Lifting Wavelet Design by Block Wavelet Transform Inversion

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Abstract-Due to its intuitive structure and efficient implementation, such as integer wavelets, lifting style wavelets gained high popularity. Following the natural correspondence between subband and lifting filters, this paper proposes a new approach to the design of wavelets indirectly through the optimisation of its corresponding block wavelet transform (BWT). BWT is a matrix transform which is generated from subbands, and it describes the relation between these two transform approaches. The BWT optimisation is achieved by making the matrix close to a particular Karhunen-Loeve transform (KLT) of interest. It has been observed that lifting-style wavelets have their constrains in the BWT matrix structure, therefore the minimisation of the difference between a KLT and a BWT derived from a lifting style wavelet becomes a non-trivial task. This paper briefly describes the vanishing moment and orthogonality constraints of the BWT, and introduces the first attempts to obtain single stage lifting wavelet filters that satisfies the constrained minimisation. Experimental results are provided.

I. INTRODUCTION

Lifting scheme was envisaged by Sweldens as a popular, successful, and flexible wavelet analysis method for the implementation of biorthogonal wavelets [1], [2]. Similar to subband filterbanks, design of lifting wavelet filters immediately became a mature field. However, the design motive was mostly the concept of compact support with maximum number of vanishing moments, as proposed by Daubechies [3].

On another track, an interesting work by Cetin and Gerek links the gap between subband decomposition structures and block matrix transforms by devising a method to generate a transform matrix (namely, the Block Wavelet Transform, BWT) from a given subband decomposition filter bank [4]. The success of the corresponding BWT matrices were tested in a typical transform coding standard, JPEG, by substituting the DCT. The direct correspondence between subbands and transform matrices inspires the idea that, if the transform matrix has an optimality in some sense, then, by inverting the BWT generation process from the subband decomposition, the corresponding wavelet filters could also be optimised, hence designed.

On the way to achieve the above mentioned BWT inversion, many intrinsic properties of the lifting-style implementation should be carefully investigated. For this purpose, these properties were researched by closely examining the *in vivo* research literature. For example, using lifting scheme,

Calderbank et al developed an integer to integer wavelet transform [5]. Li et al developed a novel algorithm that is used to lift the vanishing moments of wavelet from the general wavelet [6]. Meng et al used lifting decomposition for a different approach, namely the stationary wavelet transform [7]. Zhang et al showed a way to implement a class of infiniteimpulse-response(IIR) orthogonal wavelet filter banks by using the lifting scheme with two lifting steps [8]. Srinivasarao et al used the adaptive lifting scheme in the design of IIR orthogonal wavelets [9]. For content-based image retrieval, Quellec et al used adaptive nonseparable wavelet transform with lifting scheme [10]. Yang et al presented the lifting scheme of wavelet bi-frames along with theoretical analysis, structure, and the corresponding algorithm [11]. Fujinoki et al developed triangular biorthogonal wavelets by extending two-dimensional lifting [12]. For lossy-to-lossless image coding, Suzuki et al generalized block-lifting factorization of M-channel biorthogonal filter banks [13]. All such research outcomes provide an optimisation constraint over the lifting wavelet filters.

Our research focuses on the parametric "BWT" matrix generating properties of a flexible lifting structure, excluding the cases with cascaded prediction (P) and update (U) stages. The paper starts by re-introducing the generation of BWT from balanced subband trees. We briefly show the pattern dependence between a lower size $(2^k \times 2^k)$ BWT and the immediate larger BWT matrix of size $2^{k+1} \times 2^{k+1}$ in terms of parametric coefficients of P and U matrices.

II. BLOCK WAVELET TRANSFORM

A block wavelet transform (BWT) is a square matrix, whose coefficients are generated by feeding periodic impulse trains to tree structured subband filters [4]. By deliberately selecting the period of the input impulse train as 2^l , where *l* corresponds to the depth of the subband tree, every wavelet packet branch is set to produce a constant value, making a column vector for the whole tree leaves. By varying the phase of the input periodic signal, different output vectors are obtained. Finally, the 2^l different vectors render the columns of the desired BWT matrix. In this work, the BWT matrices are obtained by making use of lifting-style decompositions, starting from the l = 1 case in the following subsection.

A. Block Wavelet Transforms of Lifting Stages



Fig. 1. 1-level lifting structure.

Figure 1 depicts the single level lifting decomposition. For a 2×2 BWT parametrization, the degree of freedom for the prediction and update filters may not exceed 2, so a 2-tap prediction filter, i.e., $P(z) = a_0 + a_{-1}z$, and a 2-tap update filter, $U(z) = b_0 + b_1 z^{-1}$ is assumed. With the 2-periodic input:

$$x[n] = \begin{cases} \rho_0, & \text{n even} \\ \rho_1, & \text{n odd} \end{cases}$$

the output signals from the upper $(y_u[m])$ and lower $(y_d[m])$ branches become

$$y_u[m] = \rho_0 + (b_0 + b_1)[\rho_1 - \rho_0(a_0 + a_{-1})]$$

$$y_d[m] = \rho_1 - \rho_0(a_0 + a_{-1})$$
(1)

respectively. By assigning $\rho_0 = 0$ and $\rho_1 = 1$, the first output column, then by assigning $\rho_0 = 1$ and $\rho_1 = 0$, the second column is obtained for the output BWT matrix, $\mathbf{A}^{2\times 2}$ as

 $\mathbf{A}^{2\times 2} = \begin{bmatrix} f_{i,i}^{2\times 2} \end{bmatrix}$

where

f f

$$c^{2\times2}_{1,1} = 1 - (a_0 + a_{-1})(b_0 + b_1)$$

$$c^{2\times2}_{2,1} = -(a_0 + a_{-1})$$

$$f_{1,2}^{2\times2} = b_0 + b_1$$

$$f_{2,2}^{2\times2} = 1,$$
(2)

meaning that the BWT may not be varied separately by the 2-tap filters, but rather their coefficient sums determine the matrix. This low degree of freedom expands by extending the BWT matrix size, with which, the P and U filters become reasonably long.

The recursion step to obtain $2^{k+1} \times 2^{k+1}$ BWT from a $2^k \times 2^k$ version shows us that the parametric freedom of P and U filter length is limited to 2^k for a BWT matrix of size $2^{k+1} \times 2^{k+1}$. Let's notate length- 2^l filters as:

. .

$$P(z) = \sum_{i=-(2^{l-1}-1)}^{2^{l-1}} a_{-(i)_{2^{l}}} z^{i}$$
$$U(z) = \sum_{i=-(2^{l-1}-1)}^{2^{l-1}} b_{(i)_{2^{l}}} z^{-i},$$

where $(\cdot)_n$ corresponds to a modulo-n. The iteration process starts from the 2×2 case, and, while constructing the $2^{k+1} \times 2^{k+1}$ BWT matrix from the $2^k \times 2^k$ BWT, the coefficients $a_{-(i)_{2^k}}$ overlaps with a_{-i} and the coefficients $b_{(i)_{2^k}}$ overlaps with b_i for $i = 0, \ldots, 2^l - 1$. Thus in this iteration process, the P and U filter coefficients are updated according to the following assignments:

$$a_{-i} \leftarrow a_{-i} + a_{-i+2^k} + a_{-i+2^{k+1}} + \dots + a_{-i+2}$$

$$b_i \leftarrow b_i + b_{i+2^k} + b_{i+2^{k+1}} + \dots + b_{i+2^l}$$

We can generalize obtaining $2^{k+1} \times 2^{k+1}$ BWT matrix from $2^k \times 2^k$ BWT matrix as:

$$\mathbf{A}^{2^{k+1} \times 2^{k+1}} = \left[f_{i,j}^{2^{k+1} \times 2^{k+1}} \right]_{i,j=1,\dots,2^{k+1}}, \quad (3)$$

where each $f_{i,j}$ is algorithmically constructed from samples of circularly symmetric matrices that are composed of the smaller corresponding BWT matrix, $\mathbf{A}^{2^k \times 2^k}$. The derivations are omitted due to page limitations.

B. BWT case study: Daubechies 5/3

The celebrated Daubechies 5/3 lifting wavelet [3] uses the prediction and update filters of $P(z) = 0.5 \cdot (1+z)$ and $U(z) = 0.25 \cdot (1+z^{-1})$, so that

$$\mathbf{A}_{db5/3}^{4\times4} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Note that all rows of $A_{db5/3}^{4\times4}$ are orthogonal to each other despite Daubechies 5/3 wavelet being biorthogonal. The corresponding 8×8 BWT matrix can be easily expanded to

$$\begin{split} \mathbf{A}_{db5/3}^{8\times8} &= \\ \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}, \end{split}$$

which immediately shows non-orthogonal (yet invertible) row pairs. In fact, it is concluded that, in order to impose orthogonality for the 8×8 matrix, 4-tap prediction and update filters should be necessary (in case of non-cascaded lifting prediction and updates). This observation also shows that, for the shortest 2-tap prediction and update case (meaning a 4×4 transform matrix), the only useful lifting wavelet that can mimic an orthonormal KLT transform is the Daubechies 5/3. For longer lifting cases, the design flexibility exists. The imposition of the orthogonality is briefly described in the next section.

C. Orthogonality constraints on the BWTs

The orthogonality of 2×2 BWT matrix $\mathbf{A}^{2 \times 2}$ is achieved through a quadratic root formula relating sums of P and U filter coefficients:

$$\alpha = \frac{1 \pm \sqrt{1 - 4\beta^2}}{2\beta}$$
where
$$\alpha = a_0 + a_{-1} + \dots$$

$$\beta = b_0 + b_1 + \dots$$
(4)

Note that the famous wavelets db5/3, crf13/7, and swe13/7 automatically satisfy this orthogonality condition, where, for all wavelets listed, $\alpha = 1$ and $\beta = 0.5$ [14].

In our formulation, the matrix $A^{2\times 2}$ is used as a *kernel* matrix. The above mentioned iterative BWT generating algorithm may be used to develop the next BWT matrix with one degree higher. The key idea is to incorporate the below orthogonality constraints of the BWT matrix samples in the "size incrementing" process. Therefore, if the kernel matrix is orthogonal and the iterative restrictions for the orthogonality are complied with, then the BWT matrices with higher dimensions will also be orthogonal.

The orthogonality of the $2^k \times 2^k$ BWT matrix $A^{2^k \times 2^k}$ imposes that its coefficients satisfies $0 = \sum_{i=1}^{2^k} f_{l,i} f_{m,i}$ for $l \neq m$. The expansion to stage k+1 immediately adds 3 more cross-terms to the summation, each of which must all be set to zero. These conditions are imposed on the four permutation matrices that generate the BWT matrix, and the numerical results for the case of 8×8 and 16×16 BWTs (inherited from the 4×4 case) is exemplified in the following section.

III. ORTHOGONAL BWT GENERATING LIFTING EXAMPLES

The prediction and update filters of a particular biorthogonal wavelet example, where the corresponding BWT matrices become orthogonal, is given below:

$$P_{13/7}(z) = \frac{1}{2}(z^{-1} - 1 + z + z^2)$$

$$U_{13/7}(z) = \frac{1}{4}(z - 1 + z^{-1} + z^{-2})$$
(5)

The corresponding orthogonal 4×4 BWT matrix is

$$\mathbf{A}_{13/7}^{4\times4} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The achieved 8×8 BWT matrix using the 4×4 BWT matrix is

$$\mathbf{A}_{13/7}^{8\times8} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

The frequency ordering of the rows (as well as the energies) require further processing of simple row permutations and normalisation, however the orthogonality property of this construction is distinct, and could not be achieved with celebrated lifting wavelets such as db5/3, crf13/7, or swe13/7. It is now the time for the issue of back conversion of the matrix to find-back the lifting prediction and update filters. From the polyphase analysis:

$$\mathbf{H}_{p}(z) = \begin{bmatrix} H_{0,ev}(z) & H_{0,od}(z) \\ H_{1,ev}(z) & H_{1,od}(z) \end{bmatrix}$$
$$= \begin{bmatrix} 1 - P_{13/7}(z)U_{13/7}(z) & U_{13/7}(z) \\ -P_{13/7}(z) & 1 \end{bmatrix}$$

so

$$H_{13/7,0}(z) = H_{0,ev}(z^2) + zH_{0,od}(z^2)$$

$$= -\frac{1}{8}z^{-6} + \frac{1}{4}z^{-3} + \frac{1}{8}z^{-2} + \frac{1}{4}z^{-1}$$

$$+ \frac{1}{2} - \frac{1}{4}z + \frac{1}{8}z^2 + \frac{1}{4}z^3 - \frac{1}{8}z^6$$

$$H_{13/7,1}(z) = H_{1,ev}(z^2) + zH_{1,od}(z^2)$$

$$= -\frac{1}{2}z^{-2} + \frac{1}{2} + z - \frac{1}{2}z^2 - \frac{1}{2}z^4$$
(6)

Here, $H_{13/7,0}(z = 1) = 1$, $H_{13/7,0}(z = -1) = 0$ and $H_{13/7,1}(z = 1) = 0$, $H_{13/7,1}(z = -1) = -2$ assuring iterated convergence by having at least one vanishing moment. The aliasing component matrix $\mathbf{H}(z)$ is defined as [15]:

$$\mathbf{H}(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}$$

Using the prediction and update filters used in this case, the perfect reconstruction condition is the determinant of the aliasing component matrix and $\left|\det(\mathbf{H}(e^{j\omega}))\right| = 2$ where

$$\left|\det(\mathbf{H}(e^{j\omega}))\right| = |H_{13/7,0}(e^{j\omega})H_{13/7,1}(-e^{j\omega}) + H_{13/7,0}(-e^{j\omega})H_{13/7,1}(e^{j\omega})|$$
(7)

When $A^{16 \times 16}$ is desired to be orthogonal, utilisation of the prediction and update filters that are described in Equation 8

will produce the P and U filters as

$$P_{25/13}(z) = \frac{1}{2}(z^{-2} - 1 + z^2 + z^4)$$

$$U_{25/13}(z) = \frac{1}{4}(z^2 - 1 + z^{-2} + z^{-4})$$
(8)

and the corresponding subband filters turn out to be

$$H_{25/13,0}(z) = -\frac{1}{8}z^{-12} + \frac{1}{4}z^{-7} + \frac{1}{8}z^{-4} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-1} + \frac{1}{8}z^{4} + \frac{1}{4}z^{5} - \frac{1}{8}z^{12}$$
(9)
$$H_{25/13,1}(z) = -\frac{1}{2}z^{-4} + \frac{1}{2}z^{-4} + \frac{1}{2}z^{-4} - \frac{1}{2}z^{-4} + \frac{1}{$$

with $\left|\det(\mathbf{H}(e^{j\omega}))\right| = 2$ and satisfaction of necessary vanishing moments.

As a general formula, it was observed that, to achieve a $2^{k+2} \times 2^{k+2}$ BWT matrix for k = 1, 2, ..., we need to use the filters defined in Equation 10:

$$P_{12k+1/6k+1}(z) = \frac{1}{2} \left((z^{-2})^k - 1 + (z^2)^k + (z^4)^k \right)$$
$$U_{12k+1/6k+1}(z) = \frac{1}{4} \left((z^2)^k - 1 + (z^{-2})^k + (z^{-4})^k \right) \quad (10)$$
$$k = 1, 2, \dots$$

and the corresponding subband filters are

$$H_{12k+1/6k+1,0}(z) = -\frac{1}{8}z^{-6k} + \frac{1}{4}z^{-4k+1} + \frac{1}{8}z^{-2k} + \frac{1}{4}z^{-2k+1} + \frac{1}{2} - \frac{1}{4}z + \frac{1}{8}z^{2k} + \frac{1}{4}z^{2k+1} - \frac{1}{8}z^{6k} + H_{12k+1/6k+1,1}(z) = -\frac{1}{2}z^{-2k} + \frac{1}{2} + z - \frac{1}{2}z^{2k} - \frac{1}{2}z^{4k}$$
(11)

for integer k. Consequently, the class of lifting filters with orthogonal BWT properties is achieved. Since KLT matrices are orthogonal by definition, the proposed BWT inversion is limited to a search within the provided set of lifting prediction and update filters. Performance examples are provided in the next section.

IV. NUMERICAL EXPERIMENT

Numerical results will be demonstrated with two 512×512 test images, namely "Lena" and "Mandrill", due to page limitations. The 4×4 KLT matrices (generated by row-wise correlations) are obtained with two degrees of freedom for the P and U filters as:

 $P_{Lena}(z) = 0.7077 + 0.2923z, U_{Lena}(z) = 0.4271 + 0.0729z^{-1}$, and $P_{Mand}(z) = 0.7838 + 0.2162z, U_{Mand}(z) = 0.4647 + 0.0353z^{-1}$. It is noticeable that the *P* and *U* coefficients are rather different than the db5/3 filter for the same size, whilst the coefficient sums remain to be 1 and 0.5, as expected.

The new filters are applied to the Lena test image to form 1, 2 and 3-level wavelet trees. The wavelet-space variances are evaluated as the performance, where smaller variances indicate better coding gain with better energy unbalance.

Component	Daubechies 5/3	Our 5/3 Wavelet
LH	37.1815	33.6394
HL	66.3877	65.5296
HH	21.1877	26.7628
LLH	126.7025	100.7754
LHL	235.5276	193.2309
LHH	139.2559	126.7523
LLLH	282.1463	201.6424
LLHL	561.4503	446.0214
LLHH	511.1497	340.9113
	TABLE I	

VARIANCES OF THE WAVELET TREE IMAGES FOR DAUBECHIES 5/3 AND OUR WAVELET

V. CONCLUSION

In this research we have discovered several properties of the single stage lifting scheme in terms of its BWT generation abilities. The aim of the research was to find a methodology to design P and U filters which eventually produce BWT matrices that may become as close to a particular KLT matrix, as possible. Since KLT matrix is statistically optimal in terms of coding gain, the mentioned design method is expected to provide good filter outcomes. During constructing the BWT matrix corresponding to a lifting stage, it was first observed that the size of the matrix limits the degree of freedom for the size of the P and U filters with a $2^{k+1} \times 2^{k+1}$ BWT matrix leading to length -* 2^k filters.

The second observation was the systematic extension of the BWT matrix size by increasing the tree depth. The observed system enabled us to extend and inherit several nice properties of the smaller-sized BWT matrix, such as orthogonality or attainment of required vanishing moments. Eventually, an algorithm and a general rule was achieved to achieve lifting filters which can generate orthogonal BWT matrices at rather large sizes. As expected, the degree of freedom for the KLT-approximation is rather limited, once such orthogonality and vanishing moment constraints are imposed - usually in the sense of carefully distributing a total coefficient sum of 1 or 0.5 amongst filter coefficients.

An initial test with the famous Lena test image yields that the optimal length-2 P and U filters with a 4 × 4 BWT matrix which comes closest to the corresponding KLT matrix is quite different than the famous db5/3 wavelet. It is worthwhile to mention that both the generated and the db5/3 wavelet have one vanishing moment. Besides, the variances of waveletspace images provide smaller values for the produced wavelet, giving a hint of fine coding gain performance. Our research aims to provide detailed derivations of the above algorithm, together with more number of experimental examples in the future.

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