MULTI-WAVELET COHERENCE FOR POINT PROCESSES ON THE REAL LINE

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ABSTRACT

Coherence is a well established measure of linear dependency between a pair of stationary random processes in the frequency domain. Wavelet coherence measures the linear dependency between a pair of signals in time-scale space and is therefore more suitable for non-stationary processes. Until now it has only been considered in relation to regularly sampled ordinary time-series. Here, for the first time, it is applied to point processes on the real line. We consider smoothing the individual wavelet spectra by averaging over a set of orthogonal Morse wavelets and show that under the assumption of independent Poisson processes the Goodman distribution is appropriate.

Index Terms— Wavelet coherence, Morse wavelets, point processes.

1. INTRODUCTION

Point processes on the real line, for which any one realisation consists of a set of isolated points in $\mathbb{R}$ (often considered as time), are used to model a vast array of random processes. In keeping with convention we represent a point process as $\{N(t), t \in \mathbb{R}\}$ whose value at time t is a random non-negative integer indicating the number of events that have occurred in the time interval $(0, t]$. Of particular interest is if a pair of point processes, $N_1(t)$ and $N_2(t)$ say, are correlated with one another. That is, are the occurrence of events in one point process correlated with the occurrence of events in the other.

Methods for such analysis, designed for the broader class of spatial point processes (of which point processes on the real line are the 1-D subclass), include the widely studied and applied cross-K-function [1] that indicates the expected number of type 2 events within a certain time window around an arbitrary type 1 event. Such methods suffer from high computational demand and the distribution of the cross-K-function estimator is intractable in all but the most trivial scenarios with computationally expensive Monte Carlo methods often required [2].

For processes on the real line there has been considerable interest in the use of spectral methods for analysing a single (univariate) point process e.g. [3], [4], [5]. A spectral density function $S_{11}(f)$ for a point process $N_1(t)$ forms a Fourier pair with the auto-covariance density of the point process and exposes the dominant frequencies driving the arrival of events. For a pair of point process $N_1(t)$ and $N_2(t)$, in addition to the two spectral density functions $S_{11}(f)$ and $S_{22}(f)$ we can also define a cross-spectral density function $S_{12}(f)$ that exposes common frequencies between events occurring in $N_1(t)$ and events occurring in $N_2(t)$. It therefore seems sensible that any linear dependency between a pair of point processes could alternatively and conveniently be examined using the spectral domain. A normalised measure of linear dependency in the frequency domain is the coherence function (or magnitude squared coherency) and is given as [4], [6]

$$\rho^2(f) = \frac{|S_{12}(f)|^2}{S_{11}(f)S_{22}(f)}. \tag{1}$$

A value close to zero indicates there is weak correlation between the two point processes associated with that particular frequency. A value close to one indicates strong correlation.

Spectral methods inherently assume stationarity across the whole observation window — this is often impractical. Wavelets are a natural tool with which to perform time-localised spectral analysis, and the discrete wavelet transform has been use for intensity estimation of a single process [7], [8].

The continuous wavelet transform (CWT) of a continuous time signal $X(t)$ at scale $a > 0$ and translation (time) $b \in \mathbb{R}$ is defined as

$$W(a, b, X, \psi) = a^{-1/2} \int_{-\infty}^{\infty} X(t)\psi^* \left(\frac{t-b}{a}\right) \mathrm{d}t$$

where $\psi(\cdot)$ is the analyzing wavelet and $*$ denotes complex conjugation. Given a second signal $Y(t)$, Liu [9] introduced the concept of wavelet coherence (WCOH) – a time-scale measure of linear dependency,

$$\frac{|W(a, b; X, \psi)W^*(a, b; Y, \psi)|^2}{|W(a, b; X, \psi)|^2 \, |W(a, b; Y, \psi)|^2}.$$

In practical situations smoothing the individual terms is necessary to avoid this equalling unity. We therefore implement

$$\frac{|\mathcal{S}\{W(a, b; X, \psi)W^*(a, b; Y, \psi)\}|^2}{\mathcal{S}\{|W(a, b; X, \psi)|^2\}\mathcal{S}\{|W(a, b; Y, \psi)|^2\}}, \tag{2}$$

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where $\mathcal{S}\{\cdot\}$ represents a smoothing operation. For regularly sampled ordinary time series this quantity has been extensively applied (e.g. [10],[11]) and recently its distributional properties have been derived for two different smoothing methods [12], [13]. However, as yet there has been no consideration of using this measure for point processes. Importantly, with the first stage of any investigation using WCOH on a pair of processes the null hypothesis will assume stationarity and independence, therefore it is important to know the distribution of WCOH under this assumption.

In Section 2 we introduce some key definitions for point processes. In Section 3 we present some distributional results for the CWT on a point process. In Section 4 we construct a WCOH estimator using multiple orthogonal wavelets and derive the distributions under the assumption of independent homogeneous Poisson processes.

2. POINT PROCESSES

2.1. Bivariate point processes on the real line

To denote a point process let $N(A) \equiv$ number of events that occur in $A \subset \mathbb{R}$, then the integer valued process $\{N(t), t \in \mathbb{R}\}$ is $N((0,t])$ for $t > 0$ and $N(0) = 0$. A bivariate point process $\{\mathbf{N}(t) \equiv [N_1(t), N_2(t)]^T, t \in \mathbb{R}\}$ consists of a pair of component point processes defined on the same space and labelled $N_i(t)$, $i = 1, 2$.

The *differential increment* $\mathrm{d}N_i(t) \equiv N_i(t, t + \mathrm{d}t]$ is a *counting variant* that counts the number of events in the increment $\mathrm{d}t$ with starting point t. Here, we only deal with *orderly* processes where two events can not appear at the same time. In this circumstance $\mathrm{d}N_i(t)$ is a Bernoulli random variable that takes a value of 1 if an event has occurred in the interval $(t + \mathrm{d}t]$. The *intensity* $\boldsymbol{\lambda}(t)$ of a bivariate point process $\mathbf{N}(t)$ is defined as the vector $\boldsymbol{\lambda}(t) = [\lambda_1(t), \lambda_2(t)]^T$ where

$$\lambda_i(t) \equiv \lim_{\mathrm{d}t \to 0} \frac{E\{\mathrm{d}N_i(t)\}}{\mathrm{d}t}, \quad i = 1, 2$$

and where $E\{\cdot\}$ denotes the expectation operator. We can interpret $\lambda_i(t)\mathrm{d}t$ as the probability that $\mathrm{d}N_i(t) = 1$.

The second-order properties of a univariate point process $N_i(t)$ are characterized by the second-order auto-intensity function

$$\lambda_{ii}(t, s) \equiv \lim_{\mathrm{d}t, \mathrm{d}s \to 0} \frac{E\{\mathrm{d}N_i(t)\mathrm{d}N_i(s)\}}{\mathrm{d}t \cdot \mathrm{d}s} \quad t \neq s.$$

When $t = s$ it is clear that $E\{|\mathrm{d}N_i(t)|^2\} = E\{\mathrm{d}N_i(t)\}$, and we define the complete auto-intensity function [3], [5] $\lambda^c_{ii}(t, s) = \lambda_{ii}(t, s) + \lambda_i(t)\delta(t - s)$, $t, s \in \mathbb{R}$. The second-order cross-intensity function at times t and s for processes $N_1(t)$ and $N_2(t)$ is defined as

$$\lambda_{12}(t, s) \equiv \lim_{\mathrm{d}t, \mathrm{d}s \to 0} \frac{E\{\mathrm{d}N_1(t)\mathrm{d}N_2(s)\}}{\mathrm{d}t \cdot \mathrm{d}s} \quad t, s \in \mathbb{R}.$$

This allows us to define the second-order intensity matrix for the bivariate process $\mathbf{N}(t)$ as

$$\Lambda(t, s) \equiv E\{\mathrm{d}\mathbf{N}(t)\mathrm{d}\mathbf{N}^T(s)\} = \begin{bmatrix} \lambda^c_{11}(t,s) & \lambda_{12}(t,s) \\ \lambda_{21}(t,s) & \lambda^c_{22}(t,s) \end{bmatrix} \quad t, s \in \mathbb{R}.$$

A bivariate point process $\mathbf{N}(t)$ is said to be stationary if $\Lambda(t, s) = \Lambda(t - s) = \Lambda(\tau)$ for all $t, s \in \mathbb{R}$, i.e. it is dependent only on lag $\tau = t - s$. A consequence of this is that the intensity function $\boldsymbol{\lambda}(t)$ is a constant, $\boldsymbol{\lambda}$ say.

The covariance density matrix of $\mathbf{N}(t)$ is defined as

$$\Gamma(t, s) \equiv \mathrm{cov}\{\mathrm{d}\mathbf{N}(t), \mathrm{d}\mathbf{N}(s)\} = \begin{bmatrix} \gamma^c_{11}(t,s) & \gamma_{12}(t,s) \\ \gamma_{21}(t,s) & \gamma^c_{22}(t,s) \end{bmatrix}$$

where the auto-covariance density is defined as $\gamma_{ii}(t, s) \equiv \lambda_{ii}(t, s) - \lambda_i(t)\lambda_i(s)$, $t \neq s$ and completed with $\gamma^c_{ii}(t, s) = \gamma_{ii}(t, s) + \lambda_{ii}\delta(t - s)$ for $t, s \in \mathbb{R}$. The cross-covariance density is $\gamma_{12}(t, s) \equiv \lambda_{12}(t, s) - \lambda_1(t)\lambda_2(s)$, $t, s \in \mathbb{R}$. Poisson processes are orderly processes where $\gamma_{ii}(t, s) = 0$ for all $t \neq s$. Such processes are called *completely random*. Poisson processes with a constant intensity for all time are known as homogeneous Poisson processes, otherwise they are inhomogeneous. We consider a bivariate point process $\mathbf{N}(t)$ to be Poisson if both component processes are Poisson.

2.2. Integrating with point processes

We will be interested in stochastic integrals of the form $\int_{\mathcal{T}} \varphi(t)\mathrm{d}N_i(t)$ where $\mathcal{T} \subseteq \mathbb{R}$ is the observation window. If events in $N_i(t)$ occur at times $u_{i,n} \in \mathcal{T}$, $n = 1, 2, ...$, then

$$\int_{\mathcal{T}} \varphi(t)\mathrm{d}N_i(t) = \sum_n \varphi(u_{i,n}). \tag{3}$$

We will also be interested in the first- and second-order moments of the integrals. We consider the bivariate extensions. It follows from the above definitions that

$$E\left\{\int_{\mathcal{T}} \varphi(t)\mathrm{d}\mathbf{N}(t)\right\} = \int_{\mathcal{T}} \varphi(t)\boldsymbol{\lambda}(t)\mathrm{d}t$$

and

$$E\left\{\int_{\mathcal{T}} \varphi_1(t)\varphi_2(s)\mathrm{d}\mathbf{N}(t)\mathrm{d}\mathbf{N}^T(s)\right\} = \int_{\mathcal{T}} \varphi_1(t)\varphi_2(s)\Lambda(t, s)\mathrm{d}t\mathrm{d}s.$$

2.3. Spectral density and cross-spectral density functions

The spectral density function of a stationary point process $N_i(t)$ is defined as the Fourier transform of the auto-covariance sequence [3] and the cross-spectral density function defined as the Fourier transform of the cross-covariance sequence. We therefore define the spectral density matrix of a stationary bivariate point process $\mathbf{N}(t)$ as ($\tau = t - s$)

$$\mathbf{S}(f) = \int_{-\infty}^{\infty} \Gamma(\tau) \exp(-\mathrm{i}2\pi f\tau)\mathrm{d}\tau = \begin{bmatrix} S_{11}(f) & S_{12}(f) \\ S_{21}(f) & S_{22}(f) \end{bmatrix}. \tag{4}$$

The spectral density function shows the frequencies at which events occur within a process. A Poisson process exhibits complete randomness which results in all frequencies contributing equally to the spacings of events and therefore has a flat spectral density function of value λ_{ii}, as can be seen by taking the Fourier transform of $\gamma_{ii}^c(\tau) = \lambda_{ii}\delta(\tau)$. The cross-spectral density represents the covariance between the two processes in the frequency domain. Two independent point processes will have zero cross-spectral density.

The overall magnitude of the cross-spectral density function is dependent on the individual processes and therefore normalising it with respect to the spectral density functions of the individual processes becomes useful for comparison purposes. The coherence is a normalised measure of linear dependency between two stationary processes $N_1(t)$ and $N_2(t)$ at a particular frequency and is given in (1). This must be estimated using smoothed spectral estimators to avoid an estimate of unity for all frequencies, [4], [6], [14], [15], [16].

3. THE CWT FOR POINT PROCESSES

From (3), the CWT $W(a,b;N_i(t))$ (suppressing ψ in notation) for $N_i(t)$ at scale $a>0$ and translation $b\in\mathbb{R}$ is given as

$$a^{-1/2}\int_{-\infty}^{\infty}\psi^*\left(\frac{t-b}{a}\right)\mathrm{d}N_i(t) = a^{-1/2}\sum_n \psi\left(\frac{u_{i,n}-b}{a}\right),$$

where events of process $N_i(t)$ occur at times $u_{i,n}$, $n=1,2....$ The term $|W(a,b;N_i(t))|^2$ is known as the wavelet spectrum at scale a and translation b. It is a time localised measure of how events occurring across a particular scale contribute to the point process. It is typical to consider scale and frequency to have an inverse relationship.

3.0.1. Restrictions on scale and translation parameters

Consider performing a CWT on a realisation of $N_i(t)$ observed on the time window $\mathcal{T}=(0,T]$, then this gives restrictions on the scales a and translations b. The support of scaled wavelet $a^{-1/2}\psi(t/a)$ must be less than observation window $\mathcal{T}$. If ψ has support s then $a^{-1/2}\psi(t/a)$ has support sa implying that we require $a<T/s$. We also require the translated wavelet to lie in the observation window, therefore for each a we require $sa/2<b<T-sa/2$. Let $\mathcal{A}=(0,T/s)$ represent the set of values that a can take, and $\mathcal{B}(a)=(sa/2, T-sa/2)$.

3.1. Wavelet transform vector

To consider the wavelet transform of a bivariate point process $\mathbf{N}(t)=[N_1(t),N_2(t)]^T$ it makes sense to introduce the wavelet transform vector

$$\mathbf{W}(a,b)=\left[\begin{smallmatrix}W(a,b;N_1(t))\\W(a,b;N_2(t))\end{smallmatrix}\right] = a^{-1/2}\int_{-\infty}^{\infty}\psi^*\left(\frac{t-b}{a}\right)\mathrm{d}\mathbf{N}(t),$$

the expected value of which is

$$E\{\mathbf{W}(a,b)\} = a^{-1/2}\int_{-\infty}^{\infty}\psi\left(\frac{t-b}{a}\right)\boldsymbol{\lambda}(t)\mathrm{d}t$$

where $\boldsymbol{\lambda}(t)=[\lambda_1(t),\lambda_2(t)]^T$. We therefore see that the CWT of a point process is an estimator of the CWT of the process' intensity. Consequently, we are able to use it to study small and large scale changes in the intensity in a time-localized way. We note that if $\mathbf{N}(t)$ is stationary, implying $\boldsymbol{\lambda}(t)=\boldsymbol{\lambda}=[\lambda_1,\lambda_2]^T$, combined with the fact a wavelet integrates to zero by definition, then $E\{\mathbf{W}(a,b)\}=\mathbf{0}$.

Let $\Omega(a,b)\equiv\mathbf{W}(a,b)\mathbf{W}^H(a,b)$ (where ^{H} is the conjugate transpose) which we will call the wavelet spectral matrix due to the fact that the diagonal elements are the individual wavelet spectra $|W(a,b;N_i(t))|^2$ and the off-diagonal terms are the wavelet cross spectra $W(a,b;N_i(t))W^*(a,b;N_j(t))$, from which we can define the cross wavelet power spectra $|W(a,b;N_i(t))W^*(a,b;N_j(t))|$. The wavelet spectral matrix $\Omega(a,b)$ is given as

$$a^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*\left(\frac{t-b}{a}\right)\psi\left(\frac{s-b}{a}\right)\mathrm{d}\mathbf{N}(t)\mathrm{d}\mathbf{N}^T(s),$$

and $E\{\Omega(a,b)\}$ equals

$$a^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*\left(\frac{t-b}{a}\right)\psi\left(\frac{s-b}{a}\right)\Lambda(t,s)\mathrm{d}t\mathrm{d}s$$

For stationary $\mathbf{N}(t)$ we have seen $E\{\mathbf{W}(a,b)\}=0$ and therefore $\mathrm{cov}\{\mathbf{W}(a,b)\}=E\{\Omega(a,b)\}$ and is given by

$$a^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*\left(\frac{t-b}{a}\right)\psi\left(\frac{s-b}{a}\right)\Gamma(t-s)\mathrm{d}t\mathrm{d}s.$$

Let $\Psi(f)$ be the Fourier transform of $\psi(t)$, it follows from (4)

$$E\{\Omega(a,b)\} = a^{-1}\int_{-\infty}^{\infty}|\Psi(af)|^2\mathbf{S}(f)\mathrm{d}f.$$

Given $\mathbf{S}(f)$ is sufficiently smooth across the (typically tightly-peaked) function $|\Psi(af)|^2$ then wavelet spectral matrix $\Omega(a,b)$ is an approximately unbiased estimator of spectral density matrix $\mathbf{S}(f)$. In the particular case that $\mathbf{N}(t)$ consists of a pair of stationary and independent Poisson processes then $E\{\Omega(a,b)\}=\mathbf{S}(f)=\left[\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right]$.

3.2. Statistical analysis of CWT

We now present some key results for the distribution of the wavelet transform vector for a pair of stationary point processes. We let $N_d^{\mathcal{C}}(\boldsymbol{\mu},\Sigma,R)$ (and $N^{\mathcal{C}}(\mu,\sigma^2,r^2)$) denote the d-dimensional (1-dimensional) improper complex normal distribution with relation matrix R (relation parameter r) and $N_d^{\mathcal{C}}(\boldsymbol{\mu},\Sigma)$ ($N^{\mathcal{C}}(\mu,\sigma^2)$) denote the d-dimensional (1-dimensional) proper complex normal distribution ($R=0$ or $r=0$). The following theorem is given full treatment in [17].

Theorem 3.1. *Let $N(t)$ be a homogeneous Poisson process observed on the interval $\mathcal{T} = [0, T] \subset \mathbb{R}$ with intensity $\lambda > 0$, and let $\psi(t)$ be a complex valued analysing wavelet, then with respect to λ, we have $\lambda^{-1/2} W \equiv \lambda^{-1/2} W(a, b; N(t))$ is asymptotically $N^{\mathcal{C}}(0, 1, \kappa^2)$ distributed for all $a \in \mathcal{A}, b \in \mathcal{B}(a)$, where $\kappa^2 = \int_{\mathcal{T}} a^{-1} \psi((t-b)/a) \psi((t-b)/a) \mathrm{d}t$.*

The result shown above is asymptotic. We therefore assume the "large-λ" approximation $W \sim N^{\mathcal{C}}(0, \lambda, \lambda\kappa^2)$. We now consider a pair of homogeneous Poisson point processes $N_1(t)$ and $N_2(t)$. A pair of marginally normal independent random variables are jointly normal, implying for large-λ that $\mathbf{W}(a, b) \sim N_2^C(\mathbf{0}, \Sigma, R)$ where $\Sigma = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$ and $R = \left[\begin{smallmatrix} \lambda_1\kappa^2 & 0 \\ 0 & \lambda_2\kappa^2 \end{smallmatrix}\right]$ where $\kappa^2 = \int_{\mathcal{T}} a^{-1} \psi((t-b)/a) \psi((t-b)/a) \mathrm{d}t$.

4. MORSE WCOH FOR POINT PROCESSES

It is necessary we smooth the wavelet spectra to avoid WCOH estimates of unity [12]. A smoothing method explored in detail in [12] uses a set of K orthogonal Morse wavelets [18] $\{\psi_k(\cdot), k = 0, ..., K-1\}$ to give the WCOH estimator

$$\frac{\left|\sum_{k=0}^{K-1} W_k(a, b; N_1(t)) W_k^*(a, b; N_2(t))\right|^2}{\left(\sum_{k=0}^{K-1} |W_k(a, b; N_1(t))|^2\right)\left(\sum_{k=0}^{K-1} |W_k(a, b; N_2(t))|^2\right)}$$

where $W_k(a, b; N_i(t)) \equiv W(a, b; N_i(t), \psi_k)$ and $\mathbf{W}_k(a, b)$ is the associated CWT vector for the kth wavelet. The orthogonality of the K wavelets gives K approximately uncorrelated wavelet spectra across which to average. We note, for $a \in \mathcal{A}$ and $b \in \mathcal{B}(a)$, $E\{\mathbf{W}_j(a, b)\mathbf{W}_k^H(a, b)\}$ equals

$$\int_{\mathcal{T}} \int_{\mathcal{T}} \psi_j^*\left(\frac{t-b}{a}\right) \psi_k\left(\frac{s-b}{a}\right) \Lambda(t, s) \mathrm{d}s\mathrm{d}t.$$

When N_1 and N_2 are independent homogeneous Poisson processes matrix $\Lambda(t, s)$ is constant and $E\{\mathbf{W}_j(a, b)\mathbf{W}_k^H(a, b)\} = \mathbf{0}$. It is also true that for the Morse wavelets $\kappa = 0$ [17], so with $\Omega_k(a, b) \equiv \mathbf{W}_k(a, b)\mathbf{W}_k^H(a, b)$, then

$$\Omega_k(a, b) \sim W^C(1, \Lambda)$$

for all $k = 0, ..., K-1$, where $W^C(n, \Sigma)$ represents the complex Wishart distribution with n degrees of freedom and scale matrix Σ. Consequently, in the case of independent homogeneous Poisson processes we have the approximation

$$\Omega(a, b) \equiv \sum_{k=0}^{K-1} \mathbf{W}_k(a, b)\mathbf{W}_k^H(a, b) \sim W^C(K, \Lambda). \quad (5)$$

4.1. Coherence

The Morse wavelet coherence is alternatively expressed as

$$\hat{\rho}^2(a, b) = \frac{|[\Omega(a, b)]_{12}|^2}{([\Omega(a, b)]_{11}[\Omega(a, b)]_{22})}.$$

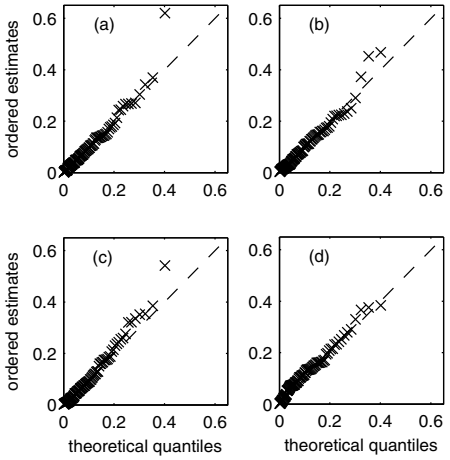

Fig. 1. QQ-plots comparing the theoretical quantiles of the Goodman distribution to the 100 WCOH estimates computed on a pair of independent homogeneous Poisson processes with intensities $\lambda_1 = \lambda_2 =$ (a) 50, (b) 200, (c) 500 and (d) 1000, at scale $a = 0.05$ and translation $b = 0.5$.

Given the distribution of $\Omega(a, b)$, $\hat{\rho}^2$ has pdf [19]

$$g(x; K, \rho^2) = (K-1)(1-\rho^2)^K \times (1-x)^{K-2} {}_2F_1(K, K; 1; \rho^2 x).$$

This is the pdf for the Goodman distribution with K degrees of freedom and 'true' parameter ρ^2. In the case of independent processes the true coherence parameter is $\rho^2 = 0$.

4.2. Simulation

We perform 100 simulations of two independent homogeneous Poisson processes on the interval $[0, 1]$ and compute the WCOH estimate for each pair using the first $K = 8$ order Morse wavelets with shape parameters $\beta = 5, \gamma = 2$ (see [18] for details). The QQ-plot in Figure 1 plots the ordered WCOH estimates for scale $a = 0.05$ and translation $b = 0.5$ against the theoretical quantiles of the proposed Goodman distribution. It is clear (with the exception of slight deviations at the tail) there is an excellent fit between the empirical and proposed large-λ distribution, even for relatively low intensities.

5. CONCLUSION

We have introduced WCOH - a measure of linear dependency in time-scale space - for a pair of point processes on the real line. Using multiple orthogonal wavelets we have shown it to be amenable to statistical study and under the assumption of homogeneous Poisson and independence the Goodman distribution is appropriate. Such an assumption can form the null hypothesis against which we search for regions in time-scale space of significant coherence.

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