbol{y}^\top (\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)}{\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1\|} \leq \lambda_\alpha, \\ \mathcal{H}_2(\boldsymbol{\theta}_2) \ \text{ else,} \end{cases} \tag{8}$$

where $\lambda_\alpha = \Phi^{-1}(1 - \alpha)$, is the most powerful test for testing $\mathcal{H}_1$ and $\mathcal{H}_2(\boldsymbol{\theta}_2)$ in $\mathcal{K}_\alpha$. Here, the standard Gaussian cumulative distribution function is denoted by $\Phi(\cdot)$ and its inverse is $\Phi^{-1}(\cdot)$. It follows that

$$\begin{aligned}
\beta^*_\alpha(\boldsymbol{\theta}_2) &= \Pr_{\boldsymbol{\theta}_2}(\Lambda_{\boldsymbol{\theta}_2}(\boldsymbol{y}) > \lambda_\alpha) \\
&= \Phi(|x| \sin(\eta) - \lambda_\alpha) = \beta^*_\alpha(x).
\end{aligned} \tag{9}$$

3.2. Proposed Test and its False Alarm Function

This subsection proposes a test for testing $\mathcal{H}_1$ and $\mathcal{H}_2$ and it studies the false alarm function. Let $\hat{\delta} : \mathbb{R}^n \mapsto \{\mathcal{H}_1, \mathcal{H}_2\}$ be the test defined by

$$\hat{\delta}(\boldsymbol{y}) = \begin{cases} \mathcal{H}_1 \text{ if } \hat{\Lambda}(\boldsymbol{y}) = \left|\boldsymbol{h}_2^\top \boldsymbol{y}\right| - \left|\cos(\eta)\boldsymbol{h}_1^\top \boldsymbol{y}\right| \leq \hat{\lambda}, \\ \mathcal{H}_2 \text{ else,} \end{cases} \quad (10)$$

where $\hat{\lambda}$ is a threshold and $\cos(\eta)$ is defined in (2). A short calculation shows that $\hat{\delta}(\boldsymbol{y})$ asymptotically coincides with a MS test when the absolute value of x is fixed and sufficiently large. This fact motivates the choice of this test as an EMS test whatever $x \neq 0$. The following propositions calculates the false alarm function of $\hat{\delta}(\boldsymbol{y})$ with respect to the scale factor x. For this purpose, let us introduce the W-function $W(\varrho, \varphi)$ defined in [21] by

$$W(\varrho, \varphi) = \int_{\xi=0}^{\infty} \int_{t=0}^{\varphi} \exp\left(-\frac{\varrho^2 + \xi^2 + 2\varrho\cos t}{2}\right) \frac{\xi}{2\pi} d\xi dt \quad (11)$$

for $\varrho \geq 0$ and $0 \leq \varphi \leq \pi$. The main properties of $W(\varrho, \varphi)$ are described in [21] and a numerical approximation to compute it is proposed in [22]. Since the distribution of $\hat{\Lambda}(\boldsymbol{y})$ is the same as the one of $\hat{\Lambda}(-\boldsymbol{y})$, it is sufficient to calculate the false alarm function and the power function for $x > 0$. Let $Q(\cdot)$ be the well-known Q-function defined by:

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt. \quad (12)$$

Proposition 1 *Assume $\hat{\lambda} \geq 0$ and $0 < \eta < \frac{\pi}{2}$. Let $\boldsymbol{\theta}_1(x) = x\boldsymbol{h}_1$ for $x > 0$. Let $R = \hat{\lambda}/\sin(\eta)$ and*

$$R_1 = R_1(x) = \sqrt{R^2 + x^2\cos^2(\eta)}.$$

Let ϕ such that $\tan(\eta) = 2\tan(\phi)$ and let

$$\phi_1 = \phi_1(x) = \arccos\left(\frac{x\sin\phi + R\cos\phi}{R_1}\right), \quad (13)$$

$$\phi_1' = \phi_1'(x) = \arccos\left(\frac{-x\sin\phi + R\cos\phi}{R_1}\right). \quad (14)$$

The false alarm function $\alpha_{\hat{\delta}}(x;\hat{\lambda}) \overset{def.}{=} \Pr_{\boldsymbol{\theta}_1(x)}(\hat{\Lambda}(\boldsymbol{y}) > \hat{\lambda})$ of $\hat{\delta}(\boldsymbol{y})$ at point $\boldsymbol{\theta}_1(x)$ is given by

$$\alpha_{\hat{\delta}}(x;\hat{\lambda}) = \begin{cases} Q(R) + W_1 + W_1' & \text{if} \quad 0 < x \leq \bar{x}, \\ Q(R) - W_1 + W_1' & \text{if} \quad \bar{x} < x, \end{cases} \quad (15)$$

with $W_1 = W(R_1, \phi_1)$, $W_1' = W(R_1, \phi_1')$ and $\bar{x} = R\tan(\eta)$.

It can be shown that choosing $\alpha \leq \frac{1}{2}$ involves that $\hat{\lambda} \geq 0$, hence the assumption $\hat{\lambda} \geq 0$ is not restrictive in practice. The closed-form expression of the false alarm depends on the known value $\bar{x}$ since the way the false alarm probability is calculated changes according to the value of x. The false alarm function is continuously differentiable with respect to x (see [19, Chapter 46]). The following corollary gives some upper and lower bounds of the false alarm function.

Corollary 1 *Assume $\hat{\lambda} \geq 0$ and $0 < \eta < \frac{\pi}{2}$. Let $\boldsymbol{\theta}_1(x) = x\boldsymbol{h}_1$ for $x > 0$. The false alarm probability of $\hat{\delta}$ satisfies*

$$Q(R) \leq \Pr_{\boldsymbol{\theta}_1(x)}(\hat{\Lambda}(\boldsymbol{y}) > \hat{\lambda}) \leq Q(R) + 2W(R, \phi) \quad (16)$$

where $R = \hat{\lambda}/\sin(\eta)$ and $\tan(\eta) = 2\tan(\phi)$.

It is interesting to note that the bounds are independents of x and it is easy to show that they are both achieved. Let $\hat{\lambda}_\alpha$ be the conservative threshold satisfying

$$Q(R_\alpha) + 2W(R_\alpha, \phi) = \alpha \quad (17)$$

with $R_\alpha = \hat{\lambda}_\alpha/\sin(\eta)$ and $\tan(\eta) = 2\tan(\phi)$. According to Corollary 1, using the threshold $\hat{\lambda}_\alpha$ guarantees that $\hat{\delta}(\boldsymbol{y}) \in \mathcal{K}_\alpha$, i.e, it satisfies the false alarm level α whatever $\boldsymbol{\theta}_1(x)$.

3.3. Power Function and EMS Optimality

The following proposition calculates the power function $\beta_{\hat{\delta}}(x;\hat{\lambda})$ of $\hat{\delta}$ as a function of x.

Proposition 2 *Assume $\hat{\lambda} \geq 0$ and $0 < \eta < \frac{\pi}{2}$. Let $\boldsymbol{\theta}_2(x) = x\boldsymbol{h}_2$ for $x > 0$, $R = \hat{\lambda}/\sin(\eta)$,*

$$R_2 = R_2(x) = \sqrt{R^2 + x^2 - 2x\sin(\eta)R},$$

$$R_2' = R_2'(x) = \sqrt{R^2 + x^2 + 2x\sin(\eta)R}.$$

Let ϕ such that $\tan(\eta) = 2\tan(\phi)$ and let

$$\phi_2 = \phi_2(x) = \arccos\left(\frac{-x\cos(\eta)}{R_2}\right), \quad (18)$$

$$\phi_3 = \phi_3(x) = \arccos\left(\frac{\cos(\phi)R - x\sin(\eta - \phi)}{R_2}\right), \quad (19)$$

$$\phi_2' = \phi_2'(x) = \arccos\left(\frac{x\cos(\eta)}{R_2'}\right), \quad (20)$$

$$\phi_3' = \phi_3'(x) = \arccos\left(\frac{\cos(\phi)R + x\sin(\eta - \phi)}{R_2'}\right). \quad (21)$$

Let $W_{i,j} = W(R_i, \phi_j)$ and $W_{i,j}' = W(R_i', \phi_j')$ for $2 \leq i, j \leq 3$. The power function $\beta_{\hat{\delta}}(x;\hat{\lambda}) = \Pr_{\boldsymbol{\theta}_2(x)}(\hat{\Lambda}(\boldsymbol{y}) > \hat{\lambda})$ of $\hat{\delta}(\boldsymbol{y})$ at point $\boldsymbol{\theta}_2(x)$ is

$$\beta_{\hat{\delta}}(x;\hat{\lambda}) = \begin{cases} W_{2,2} + W_{2,3} + W_{2,2}' + W_{2,3}' & \text{if} \;\; 0 < x \leq \bar{x}_1, \\ W_{2,2} - W_{2,3} + W_{2,2}' + W_{2,3}' & \text{if} \;\; \bar{x}_1 < x \leq \bar{x}_2, \\ 1 - W_{2,2} - W_{2,3} + W_{2,2}' + W_{2,3}' & \text{if} \;\; \bar{x}_2 < x, \end{cases}$$

where

$$\bar{x}_1 = \frac{R\sin(\phi)}{\cos(\eta - \phi)}, \quad \bar{x}_2 = \frac{R}{\sin(\eta)}. \quad (22)$$

It is straightforward to verify that the power function $\beta_{\hat{\delta}}(x;\hat{\lambda})$ is continuously differentiable with respect to x. Then, the following theorem establishes that $\hat{\delta}(\boldsymbol{y})$ is an EMS test for testing $\mathcal{H}_1$ and $\mathcal{H}_2$ given in (1). The proof is based on the comparison between the power function given in Proposition 2 and the envelope power function (9).

Theorem 1 *Let $0 < \alpha \leq \frac{1}{2}$ and $0 < \eta \leq \frac{\pi}{2}$. Then, the test $\hat{\delta}(\boldsymbol{y})$ is an EMS test in the class $\mathcal{K}_\alpha$ for testing $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Deriving a closed-form upper-bound for the loss of optimality ε is difficult. Hopefully, it is very easy to compute it numerically since the power function function of $\hat{\delta}$ is given in Proposition 2 and the envelope power function is given in (9).

4. NUMERICAL RESULTS

Fig. 1 presents the EMS power function, given in Proposition 2, and the envelope power function, given in (9), with respect to x for four several angles $10°$, $30°$, $50°$ and $89°$ for $\alpha = 10^{-3}$. When the angle is small, the difference between these two power functions, i.e., the shortcoming $\gamma_{\hat{\delta},\alpha}(x)$, is small. The difference increases as the angle increases but, even for a very large angle close to $90°$, this difference remains negligible for almost all $x \neq 0$. The loss of optimality is quite acceptable for almost all $x \neq 0$. Changing the level α does not change the results interpretation.

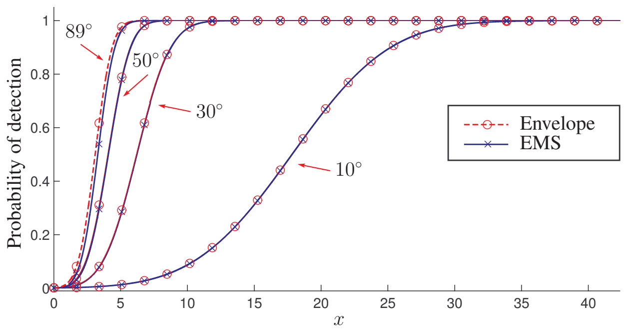

Fig. 1. The power function of the EMS test and the envelope power function with respect to x for the prescribed level $\alpha = 10^{-3}$ and the angles $10°$, $30°$, $50°$ and $89°$.

A calculation shows that the GLRT $\delta_{\text{GLRT}}(\boldsymbol{y})$ is given by:

$$\delta_{\text{GLRT}}(\boldsymbol{y}) = \begin{cases} \mathcal{H}_1 & \text{if } \Lambda_{\text{GLRT}}(\boldsymbol{y}) = \left(\boldsymbol{h}_2^\top \boldsymbol{y}\right)^2 - \left(\boldsymbol{h}_1^\top \boldsymbol{y}\right)^2 \leq \lambda_{\text{GLRT}}, \\ \mathcal{H}_2 & \text{else,} \end{cases}$$

where λ_{GLRT} is the threshold (a general definition of the GLRT is given in [19]). Fig. 2 compares the false alarm function and the shortcoming function of the EMS test and the GLRT for the prescribed level $\alpha = 10^{-2}$ and the angle $\eta = 70°$. The variable x is discretized into a finite set $\mathcal{X}$ of 200 equally spaced values between 0.01 and 12. The false alarm and shortcoming functions are computed by using a Monte-Carlo simulation with 10^7 samples for each point x. To compute the threshold of the GLRT, the decision function $\Lambda_{\text{GLRT}}(\boldsymbol{y})$ is computed under hypothesis $\mathcal{H}_1$ with x fixed. The threshold $\lambda_{\text{GLRT}}(x)$ is computed to warrant that the false alarm of the GLRT is α for the chosen value $x \in \mathcal{X}$. Finally, λ_{GLRT} is chosen as $\lambda_{\text{GLRT}} = \max_{x \in \mathcal{X}} \lambda_{\text{GLRT}}(x)$. From this way, the GLRT

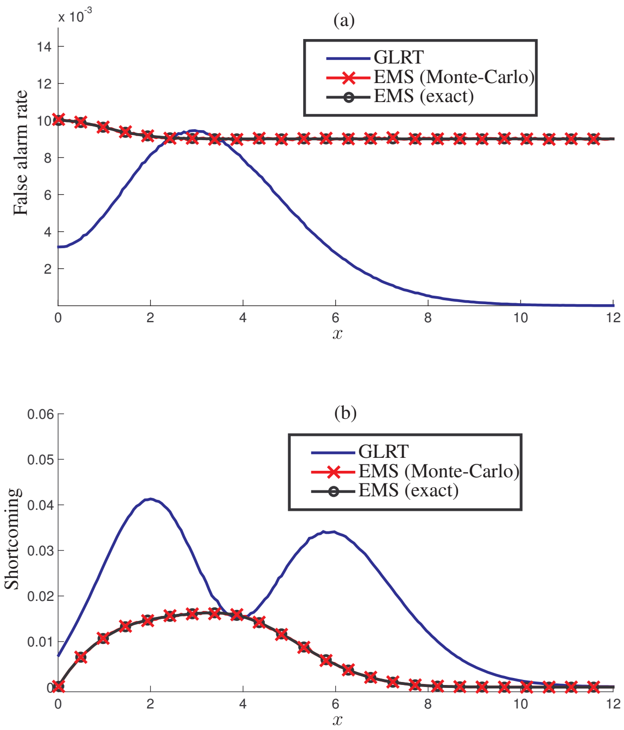

Fig. 2. Comparison between the false alarm function (a) and the shortcoming function (b) of the EMS test and the GLRT for the prescribed level $\alpha = 10^{-2}$ and the angle $\eta = 70°$.

belongs to $\mathcal{K}_\alpha$ for testing $\mathcal{H}_1$ and $\mathcal{H}_2$ whatever the value of $x \in \mathcal{X}$. The threshold of the EMS test is computed as described in (17). The statistical performances of the EMS are computed in two different ways: 1) by using the Monte-Carlo simulation as the GLRT does and 2) by using the theoretical formulas given in Section 3. Fig. 2.(a) clearly shows that the false alarm function of the GLRT significantly varies with respect to x. As established by Corollary 1, the EMS test achieves the prescribed false alarm level for $x = 0$. The false alarm function of the EMS test, which is computed by using theoretical formulas, coincides with the false alarm function obtained by the Monte-Carlo simulation. Fig. 2.(b) shows that the shortcoming function of the GLRT always exceeds the shortcoming function of the EMS test.

5. CONCLUSION

This paper deals with the discrimination of two vector lines under a constrained false alarm probability. The proposed EMS test minimizes, up to a small loss of optimality, the maximum shortcoming between its power function and the envelope power function which defines an ideal maximum power function. This test outperforms the GLRT which could be considered as the standard suboptimal solution.

6. REFERENCES

[1] H. L. V. Trees, *Detection, Estimation, and Modulation Theory: Radar-Sonar Signal Processing and Gaussian Signals in Noise*. Melbourne, FL, USA: Krieger Publishing Co., Inc., 1992.

[2] H. V. Poor, *An introduction to signal detection and estimation (2nd ed.)*. New York, NY, USA: Springer-Verlag, 1994.

[3] A. Frakt, W. Karl, and A. Willsky, "A multiscale hypothesis testing approach to anomaly detection and localization from noisy tomographic data," *IEEE Trans. Image Process.*, vol. 7, no. 6, pp. 825–837, 1998.

[4] N. Merhav and Y. Ephraim, "A bayesian classification approach with application to speech recognition," *IEEE Trans. Signal Process.*, vol. 39, no. 10, pp. 2157–2166, 1991.

[5] A. Abramson and I. Cohen, "Simultaneous detection and estimation approach for speech enhancement," *IEEE Trans. on Audio, Speech, and Language Processing*, vol. 15, no. 8, pp. 2348–2359, 2007.

[6] L. Fillatre and I. Nikiforov, "Non-bayesian detection and detectability of anomalies from a few noisy tomographic projections," *IEEE Trans. Signal Process.*, vol. 55, no. 2, pp. 401–413, 2007.

[7] A. Lakhina, M. Crovella, and C. Diot, "Diagnosing network-wide traffic anomalies," in *ACM SIGCOMM*, 2004.

[8] L. Fillatre and I. Nikiforov, "Asymptotically Uniformly Minimax Detection and Isolation in Network Monitoring," *IEEE Trans. Signal Process.*, vol. 60, no. 7, pp. 3357–3371, 2012.

[9] J. G. Proakis, *Digital Communications*. New York, USA: McGraw-Hill, 1983.

[10] R. O. Duda, P. E. Hart, and D. G. Stork, *Pattern Classification*, 2nd ed. New York, NY: John Wiley & Sons, 2001.

[11] K. Fukunaga, *Introduction to Statistical Pattern Recognition*, 2nd ed. Academic Press, 1990.

[12] B. Baygün and A. O. Hero, "Optimal simultaneous detection and estimation under a false alarm constraint," *IEEE Trans. Inf. Theory*, vol. 41, no. 3, pp. 688–703, 1995.

[13] L. Fillatre, "Constrained epsilon-minimax test for simultaneous detection and classification," *IEEE Trans. Inf. Theory*, vol. 57, no. 12, pp. 8055–8071, dec. 2011.

[14] T. Ferguson, *Mathematical Statistics: A Decision Theoretic Approach*. Academic Press, 1967.

[15] L. L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time-series Analysis*. Addison-Welsey, 1993.

[16] E. L. Lehmann and J. P. Romano, *Testing statistical hypotheses*, 3rd ed., ser. Springer Texts in Statistics. New York: Springer, 2005.

[17] W. Schaafsma, "A comparison of the most stringent and the most stringent somewhere most powerful test for certain problems with restricted alternative," *The Annals of Mathematical Statistics*, vol. 39, no. 2, pp. 531–546, April 1968.

[18] C. Morris, "A class of epsilon bayes tests of a simple null hypothesis on many parameters in exponential families," Stanford University, Department of statistics, Technical report, 1967.

[19] A. A. Borovkov, *Mathematical Statistics*. Gordon and Breach Sciences Publishers, Amsterdam, 1998.

[20] A. Wald, "Tests of statistical hypotheses concerning several parameters when the number of observations is large," *Trans. Amer. Math. Soc.*, vol. 54, pp. 426–482, 1943.

[21] H. Ruben, "Probability Content of Regions Under Spherical Normal Distributions, III: The Bivariate Normal Integral," *The Annals of Mathematical Statistics*, vol. 32, no. 1, pp. 171–186, 1961.

[22] D. R. Divgi, "Calculation of univariate and bivariate normal probability functions," *The Annals of Statistics*, vol. 7, no. 4, pp. 903–910, 1979.