EFFICIENT KERNEL-BASED FORMULATIONS OF SPATIO-SPECTRAL AND RELATED TRANSFORMATIONS ON THE 2-SPHERE

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ABSTRACT

In this paper we show that the spatially localized spherical harmonic transform (SLSHT), which represents a signal on the 2-sphere in the spatio-spectral domain, can be efficiently computed using new kernel-based formulations. In addition to the standard spatio-spectral domain, we show there are three other related transforms that provide alternative representations in the spatio-spatial, spectro-spatial and spectro-spectral domains. We provide inversion results that extend available results for the SLSHT. We show that for signals on the 2-sphere band-limited to degree L, the computational complexity using our class of kernel-based SLSHT transforms is $O(L^4)$ and outperforms the previous best known fast methods, which have complexity $O(L^5)$.

Index Terms— 2-sphere; unit sphere; spherical harmonic transform; spatio-spectral domain; spatially localized spherical harmonic transform; fast transforms.

1. INTRODUCTION

For some decades the short-time Fourier transform (STFT) and its variants have been used to generalize classical filtering to the joint time-frequency domain and this provides the ability to better handle signals arising from time-varying and non-stationary systems [1]. Developing methods analogous to the STFT to deal with processing of signals defined on the 2-sphere (domain) is similarly of interest because signal features may be localized spatially (on the 2-sphere) and spectrally (where the spectral representation is obtained from the spherical harmonic transform [2]). These signals on the 2-sphere arise in a number of applications including geodesy [3], cosmology [4], spherical harmonic computerized lighting [5], medical image analysis [2], and wireless channel modeling [6].

1.1. Relation to Prior Work

On the 2-sphere the analogue of the STFT is called the spatially localized spherical harmonic transform (SLSHT) [7]. It has enabled the use of spatially varying spectral filtering on the 2-sphere for localized analysis using the joint spatio-spectral domain [8]. Along with a directional-SLSHT generalization, fast spectral computational methods have been developed [9]. In [7] the original signal (in spectral form) was shown to be recoverable by spatial marginalization of the SLSHT. In this paper we show how to recover spatial representation of the original signal by spectral marginalization. The invertibility window conditions differ to those required for spatial marginalizing the SLSHT [7] and lead to new insights into the design of window functions used for the localized spatial analysis. This answers the question in the positive of whether there are two SLSHT marginalization strategies to recover the signal as there are for the STFT case.

This paper deviates most markedly from the standard SLSHT formulations in the use of an isotropic kernel function that fully captures the symmetry conditions of the SLSHT definition [7]. This new formulation significantly simplifies the derivation of key known identities and reveals new SLSHT-inversion results. This isotropic kernel also plays a direct role in defining three new transforms, closely related to the SLSHT, that provide representations in the spatio-spatial, spectro-spatial and spectro-spectral domains. The corresponding inversion results, which show how to recover the original (or processed) signal from these joint domains, are also given. Some of these results are known or have been implicitly used in other works, but this paper introduces new identities and a more complete theory.

Wavelets provide an alternative approach to localized signal processing on the 2-sphere and enable filtering at different scales [10–14]. These lead to a space-scale representation and are an alternative to the SLSHT approach. However, unlike the Euclidean case, dilations on the sphere or the spherical harmonic spectrum are not straightforward or completely natural. In many cases, one is still interested in a classical spherical harmonic spectral interpretation of signals and their processing and this is furnished through the SLSHT methods (and related transforms defined herein). Further, the existence of fast processing methods for the standard SLSHT and its directional extension [9] and presentation of even faster processing methods based on a new spatio-spatial domain in this paper, strengthens its case for processing in applications.

1.2. Paper Structure

In Section 2 we show how to reformulate the standard SLSHT using an isotropic kernel. In Section 3 we develop new marginalization methods for SLSHT inversion. In Section 4 we introduce new spatio-spatial, spectro-spectral and spectro-spatial variants of the SLSHT. Section 5 shows these new domains and isotropic kernel techniques yield state of the art computational performance.

2. FORMULATION WITH AN ISOTROPIC KERNEL

Relevant background to signals on the 2-sphere $\mathbb{S}^2$, inner products and spherical harmonics are given in the Appendix. For a complex-valued spatial signal on the 2-sphere, $f(\widehat{\boldsymbol{x}})$, the spherical harmonic transform (SHT) is given by

$$(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{y}}) \overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}). \tag{1}$$

This work was supported under the Australian Research Council's Linkage Projects funding scheme (project no. LP100100588).

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The discrete components, $(f)_\ell^m$ for all possible values of the degree $\ell \in \{0, 1, 2, \ldots\}$ and order $m \in \{-\ell, \ldots, \ell\}$ give the spectral representation of the signal. The inverse spherical harmonic transform (ISHT), is given in (18) in the Appendix. This spectrum is the generalization of classical Fourier series for 1D periodic signals.

2.1. Standard Formulation of SLSHT

The central equation in the SLSHT [7–9, 15] is the following:

Definition 1 (Spatio-Spectral SLSHT [7]) *For a spatial signal on the 2-sphere, $f(\widehat{\boldsymbol{x}})$, the spatio-spectral SLSHT is given by*

$$g(\widehat{\boldsymbol{x}}; \ell, m) \triangleq \int_{\mathbb{S}^2} \big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}}) f(\widehat{\boldsymbol{y}}) \overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}), \tag{2a}$$

where $h(\widehat{\boldsymbol{y}})$ is an azimuthally symmetric window function satisfying

$$\langle h, Y_\ell^m \rangle = 0, \quad \forall m \neq 0. \tag{2b}$$

In this definition $\big(\mathcal{D}(\widehat{\boldsymbol{x}})h\big)(\widehat{\boldsymbol{y}})$ is the window $h(\widehat{\boldsymbol{y}})$ rotated such that it is rotationally symmetric about the point $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$.

2.2. Interpretation of SLSHT

The SLSHT $g(\widehat{\boldsymbol{x}}; \ell, m)$ has straightforward interpretations. The window $h(\cdot)$ is typically chosen to concentrate analysis into a local region on the sphere centered at an arbitrary point $\widehat{\boldsymbol{x}}$. For the SLSHT $g(\widehat{\boldsymbol{x}}; \ell, m)$: (i) by fixing the spatial argument $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$ and varying the spectral degree ℓ and order m we get information of which spherical harmonics contribute most to explain that localized portion of the spatial signal $f(\cdot)$ within the windowed region centered at point $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$; and (ii) by fixing both the spectral degree ℓ and order m and varying $\widehat{\boldsymbol{x}} \in \mathbb{S}^2$ we can infer from which parts of the sphere the signal most strongly contribute to the (global) ℓ, m-spherical harmonic coefficient.

In both interpretations we have the quantities in the form of a distribution, not necessarily normalized, across the spatio-spectral representation. These interpretations are extended further in Section 4 so that we can develop new transforms that go beyond the SLSHT.

2.3. SLSHT Reformulation with Isotropic Convolution Kernel

We can expand the azimuthally symmetric window function $h(\widehat{\boldsymbol{x}})$ in the order $m = 0$ spherical harmonics, $Y_\ell^0(\widehat{\boldsymbol{x}})$, with coefficients $(h)_\ell^0 \triangleq \langle h, Y_\ell^0 \rangle$,

$$h(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty} (h)_\ell^0 Y_\ell^0(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty} (h)_\ell^0 \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\eta}}),$$

where $\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{\eta}} = \cos\theta$ ($\widehat{\boldsymbol{\eta}}$ is the north pole). Then it can be shown that

$$H(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) \triangleq \big(\mathcal{D}(\widehat{\boldsymbol{y}})h\big)(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty} (h)_\ell^0 \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) \tag{3}$$

where the newly defined function $H(z)$ is defined on the interval $z \in [-1, +1]$ and synthesizes the window in terms of a Legendre polynomial expansion. To recover $(h)_\ell^0$ from the window $h(\widehat{\boldsymbol{x}})$ we then need to perform the spherical harmonic transform

$$\begin{aligned}(h)_\ell^0 = \langle h, Y_\ell^0 \rangle &= \int_{\mathbb{S}^2} h(\widehat{\boldsymbol{x}}) \overline{Y_\ell^0(\widehat{\boldsymbol{x}})}\, ds(\widehat{\boldsymbol{x}}) \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} H(\cos\theta) \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)\, \sin\theta\, d\theta \\ &= \sqrt{\pi(2\ell+1)} \int_{-1}^{+1} H(z) P_\ell(z)\, dz, \end{aligned} \tag{4}$$

which shows how to recover $(h)_\ell^0$ using the Legendre transform on $H(z) \equiv H(\cos\theta)$.

So, using (3), the SLSHT, (2a) with (2b), can be reformulated leading to our first result:

Property 1 (Kernel-Based Spatio-Spectral SLSHT) *The spatio-spectral SLSHT [7] given by* (2a) *can be written*

$$g(\widehat{\boldsymbol{x}}; \ell, m) = \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) f(\widehat{\boldsymbol{y}}) \overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}), \tag{5}$$

where

$$H(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) = \sum_{\ell=0}^{\infty} (h)_\ell^0 \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) \tag{6}$$

and coefficients $\{(h)_\ell^0 \colon \ell = 0, 1, 2, \ldots\}$ *given by* (4).

This reveals the SLSHT as being a hybrid of isotropic convolution [16] and the SHT (1). Because of the symmetry of the window, convolution present in the reformulation (5) and direct spatial weighting present in the original (2a) are equivalent. The function $H(\cdot)$ is referred to as the *isotropic convolution kernel* function derived from the window [16].

3. SPATIO-SPECTRAL SLSHT-INVERSION

A natural objective is to recover the original signal, either in spatial form $f(\widehat{\boldsymbol{x}})$ or spectral form $(f)_\ell^m$, from the (spatio-spectral) SLSHT $g(\widehat{\boldsymbol{x}}; \ell, m)$. This is called (spatio-spectral) SLSHT-inversion. Perfect recovery is tantamount to the statement that forming the SLSHT is information preserving, and therefore, a feasible setting to perform spatio-spectral signal processing. In this section we advance the theory of (spatio-spectral) SLSHT-inversion and improve on the findings in [7].

3.1. Spatial Marginal SLSHT-Inversion

Using the new SLSHT expression (5) the SLSHT-inversion given in [7] can be derived in a simpler way. Consider the spatial marginalization of (5)

$$\begin{aligned}&\int_{\mathbb{S}^2} g(\widehat{\boldsymbol{x}}; \ell, m)\, ds(\widehat{\boldsymbol{x}}) \\ &\qquad = \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}})\, ds(\widehat{\boldsymbol{x}})\, f(\widehat{\boldsymbol{y}}) \overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\, ds(\widehat{\boldsymbol{y}}),\end{aligned}$$

where using (3) we have

$$\begin{aligned}\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}})\, ds(\widehat{\boldsymbol{x}}) &= \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} (h)_\ell^0 \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}})\, ds(\widehat{\boldsymbol{x}}) \\ &= \sqrt{4\pi} (h)_0^0,\end{aligned}$$

since $\int_{\mathbb{S}^2} P_\ell(\widehat{\boldsymbol{x}} \cdot \widehat{\boldsymbol{y}}) ds(\widehat{\boldsymbol{x}}) = 4\pi\delta_{\ell,0}$, where $\delta_{\ell,p}$ denotes the Kronecker delta function. Therefore, we have shown:

Property 2 (SLSHT-Inversion [7]) *The spectral signal $(f)_\ell^m$ can be recovered from the SLSHT $g(\widehat{\boldsymbol{x}}; \ell, m)$ by spatial marginalization:*

$$(f)_\ell^m = \frac{1}{\sqrt{4\pi}(h)_0^0} \int_{\mathbb{S}^2} g(\widehat{\boldsymbol{x}}; \ell, m)\, ds(\widehat{\boldsymbol{x}}), \quad \textit{where } (h)_0^0 \neq 0 \tag{7}$$

and $(h)_0^0$ is given in (4) *with $\ell = 0$.*

This shows spectral recovery (SLSHT-inversion) is possible whenever $(h)_0^0 \neq 0$, that is, when the window $h(\widehat{\boldsymbol{x}})$ has a non-zero DC value, (4). In summary, this inversion result (7) is not new, but the derivation is greatly simplified from that given in [7]. The next inversion result, in contrast, is new.

3.2. Spectral Marginal SLSHT-Inversion

This time we marginalize the spectral argument of the SLSHT with a spherical harmonic weighting to reveal that we can recover the signal in spatial form, $f(\widehat{\boldsymbol{x}})$:

$$\begin{aligned}\sum_{\ell,m} g(\widehat{\boldsymbol{x}};\ell,m)Y_\ell^m(\widehat{\boldsymbol{x}}) &= \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}})\sum_{\ell,m} Y_\ell^m(\widehat{\boldsymbol{x}})\overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,ds(\widehat{\boldsymbol{y}}) \\ &= \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}})\,\delta(\widehat{\boldsymbol{x}}-\widehat{\boldsymbol{y}})\,d\theta(\widehat{\boldsymbol{y}})\,d\varphi(\widehat{\boldsymbol{y}}),\end{aligned}$$

where $\delta(\widehat{\boldsymbol{x}}-\widehat{\boldsymbol{y}})$ denotes the 2-sphere Dirac delta function [15, p.95].

We summarize this as our second novel result:

Property 3 (SLSHT-Inversion) *The spatial signal $f(\widehat{\boldsymbol{x}})$ can be recovered from the SLSHT $g(\widehat{\boldsymbol{x}};\ell,m)$ by weighted spectral marginalization in the form*

$$f(\widehat{\boldsymbol{x}}) = \frac{1}{H(1)}\sum_{\ell,m} g(\widehat{\boldsymbol{x}};\ell,m)Y_\ell^m(\widehat{\boldsymbol{x}}), \quad \textit{where } H(1)\neq 0$$

and $H(\cdot)$ is the isotropic convolution kernel given in (3).

Since $H(\cos 0) = H(1)$ then this corresponds to the value of the azimuthally symmetric window at its center (or north pole $\widehat{\boldsymbol{\eta}}$).

4. FOUR VARIANTS OF SLSHT REPRESENTATION

The spatio-spectral domain of the SLSHT is natural with the interpretation we gave earlier in Section 2.2. However, here we show new representations, spatio-spatial, spectro-spectral and spectro-spatial, which are equivalent but offer advantages in different signal processing contexts depending on the structure of signals, representations and the computational requirements for signal processing.

4.1. Spatio-Spatial Domain

The spectral argument (being both the degree and order) of the (spatio-spectral) SLSHT can be mapped to its own distinct spatial argument in an elementary way to obtain a spatio-spatial representation defined on $\mathbb{S}^2\times\mathbb{S}^2$ as follows:

Definition 2 (Spatio-Spatial SLSHT) *The Spatio-Spatial SLSHT is defined by*

$$g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}}) \triangleq \sum_{\ell,m} g(\widehat{\boldsymbol{x}};\ell,m)Y_\ell^m(\widehat{\boldsymbol{z}}), \quad \widehat{\boldsymbol{x}},\widehat{\boldsymbol{z}}\in\mathbb{S}^2, \tag{8}$$

which is the ISHT in the second argument of the SLSHT.

To recover the standard (spatio-spectral) SLSHT from $g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})$ one takes the SHT in the second argument $\widehat{\boldsymbol{z}}$:

$$\int_{\mathbb{S}^2} g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})\overline{Y_\ell^m(\widehat{\boldsymbol{z}})}\,ds(\widehat{\boldsymbol{z}}) = g(\widehat{\boldsymbol{x}};\ell,m). \tag{9}$$

These transforms, (8)–(9), are depicted, along with later definitions and results, in the upper portion of Fig. 1.

Now using analysis that closely mimics what was done in Section 3.2, we have

$$\sum_{\ell,m} g(\widehat{\boldsymbol{x}};\ell,m)Y_\ell^m(\widehat{\boldsymbol{z}}) = \int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}})\delta(\widehat{\boldsymbol{z}}-\widehat{\boldsymbol{y}})\,d\theta(\widehat{\boldsymbol{y}})\,d\varphi(\widehat{\boldsymbol{y}}).$$

We incorporate this into the following statement:

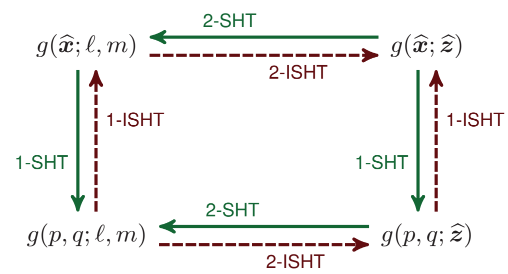

Fig. 1: Transformations between the four variants of the SLSHT. In notation, 1-SHT/2-SHT denotes the spherical harmonic transform (SHT) on the 1st/2nd argument. Similarly 1-ISHT/2-ISHT denotes the inverse spherical harmonic transform (ISHT) on the 1st/2nd argument, as given in the Appendix (18).

Property 4 (Kernel-Based Spatio-Spatial SLSHT) *The spatio-spatial SLSHT transform,* (8)*, of a signal $f(\cdot)$ can be written*

$$g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}}) \triangleq H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})f(\widehat{\boldsymbol{z}}), \quad \widehat{\boldsymbol{x}},\widehat{\boldsymbol{z}}\in\mathbb{S}^2, \tag{10}$$

using the isotropic convolution kernel $H(\cdot)$, given in (3).

Using (10) we can see that (9) is equivalent to the SLSHT reformulation (5). The spatio-spatial SLSHT is clearly equivalent to the (spatio-spectral) SLSHT but provides an interpretation and processing in the $\mathbb{S}^2\times\mathbb{S}^2$ domain. In fact, as defined, it is independent of any spherical harmonics because it has no spectral components.

As a corollary of Result 4, letting $\widehat{\boldsymbol{x}}=\widehat{\boldsymbol{z}}$, we have:

Property 5 (Spatio-Spatial SLSHT-Inversion) *The spatial signal $f(\widehat{\boldsymbol{x}})$ can be recovered from the spatio-spatial SLSHT, $g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})$, as follows:*

$$f(\widehat{\boldsymbol{x}}) = \frac{1}{H(1)}g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{x}}), \quad \textit{where } H(1)\neq 0$$

and $H(\cdot)$ is the isotropic convolution kernel given in (3).

This is presented as a simple method for inverting the spatio-spatial SLSHT (10), but is not the only inversion expression possible.

Property 6 (Spatio-Spatial SLSHT Energy) *The total energy in the spatio-spatial SLSHT $g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})$ is the product of energy of the signal $f(\widehat{\boldsymbol{x}})$ and energy of the symmetric window function $h(\widehat{\boldsymbol{x}})$:*

$$\|g\|^2 \triangleq \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})\overline{g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})}\,ds(\widehat{\boldsymbol{x}})\,ds(\widehat{\boldsymbol{z}}) = \|h\|^2\|f\|^2. \tag{11}$$

To show this we use (10) in conjunction with (3) and the orthogonality of the Legendre polynomials. From the definition of $\|g\|^2$ in (11) we have:

$$\begin{aligned}\|g\|^2 &= \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}\frac{\sqrt{(2\ell+1)(2\ell'+1)}}{4\pi}(h)_\ell^0\overline{(h)_{\ell'}^0} \\ &\quad\times\int_{\mathbb{S}^2} f(\widehat{\boldsymbol{z}})\overline{f(\widehat{\boldsymbol{z}})}\Big(\int_{\mathbb{S}^2} P_\ell(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})P_{\ell'}(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})\,ds(\widehat{\boldsymbol{x}})\Big)ds(\widehat{\boldsymbol{z}}) \\ &= \sum_{\ell=0}\big|(h)_\ell^0\big|^2\int_{\mathbb{S}^2} f(\widehat{\boldsymbol{z}})\overline{f(\widehat{\boldsymbol{z}})}\,ds(\widehat{\boldsymbol{z}}) = \|h\|^2\|f\|^2.\end{aligned}$$

4.2. Spectro-Spectral Domain

The spatial argument of the (spatio-spectral) SLSHT can be mapped to its own distinct spectral argument of degree p and order q to yield a spectro-spectral representation defined on $\ell^2\times\ell^2$:

Definition 3 (Spectro-Spectral SLSHT) *The spectro-spectral SLSHT is defined by*

$$g(p,q;\ell,m) \triangleq \int_{\mathbb{S}^2} g(\widehat{\boldsymbol{x}};\ell,m)\overline{Y_p^q(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}). \quad (12)$$

which is the SHT in the first argument of the SLSHT.

This (and inverse) are depicted in the left portion of Fig. 1. Then we have, using (5):

Property 7 (Kernel-Based Spectro-Spectral SLSHT) *The spectro-spectral SLSHT transform,* (12)*, of a signal $f(\cdot)$ using the isotropic convolution kernel $H(\cdot)$, given in* (3)*, can be written*

$$g(p,q;\ell,m) \triangleq \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{y}})f(\widehat{\boldsymbol{y}}) \times \overline{Y_\ell^m(\widehat{\boldsymbol{y}})}\,\overline{Y_p^q(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{y}})\,ds(\widehat{\boldsymbol{x}}), \quad (13)$$

using the isotropic convolution kernel $H(\cdot)$, given in (3).

In this case spectral recovery (SLSHT-inversion) is particularly simple from the spectro-spectral SLSHT (12) or (13)

$$(f)_\ell^m = \frac{g(0,0;\ell,m)}{(h)_0^0}, \quad \text{where } (h)_0^0 \neq 0 \quad (14)$$

and $(h)_0^0$ is given in (4) with $\ell = 0$. This is established by comparing (7) with (12) for $p = q = 0$. The spectro-spectral SLSHT transform being defined on $\ell^2 \times \ell^2$ has made it a natural computational domain for fast spectral-based algorithms [9].

4.3. Spectro-Spatial Domain

Definition 4 (Spectro-Spatial SLSHT) *The spectro-spatial SLSHT is defined by*

$$g(p,q;\widehat{\boldsymbol{z}}) \triangleq \int_{\mathbb{S}^2} g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})\overline{Y_p^q(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}), \quad \widehat{\boldsymbol{z}}\in\mathbb{S}^2, \quad (15)$$

which is the SHT in the first argument of the spatio-spatial SLSHT.

This is represented on the right portion of Fig. 1. This figure also can be used to determine the two transforms to obtain the spectro-spatial SLSHT $g(p,q;\widehat{\boldsymbol{z}})$ from the spatio-spectral SLSHT $g(\widehat{\boldsymbol{x}};\ell,m)$.

Combining (15) with (10) we obtain:

Property 8 (Kernel-Based Spectro-Spatial SLSHT) *The spectro-spatial SLSHT transform,* (15)*, of a signal $f(\cdot)$ can be written*

$$g(p,q;\widehat{\boldsymbol{z}}) \triangleq f(\widehat{\boldsymbol{z}})\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})\overline{Y_p^q(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}), \quad \widehat{\boldsymbol{z}}\in\mathbb{S}^2, \quad (16)$$

using the isotropic convolution kernel $H(\cdot)$, given in (3).

To obtain spatial recovery (SLSHT-Inversion) we have

$$g(0,0;\widehat{\boldsymbol{z}}) = \frac{f(\widehat{\boldsymbol{z}})}{\sqrt{4\pi}}\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})\,ds(\widehat{\boldsymbol{x}})$$

but by straightforward considerations

$$\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})\,ds(\widehat{\boldsymbol{x}}) = \sqrt{4\pi}(h)_0^0.$$

Therefore, we have shown:

Property 9 (Spectro-Spatial SLSHT-Inversion) *The spatial signal $f(\widehat{\boldsymbol{z}})$ can be recovered from the spectro-spatial SLSHT, $g(p,q;\widehat{\boldsymbol{z}})$, as follows:*

$$f(\widehat{\boldsymbol{z}}) = \frac{g(0,0;\widehat{\boldsymbol{z}})}{(h)_0^0}, \quad \textit{where } (h)_0^0 \neq 0$$

and $(h)_0^0$ is given in (4) *with $\ell = 0$.*

Alternatively, we can do spectral marginalization,

$$\sum_{p,q} g(p,q;\widehat{\boldsymbol{z}})Y_p^q(\widehat{\boldsymbol{z}}) = f(\widehat{\boldsymbol{z}})\int_{\mathbb{S}^2} H(\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{z}})\sum_{p,q} Y_p^q(\widehat{\boldsymbol{z}})\overline{Y_p^q(\widehat{\boldsymbol{y}})}\,ds(\widehat{\boldsymbol{y}}) = H(1)f(\widehat{\boldsymbol{z}}),$$

which recovers $f(\widehat{\boldsymbol{z}})$ up to constant $H(1) \neq 0$.

5. APPLICATION OF SPATIO-SPATIAL SLSHT

We show that the SLSHT (and any of the four variants) can be efficiently computed using the kernel-based formulation of SLSHT presented here (more specifically exploiting our newly introduced spatio-spatial SLSHT) as compared to the standard formulation [9].

For our comparison we assume that both the signal f and window h are band-limited to degree L in the sense that $(f)_\ell^m = (h)_\ell^m = 0$ for all $\ell \geq L$. Thereby the SHT can be computed exactly using $2L^2$ number of samples on the sphere with computational complexity of the order $O(L^3)$ [17]. Using the standard formulation of SLSHT, $g(\widehat{\boldsymbol{x}};\ell,m)$ can be computed for ℓ, m and all number of samples on the sphere with computational complexity of $O(L^5)$ [9]. For the kernel-based *spatio-spectral* formulation in (5), the computation of $g(\widehat{\boldsymbol{x}};\ell,m)$ is again of the order $O(L^5)$, since we need to take the SHT for each sample $\widehat{\boldsymbol{x}}\in\mathbb{S}^2$ on the sphere and the number of samples are of the order $O(L^2)$. However, the kernel-based *spatio-spatial* SLSHT representation $g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})$ in the spatio-spatial domain given in (10) can be computed in $O(L^4)$ as it *only* requires the product of the kernel $H(\widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{z}})$ and $f(\widehat{\boldsymbol{z}})$ for each $\widehat{\boldsymbol{x}},\widehat{\boldsymbol{z}}\in\mathbb{S}^2$. Once the spatio-spatial representation $g(\widehat{\boldsymbol{x}};\widehat{\boldsymbol{z}})$ is computed, each representation in other three domains can be computed in $O(L^3)$ by taking the SHT in arguments $\widehat{\boldsymbol{x}}$ or $\widehat{\boldsymbol{z}}$ or both.

APPENDIX: SPHERICAL HARMONICS

Define the 2-sphere by $\mathbb{S}^2 \triangleq \{\boldsymbol{x}\in\mathbb{R}^3 \colon |\boldsymbol{x}| = 1\}$ and the inner product of two functions whose domain is the 2-sphere

$$\langle f,g\rangle \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\overline{g(\widehat{\boldsymbol{x}})}\,ds(\widehat{\boldsymbol{x}}), \quad (17)$$

where $\widehat{\boldsymbol{x}} \triangleq (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)' \in \mathbb{S}^2 \subset \mathbb{R}^3$ and $ds(\widehat{\boldsymbol{x}}) = \sin\theta\,d\theta\,d\varphi$ is the uniform surface measure satisfying $\int_{\mathbb{S}^2} ds(\widehat{\boldsymbol{x}}) = 4\pi$. Finite energy functions are those that satisfy

$$f\in L^2(\mathbb{S}^2) \iff \|f\| \triangleq \langle f,f\rangle^{1/2} < \infty.$$

The spherical harmonics are defined through

$$Y_\ell^m(\theta,\varphi) \triangleq \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\varphi} \equiv Y_\ell^m(\widehat{\boldsymbol{x}}),$$

where $\ell\in\{0,1,\ldots\}$ is the degree, $m\in\{-\ell,-\ell+1,\ldots,\ell\}$ is the order, the associated Legendre functions are [15]

$$P_\ell^m(z) \triangleq \frac{(-1)^m}{2^\ell\ell!}(1-z^2)^{m/2}\frac{d^{\ell+m}}{dz^{\ell+m}}(z^2-1)^\ell,$$

for $m\in\{0,1,\ldots,\ell\}$, and satisfy

$$P_\ell^{-m}(z) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(z), \quad m\in\{0,1,\ldots,\ell\}.$$

The inverse spherical harmonic transform (ISHT) is given by

$$f(\widehat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(f)_\ell^m\,Y_\ell^m(\widehat{\boldsymbol{x}}), \quad (18)$$

where the coefficients are $(f)_\ell^m \triangleq \langle f, Y_\ell^m\rangle$.

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