ESTIMATING THE NUMBER OF SIGNALS IN THE PRESENCE OF NONUNIFORM NOISE

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ABSTRACT

An important problem in sensor array processing is the estimation of the number of transmitted signals. Most of the proposed solutions rely on the assumption of uniform additive white noise on the measured signals. In this paper, an approach for estimating the number of sources in the presence of nonuniform white noise is proposed. The method is based on the computation of the maximal corank of the covariance matrix of the noisy data in the Frisch scheme context. The effectiveness of the method is tested by means of Monte Carlo simulations.

Index Terms— Source number estimation, sensor array processing, nonuniform additive noise, Frisch scheme.

1. INTRODUCTION

A problem of great relevance in radar, sonar, navigation, geophysics and acoustics applications is the estimation of the Direction–of–Arrival (DOA) of multiple signals on the basis of the measures performed by means of arrays of narrow–band sensors. Several solutions of this problem have been proposed and compared during the last decades, among them MUSIC and the maximum likelihood (ML) method [1–3].

An important issue in applying these methods concerns the estimation of the number of signals, because the covariance matrix of the observations is partitioned into two parts associated with the signal and noise spaces. A common approach, based on information theoretic criteria (ITC), has been described in [4] and, as many other solutions [5–8], relies on the assumption of uniform white noise, i.e. the sensor errors are modeled as spatially uncorrelated white Gaussian noises with equal and unknown variance.

In many applications this assumption can be unrealistic and the sensor noise should be considered as a colored process, as discussed in [9, 10]. Nevertheless, in some important real applications, for example when reverberating or seismic problems are modeled on the basis of measures obtained from sparse arrays, the general colored noise assumption can be relaxed by assuming the sensor noises as spatially white with unequal noise variances [9–13]. Very few papers deal with the problem of source detection in the presence of nonuniform noise [14, 15].

In this paper, the problem of estimating the number of signals in the presence of spatially nonuniform independent sensor noise is solved by applying an approach based on the Frisch scheme [16]. In particular, the problem of estimating the number of sources is mapped into the problem of evaluating the maximal corank of the covariance matrix of the noisy data in the Frisch scheme context.

The organization of the paper is the following. Section 2 describes the DOA framework and defines the considered problem. Section 3 recalls the properties of the Frisch scheme. Section 4 first recalls some important results concerning the evaluation, on the basis of a geometric approach, of the maximal corank of a covariance matrix in the context of the Frisch scheme and then, shows how these results can be used for solving the problem of estimating the number of signals in the DOA environment. In Section 5 the performance of the proposed estimation method is tested and compared with that of the procedures introduced in [4, 14] by means of some Monte Carlo simulations. Some concluding remarks are finally reported in Section 6.

2. PROBLEM STATEMENT

Consider an array of n sensors receiving p narrow–band signals from sources with directions of arrival θ_i $(i = 1, \dots, p)$. The sensor array outputs are collected in a n–dimensional vector $y(t)$ and modeled by the following equation

$$y(t) = A(\theta)\, x(t) + e(t), \qquad t = 1, \dots, N \tag{1}$$

where

$$\theta = [\,\theta_1, \theta_2, \dots, \theta_p\,]^T, \tag{2}$$

N is the number of observations, $A(\theta)$ is the $(n \times p)$ array transfer matrix, $x(t)$ is the p–dimensional vector of source signals and $e(t)$ is the n–dimensional vector of the noises affecting the measures. The additive noise $e(t)$ is assumed as a zero–mean ergodic spatially and temporally white complex process with unknown diagonal covariance matrix

$$\tilde{\Sigma} = E[\,e(t)e^H(t)\,] = \text{diag}\,[\,\tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \dots, \tilde{\sigma}_n^2\,]\,, \tag{3}$$

where $(\cdot)^H$ denotes Hermitian transpose and $E[\,\cdot\,]$ is the expectation operator. The source signal $x(t)$ is a zero–mean, second–order ergodic complex random vector with non–singular $(p \times p)$ covariance matrix

$$\Sigma_x = E[\,x(t)x^H(t)\,]\,. \tag{4}$$

The signal $x(t)$ is also assumed to be uncorrelated with the noise $e(t)$, so that the $(n \times n)$ array covariance matrix is given by

$$\Sigma = E[\,y(t)\,y^H(t)\,] = \Sigma_0 + \tilde{\Sigma}, \tag{5}$$

where

$$\Sigma_0 = A(\theta)\,\Sigma_x A(\theta)^H. \tag{6}$$

The matrix $A(\theta)$ is assumed with full column rank so that the rank of Σ_0 is p, i.e. Σ_0 has its $n - p$ smallest eigenvalues equal to zero.

The problem under investigation consists in estimating the number of signal sources p starting from a set of N observations $y(1), y(2), \dots, y(N)$.

In the next sections, it will be shown that this problem can be mapped into the problem of determining the maximal corank of the covariance matrix Σ in the Frisch scheme context.

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3. FRISCH SCHEME PROPERTIES

In the following, the $(n \times n)$ symmetric, positive definite covariance matrix of the noisy data Σ will be denoted as Σ_n for notation convenience. Given the matrix Σ_n, consider now the problem of finding all diagonal matrices $\tilde{\Sigma}(P) = \text{diag}\,[\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2]$ with nonnegative elements σ_i^2 $(i = 1, \ldots, n)$ such that the matrix $\Sigma_n - \tilde{\Sigma}(P)$ is singular and nonnegative definite, i.e.

$$\Sigma_0(P) = \Sigma_n - \tilde{\Sigma}(P) \geq 0 \quad \text{and} \quad \det \Sigma_0(P) = 0\,, \qquad (7)$$

where $P = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ is a point belonging to the positive orthant of $\mathcal{R}^n$ [16,17]. Every positive definite or semidefinite diagonal matrix $\tilde{\Sigma}$ satisfying (7) is a *solution* of the Frisch Scheme. The corresponding point P can be considered as an admissible solution in the *noise space*. The locus of all admissible solutions is described by the following theorem [17].

Theorem 1. All admissible solutions in the noise space lie on a convex (hyper)surface $\mathcal{S}(\Sigma)$ whose concavity faces the origin and whose intersections with the coordinate axes are the points $(0, \ldots, \sigma_i^2, \ldots, 0)$ corresponding to the n least squares solutions.

Definition 1. [18] The (hyper)surface $\mathcal{S}(\Sigma_n)$ will be called *singularity (hyper)surface* of Σ_n because every point P of $\mathcal{S}(\Sigma_n)$ defines a noise covariance matrix $\tilde{\Sigma}(P)$ that leads to a singular matrix $\Sigma_0(P)$.

As an example, Fig. 1 shows the singularity surface $\mathcal{S}(\Sigma_3)$ for a (3×3) noisy covariance matrix.

The corank of the singular matrix $\Sigma_0(P) = \Sigma_n - \tilde{\Sigma}(P)$ is defined as the dimension m of the null space of $\Sigma_0(P)$ so that it coincides with the number of null eigenvalues of $\Sigma_0(P)$. It is worth to highlight that the corank of $\Sigma_0(P)$ can change by moving P on $\mathcal{S}(\Sigma_n)$ i.e., by varying $\tilde{\Sigma}(P)$ according to condition (7) [19]. The maximum value that m can assume for $P \in \mathcal{S}(\Sigma_n)$ is defined as the maximal corank of Σ_n in the Frisch scheme context

$$\text{Maxcor}_F(\Sigma_n) = \max_{\tilde{\Sigma}(P)}\,[\,n - \text{rank}(\Sigma_n - \tilde{\Sigma}(P))\,]. \qquad (8)$$

Note that relations (5), (6) and the assumption on $A(\theta)$ lead easily to the following result.

Corollary 1. The point $\tilde{P} = (\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2)$ corresponding to the actual array noise variances belongs to $\mathcal{S}(\Sigma_n)$ and is associated with a singular matrix $\Sigma_0(P) = \Sigma_n - \tilde{\Sigma}(\tilde{P})$ whose null space has dimension $n - p$, i.e. corank $(\Sigma_0(P)) = n - p$.

We will assume that $\tilde{P}$ is the only point of $\mathcal{S}(\Sigma_n)$ leading to a singular matrix with corank $n - p$, that is

$$\text{corank}\,(\Sigma_0(P)) < n - p \quad \forall P \neq \tilde{P}, \qquad (9)$$

that leads immediately to

$$\text{Maxcor}_F(\Sigma_n) = n - p. \qquad (10)$$

For a discussion concerning the above assumption see [20]. Because of (10), the problem of estimating the number of signal sources p can thus be seen as the problem of determining the maximal corank of the covariance matrix Σ_n in the Frisch scheme context. The solution of this problem is recalled in the next Section.

4. COMPUTATION OF THE MAXIMAL CORANK OF A COVARIANCE MATRIX IN THE FRISCH SCHEME CONTEXT

Some important results concerning the evaluation of the maximal corank of Σ_n in the Frisch scheme context are the following.

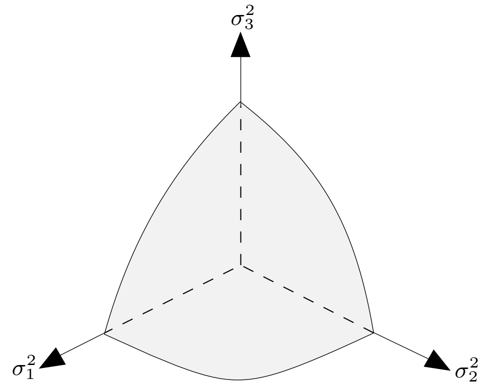

Fig. 1. Typical shape of $\mathcal{S}(\Sigma_3)$.

Theorem 2. $\text{Maxcor}_F(\Sigma_n) = 1$ if and only if all entries of Σ_n^{-1} are positive or can be made positive (Frobenius–like according to the definition of Kalman [21]) by changing the sign of some variables.

Theorem 3. [22] When $\text{Maxcor}_F(\Sigma_n) > 1$, $\mathcal{S}(\Sigma_n)$ is nonuniformly convex.

Theorem 4. [19] All points of $\mathcal{S}(\Sigma_n)$ where $\text{corank}\,(\Sigma_n) = k\,(k > 1)$ are accumulation points for those where $\text{corank}\,(\Sigma_n) = k - 1$.

Despite its simple formulation, the problem of determining $\text{Maxcor}_F(\Sigma_n)$ remained unsolved for many years. One of the reasons is probably due to the focus of many researches on the locus of the solutions in the parameter space and to the practical impossibility of describing this locus, when $\text{Maxcor}_F(\Sigma_n) > 1$, except than in elementary cases. An upper bound to $\text{Maxcor}_F(\Sigma_n)$ has been given in [23]; geometric conditions to evaluate $\text{Maxcor}_F\,(\Sigma_n)$ have, instead, been given in [19] on the basis of the analysis of the properties of the locus of noise space solutions.

Define, to this purpose, the singularity (hyper)surface $\mathcal{S}(\Sigma_{n/r})$ as the locus of the points $(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_r^2) \in \mathcal{R}^r$ such that

$$\Sigma_n - \text{diag}\left[\,\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0\,\right] \qquad (11)$$

is singular and nonnegative definite. Define also Σ_r as the $(r \times r)$ upper left corner of Σ_n and $\mathcal{S}(\Sigma_r)$ as the locus of the points $(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_r^2) \in \mathcal{R}^r$ such that

$$\Sigma_r - \text{diag}\left[\,\sigma_1^2, \ldots, \sigma_r^2\,\right] \qquad (12)$$

is singular and nonnegative definite. As an example, Fig. 2 shows the locus $\mathcal{S}(\Sigma_{3/2})$ and $\mathcal{S}(\Sigma_2)$ related to the covariance matrix Σ_3 already considered in Fig. 1. The following geometric relations hold.

Theorem 5. [24] $\mathcal{S}(\Sigma_{n/r})$ lies always under or on $\mathcal{S}(\Sigma_r)$.

Theorem 6. [19] $\text{Maxcor}_F(\Sigma_n) \geq q$ if and only if $\mathcal{S}(\Sigma_{n-q+1}) \cap \mathcal{S}(\Sigma_{n/n-q+1}) \neq \{0\}$ for every subset of $n - q + 1$ variables, i.e. for every permutation of the data leading to different subgroups in the first $n - q + 1$ positions.

Theorem 6 allows the straightforward formulation of an algorithm for computing whether $\text{Maxcor}_F\,(\Sigma_n) \geq 2, 3, \ldots$ until the required conditions are no longer satisfied.

The existence of common points between different singularity hypersurfaces can be easily and efficiently verified by relying on the radial parameterization of these surfaces introduced in [25], based on the following theorem.

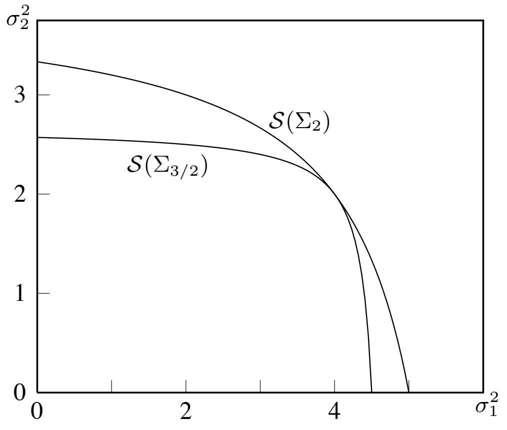

Fig. 2. Common points between $\mathcal{S}(\Sigma_2)$ and $\mathcal{S}(\Sigma_{3/2})$ in a (3×3) covariance matrix with $\mathrm{Maxcor}_F(\Sigma_3) = 2$.

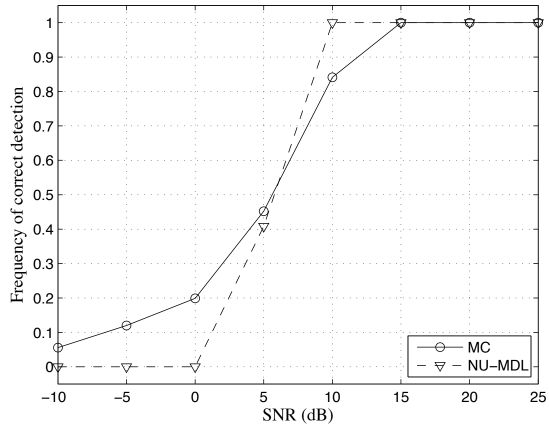

Fig. 3. Frequency of correct detection versus SNR, $N = 100$.

Theorem 7. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a generic point in the positive orthant of $\mathcal{R}^n$ and ρ the straight line from the origin through ξ. The intersection $P = (\sigma_1^2, \ldots, \sigma_n^2)$ between ρ and $\mathcal{S}(\Sigma_n)$ is given by

$$P = \frac{\xi}{\lambda_M}, \tag{13}$$

where

$$\lambda_M = \max \mathrm{eig} \left(\Sigma_n^{-1} \tilde{\Sigma}^\xi \right) \tag{14}$$

and

$$\tilde{\Sigma}^\xi = \mathrm{diag} \left[\xi_1, \ldots, \xi_n \right]. \tag{15}$$

This result introduces a parameterization of the singularity (hyper)surface of a covariance matrix, characterized by its intersections with a sheaf of straight lines through the origin, and has become a standard tool for the efficient solution of Frisch identification problems [16,26]. In particular, note that the intersections P' and P'' of the same line ρ with two singularity hypersurfaces $\mathcal{S}(\Sigma_n')$ and $\mathcal{S}(\Sigma_n'')$, can be easily computed by means of Theorem 7 and $\|P' - P''\|$ gives the distance between $\mathcal{S}(\Sigma_n')$ and $\mathcal{S}(\Sigma_n'')$. A simple search procedure allows then to compute the minimal distance between $\mathcal{S}(\Sigma_n')$ and $\mathcal{S}(\Sigma_n'')$ and, consequently, to evaluate the presence of common points.

Of course the same procedure can be used to operate on singularity surfaces belonging to the same vector space but associated with matrices with different dimensions like, for instance, $\mathcal{S}(\Sigma_r)$ and $\mathcal{S}(\Sigma_{n/r})$. This allows the solution of the problem of evaluating $\mathrm{Maxcor}_F(\Sigma_n)$ and, as a consequence, the determination of the number of signal sources p. In practice, the covariance matrix Σ_n is replaced by the sample estimate

$$\hat{\Sigma}_n = \frac{1}{N} \sum_{t=1}^{N} y(t)\, y^H(t). \tag{16}$$

In this case, it is still possible to define singularity hypersurfaces like $\mathcal{S}(\hat{\Sigma}_n)$ and $\mathcal{S}(\hat{\Sigma}_{n/r})$ so that the above mentioned search procedure can still be employed to evaluate the maximal corank of $\hat{\Sigma}_n$.

5. SIMULATION RESULTS

The proposed approach has been tested by considering a uniform linear array with omnidirectional sensors and half-wavelength interelement spacing. We assume that there are $p = 2$ sources and $n = 5$ sensors. The two sources are mutually uncorrelated complex white gaussian processes with unit variance. The directions of arrival are

$$\theta_1 = 7^\circ \qquad \theta_2 = 13^\circ,$$

so that the array transfer matrix is given by [27]

$$A = \begin{bmatrix} 1 & 1 \\ e^{i\pi \sin 7} & e^{i\pi \sin 13} \\ \vdots & \vdots \\ e^{i4\pi \sin 7} & e^{i4\pi \sin 13} \end{bmatrix}.$$

In the first example, the sensor noise $e(t)$ has the covariance matrix

$$\tilde{\Sigma} = \mu\, \mathrm{diag} \left[1,\ 3,\ 4,\ 2,\ 5 \right], \tag{17}$$

where the scalar $\mu > 0$ is adjusted in order to set the desired array signal to noise ratios defined as [10]

$$\mathrm{SNR}_i = \frac{E[x_i(t)^2]}{5} \sum_{j=1}^{5} \frac{1}{\sigma_j^2}, \qquad i = 1, 2.$$

Since the signal sources have equal variances we have $\mathrm{SNR}_1 = \mathrm{SNR}_2 = \mathrm{SNR}$. Note that the worst noise power ratio $\mathrm{WNPR} = \sigma_{max}^2 / \sigma_{min}^2$ is 5 [10]. The proposed algorithm, based on the computation of the maximal corank (MC), has been tested by considering $N = 100$ data samples and a SNR ranging from -10 dB to 25 dB. For each value of the array SNR a Monte Carlo simulation of 1000 independent runs has been performed. The performance of the MC method is compared to that of the averaged version of the Non-Uniform MDL (NU-MDL) criterion proposed in [14].

Fig. 3 reports the frequency of correct estimation of p versus the SNR. The MC criterion exhibits a better performance for low signal to noise ratios.

Figs. 4 and 5 compare the performances of MC and NU-MDL with respect to the number of samples for $\mathrm{SNR} = 5$ dB and $\mathrm{SNR} = 10$ dB respectively. It can be observed that MC outperforms NU-MDL when the number of samples in not too small.

In the second example, in order to compare the considered estimation algorithms with respect to the non uniformity of the noise, we introduce the following noise covariance matrix

$$\tilde{\Sigma}^k = \mu\, \mathrm{diag} \left[1,\ 1+2k,\ 1+3k,\ 1+k,\ 1+4k \right],$$

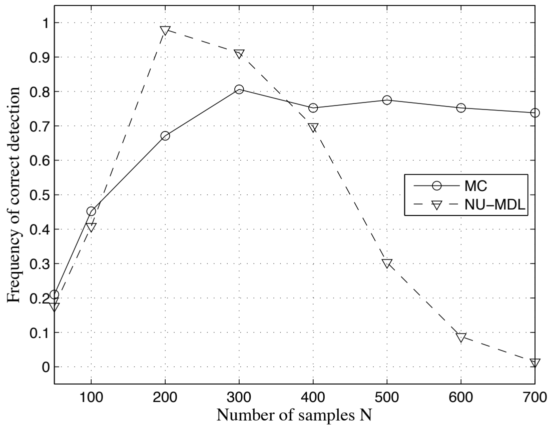

Fig. 4. Frequency of correct detection versus the number of samples N, $\mathrm{SNR} = 5$ dB.

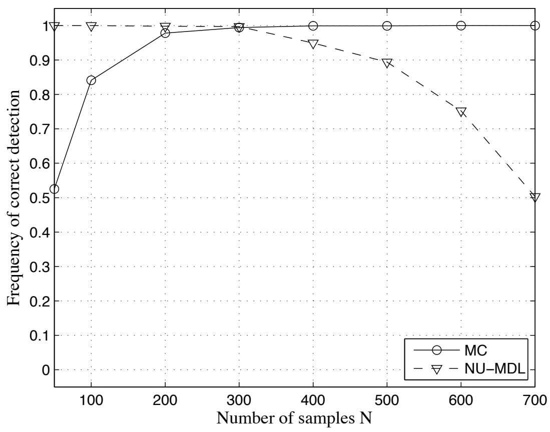

Fig. 5. Frequency of correct detection versus the number of samples N, $\mathrm{SNR} = 10$ dB.

where $0 \leq k \leq 4$ and μ is selected in order to obtain a SNR of 5 dB. By varying k from $k = 0$ to $k = 4$, the WNPR ranges from 1 to 17. Note that $k = 0$ corresponds to the uniform noise case (all variances are equal) whereas $k = 1$ leads to the noise covariance matrix (17) of the previous example. In practice, the index k can be considered as a measure of the noise nonuniformity. For each value of k (of WNPR) a Monte Carlo simulation of 1000 independent runs has been carried out. The MC and NU-MDL criteria have been compared with the frequently adopted Akaike Information Criterion (AIC) and Minimum Description Length (MDL) criterion described in [4].

Figs. 6 and 7 report the frequency of correct estimation of p versus WNPR for $N = 100$ and $N = 300$ respectively. The obtained results show that the proposed algorithm is less sensitive to the noise unbalance compared to the NU-MDL criterion. The performance of AIC and MDL are very good for small values of WNPR but decreases fastly with WNPR. This is not surprising since both AIC and MDL are based on the assumption of uniform noise.

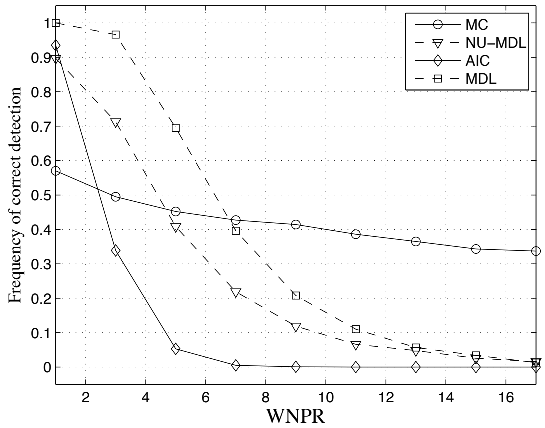

Fig. 6. Frequency of correct detection versus WNPR, $\mathrm{SNR} = 5$ dB, $N = 100$.

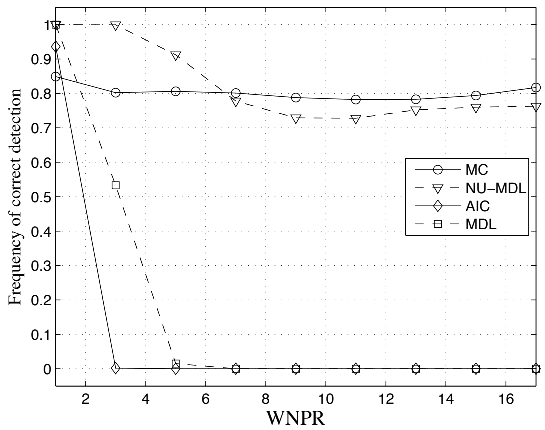

Fig. 7. Frequency of correct detection versus WNPR, $\mathrm{SNR} = 5$ dB, $N = 300$.

6. CONCLUSIONS

An approach for estimating the number of source signals in the presence of nonuniform additive white noise has been proposed. In particular, the problem of estimating the number of sources has been mapped into the problem of evaluating the maximal corank of the covariance matrix of the noisy data in the Frisch scheme context. It is worth stressing that, unlike the criteria described in [4, 14], the assumption of gaussianity of the source signals and of the sensor noises is not required by the proposed method.

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