M-CHANNEL OVERSAMPLED PERFECT RECONSTRUCTION FILTER BANKS FOR GRAPH SIGNALS

Yuichi Tanaka and Akie Sakiyama

Graduate School of BASE, Tokyo University of Agriculture and Technology
Koganei, Tokyo, 184-8588 Japan
Email: ytnk@cc.tuat.ac.jp, akiesakiyama@msp.lab.tuat.ac.jp

ABSTRACT

This paper proposes M-channel oversampled filter banks for graph signals. The filter set satisfies the perfect reconstruction condition. A method of designing oversampled graph filter banks is presented which allows us to design filters with arbitrary parameters, unlike the conventional critically-sampled graph filter banks. The practical performance of the proposed filter banks is validated through graph signal denoising experiments.

Index Terms— Graph signal processing, graph filter banks, graph wavelets, oversampled filter banks, graph signal denoising

1. INTRODUCTION

Signal processing on graphs is one of the emerging topics in signal processing [1–11]. Unlike regular signal processing[1], graph signal processing must explicitly consider the structure of the signal. In this context, well-studied signals, e.g., acoustic, image, and video signals, can be considered to be graph signals with very simple structures, and for this reason, tools for graph signal processing have received much attention.

Graph wavelet transforms have been developed for signals with structures. Some transforms require simplifications to be made to the graph, such as decomposition into even and odd nodes [5,6]. Recently, wavelet transforms without graph simplifications have been proposed [2–4]. A key technique of (digital) signal processing is down and upsampling, and it has been also studied for graph signals in the context of the *spectral folding phenomenon* [2, 3], which is analogous to the *aliasing* effect of regular signal processing. Since the spectral folding phenomenon affects bipartite graphs, the filter banks are designed to be *two-channel critically-sampled* ones. Unfortunately, these designs have strong limitations imposed on them if they are to be used to obtain critically-sampled perfect reconstruction filter banks. In contrast, M-channel ($M > 2$) filter banks would be useful for graph signals, since oversampled filter banks for regular signals have more freedom in their design and it has been shown that they outperform critically-sampled systems in several applications [12–17].

In this paper, M-channel oversampled graph filter banks are studied. For instance, perfect reconstruction is possible even if we use an *arbitrary* lowpass filter, and the filters we design have good stopband attenuation. As a possible application, we show how our oversampled graph filter bank can be used to *denoise graph signals*. A comparison of implementations of a simple hard-thresholding technique shows that the proposed filter bank outperforms the critically-sampled graph filter bank and the wavelet transform for regular signals.

The remaining of this paper is organized as follows. Preliminaries and notations are summarized in the rest of this section. Section 2 gives reviews of the existing works. Section 3 gives the perfect reconstruction condition of oversampled graph filter banks and their detailed design method is presented in Section 4. The design examples and experimental results for graph signal decomposition are shown in Section 5. Finally, Section 6 concludes the paper.

Preliminaries and Notations: In this paper, we consider a finite undirected graph $G = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V}$ and $\mathcal{E}$ represent sets of nodes and edges in the graph, respectively. Similar to [2, 3], we assume a graph without self-loops and multiple connections. The number of nodes is $N = |\mathcal{V}|$ unless specified. The (m, n)-th element in the $N \times N$ adjacency matrix $\mathbf{A}$ is w_{mn} if nodes m and n are connected, and 0 otherwise. The diagonal degree matrix $\mathbf{D}$ contains $d_{mm} = \sum_n a_{mn}$. With $\mathbf{A}$ and $\mathbf{D}$, the unnormalized graph Laplacian matrix (GLM) is defined as $\mathbf{L} := \mathbf{D} - \mathbf{A}$. In this paper, we consider the symmetric normalized GLM (SNGLM) $\boldsymbol{\mathcal{L}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$. As mentioned in [3], our filter bank is also applicable to the random-walk GLM $\boldsymbol{\mathcal{L}}_r = \mathbf{D}^{-1}\mathbf{L}$, without any changes.

The important symbols for the paper are listed below:

1. $\boldsymbol{f}$: Graph signal ($\boldsymbol{f} \in \mathbb{R}^N$)
2. $\boldsymbol{u}_{\lambda_i}$: i-th eigenvector of $\boldsymbol{\mathcal{L}}$
3. λ_i: i-th eigenvalue of $\boldsymbol{\mathcal{L}}$ ($\boldsymbol{\mathcal{L}}\boldsymbol{u}_{\lambda_i} = \lambda_i \boldsymbol{u}_{\lambda_i}$), where $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{N-1} = 2$ for bipartite graphs.
4. $\sigma(G)$: Spectrum of the graph, i.e., $\sigma(G) := \{\lambda_0, \ldots, \lambda_{N-1}\}$.

2. REVIEW

In this section, we briefly review the existing approaches of graph wavelets and filter banks.

2.1. Filter in Graph Spectral Domain

The eigenspace projection matrix of a graph is defined as follows:

$$\mathbf{P}_{\lambda_i} = \sum_{\lambda=\lambda_i} \boldsymbol{u}_\lambda \boldsymbol{u}_\lambda^T \tag{1}$$

where $\cdot^T$ is the transpose of the matrix. Note that $\boldsymbol{u}_{\lambda_i}$ is orthogonal to each other. That is,

$$\mathbf{P}_{\lambda_i}\mathbf{P}_{\lambda_j} = \delta(\lambda_i - \lambda_j)\mathbf{P}_{\lambda_i}, \tag{2}$$

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[1]In this paper, we often use the word *regular* signals/transforms to distinguish discrete signals on uniform points without explicit structures from *graph* signals discussed in this paper.

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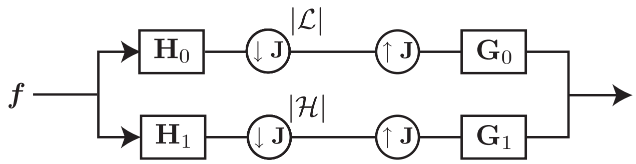


Fig. 1. Two-channel critically-sampled graph filter bank.

where $\delta(\cdot)$ is the Kronecker delta function. By using $\mathbf{P}_{\lambda_i}$, a spectral domain filter for graph signals can be defined as follows:

$$\mathbf{H} = \sum_{\lambda_i \in \sigma(G)} h(\lambda_i)\mathbf{P}_{\lambda_i} \tag{3}$$

where $h(\lambda)$ is the kernel of $\mathbf{H}$.

2.2. Critically-Sampled Graph Filter Banks

The downsampling and upsampling effects for the SNGLM of a bipartite graph have been studied in [2]. Note that an arbitrary graph can be decomposed into K_G disjoint bipartite graphs. Let us define a bipartite graph $G = \{\mathcal{L}, \mathcal{H}, \mathcal{E}\}$ where nodes in G are divided into two disjoint sets $\mathcal{L}$ and $\mathcal{H}$. We call the nodes in $\mathcal{L}$ the lowpass channel and those in $\mathcal{H}$ the highpass channel, for the sake of convenience.

Similar to the regular signals, the downsampling-then-upsampling operation can be defined as follows:

$$\mathbf{D}_{du,\mathcal{L}} = \frac{1}{2}(\mathbf{I}_N - \mathbf{J}), \quad \mathbf{D}_{du,\mathcal{H}} = \frac{1}{2}(\mathbf{I}_N + \mathbf{J}) \tag{4}$$

where $\mathbf{J}$ is a diagonal matrix with *antipodal* coefficients, i.e., $+1$ or -1, with the following form:

$$\mathbf{J}_{mm} = \begin{cases} +1 & \text{if } f(m) \text{ belongs to } \mathcal{H}, \\ -1 & \text{if } f(m) \text{ belongs to } \mathcal{L}. \end{cases} \tag{5}$$

Fig. 1 illustrates the entire transformation for one bipartite graph. It is similar to the regular filter banks [18–20], but the number of signals in the lowpass and highpass channels are no longer $N/2$ (or, more formally, $\lfloor N/2 \rfloor$ or $\lceil N/2 \rceil$): the lowpass channel contains $|\mathcal{L}|$ signals whereas the highpass channel retains $|\mathcal{H}|$ signals, where $|\mathcal{L}| + |\mathcal{H}| = N$. The number of signals in each channel is determined on the basis of the graph-coloring result.

The critically-sampled wavelet filter banks [2, 3] are designed to satisfy the following perfect reconstruction property:

$$\begin{aligned} \mathbf{T} &= \frac{1}{2}\mathbf{G}_0(\mathbf{I}-\mathbf{J})\mathbf{H}_0 + \frac{1}{2}\mathbf{G}_1(\mathbf{I}+\mathbf{J})\mathbf{H}_1 \\ &= \frac{1}{2}(\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1) + \frac{1}{2}(\mathbf{G}_1\mathbf{J}\mathbf{H}_1 - \mathbf{G}_0\mathbf{J}\mathbf{H}_0) = \mathbf{I}_N. \end{aligned} \tag{6}$$

where $\mathbf{H}_k = \sum_{\lambda_i \in \sigma(G)} h_k(\lambda_i)\mathbf{P}_{\lambda_i}$ is a filter in the analysis bank and $\mathbf{G}_k = \sum_{\lambda_i \in \sigma(G)} g_k(\lambda_i)\mathbf{P}_{\lambda_i}$ is one in the synthesis bank. In (6), the second term is called the *spectral folding* term, and it corresponds to *aliasing* in regular signals. Therefore, this spectral folding term must be zero. As a result, the critically-sampled perfect reconstruction graph filter bank must satisfy the following conditions:

$$\begin{aligned} g_0(\lambda)h_0(\lambda) + g_1(\lambda)h_1(\lambda) = 2 \\ g_0(\lambda)h_0(2-\lambda) + g_1(\lambda)h_1(2-\lambda) = 0. \end{aligned} \tag{7}$$

For the orthogonal solution [2], one prototype lowpass filter $h_0(\lambda)$ is used to yield the remaining $h_1(\lambda)$, $g_0(\lambda)$, and $g_1(\lambda)$. Moreover, $h_0(\lambda)$ has to satisfy $h_0^2(\lambda) + h_0^2(2-\lambda) = c^2$, where c is some constant. However, such an $h_0(\lambda)$ cannot be an exact polynomial, hence, exact perfect reconstruction and orthogonality are not possible [2, 3]. Instead, the biorthogonal solution [3] (hereafter, it is referred to as *graphBior*) is based on the spectral factorizations of the maximally-flat filter pair, which is the similar approach to that of the Cohen-Daubechies-Fauveau wavelet transform [21], and the designed filter bank strictly satisfies (6).

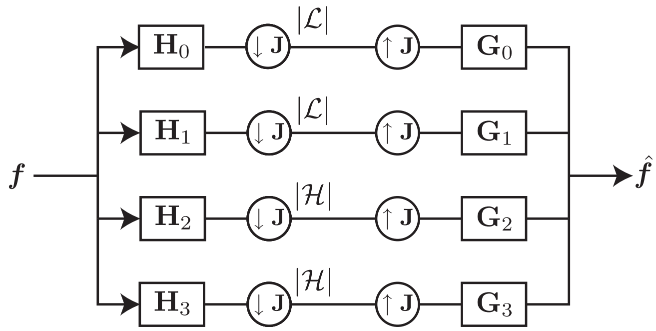


Fig. 2. Oversampled graph filter bank.

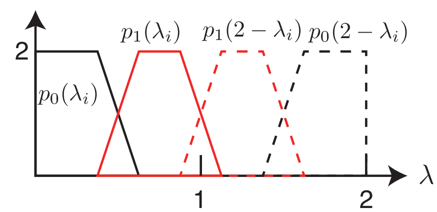


Fig. 3. Four-channel product filter example.

3. PERFECT RECONSTRUCTION CONDITION OF OVERSAMPLED GRAPH FILTER BANKS

The details of the perfect reconstruction condition is discussed in this section. For clearer understanding, first we present the case of $M = 4$, and it is extended to the M-channel case.

3.1. Four-Channel Case

Consider a four-channel graph filter bank shown in Fig. 2. After filtering by $\mathbf{H}_k$, the zeroth and first channels pass $|\mathcal{L}|$ signals, whereas the second and third ones keep $|\mathcal{H}|$ signals. $\hat{\boldsymbol{f}}_k$ is represented as

$$\hat{\boldsymbol{f}}_k = \begin{cases} \frac{1}{2}\mathbf{G}_k(\mathbf{I}-\mathbf{J})\mathbf{H}_k\boldsymbol{f} & k = 0, 1 \\ \frac{1}{2}\mathbf{G}_k(\mathbf{I}+\mathbf{J})\mathbf{H}_k\boldsymbol{f} & k = 2, 3. \end{cases} \tag{8}$$

Therefore, the overall transfer function $\mathbf{T}$ is

$$\begin{aligned} \mathbf{T} = &\frac{1}{2}\sum_{\lambda_i}\sum_{k=0}^{3} g_k(\lambda_i)h_k(\lambda_i)\mathbf{P}_{\lambda_i} \\ &+ \frac{1}{2}\sum_{\lambda_i}\{g_2(\lambda_i)h_2(2-\lambda_i) + g_3(\lambda_i)h_3(2-\lambda_i) \\ &\quad - g_0(\lambda_i)h_0(2-\lambda_i) - g_1(\lambda_i)h_1(2-\lambda_i)\}\mathbf{P}_{\lambda_i}\mathbf{J} \end{aligned} \tag{9}$$

which is similar to [2, 3]. As a result, the perfect reconstruction condition becomes

$$\sum_{k=0}^{3} g_k(\lambda)h_k(\lambda) = 2 \tag{10}$$

and

$$\begin{aligned} &g_2(\lambda)h_2(2-\lambda) + g_3(\lambda)h_3(2-\lambda) \\ &- g_0(\lambda)h_0(2-\lambda) - g_1(\lambda)h_1(2-\lambda) = 0 \end{aligned} \tag{11}$$

for any λ. (11) is satisfied if we use the constraints $g_0(\lambda) = h_2(2-\lambda)$, $g_1(\lambda) = h_3(2-\lambda)$, $g_2(\lambda) = h_0(2-\lambda)$, and $g_3(\lambda) = h_1(2-\lambda)$ (similar to what is done in [3]). Let us define a product filter as $p_k(\lambda) = g_k(\lambda)h_k(\lambda)$. Finally, (10) can be rewritten as

$$p_0(\lambda) + p_0(2-\lambda) + p_1(\lambda) + p_1(2-\lambda) = 2. \tag{12}$$

By this perfect reconstruction condition, we can select *four-channel* product filters instead of two-channel systems of the critically-sampled graph filter bank. It is shown in Fig. 3.

3.2. General M-Channel Case

Let us assume that an oversampled graph filter bank has M channels where M is even. Additionally, $M/2$ filters keeps $|\mathcal{L}|$ signals and the other ones keep $|\mathcal{H}|$ signals. With the similar derivations of the previous subsection, the perfect reconstruction condition is

$$\sum_{k=0}^{M-1} g_k(\lambda)h_k(\lambda) = 2 \tag{13}$$

$$\sum_{k=0}^{M/2-1} g_{k+M/2}(\lambda)h_{k+M/2}(2-\lambda) - g_k(\lambda)h_k(2-\lambda) = 0 \tag{14}$$

for any λ. The latter equation is satisfied when we choose $g_k(\lambda) = h_{k+M/2}(2-\lambda)$ and $g_{k+M/2}(\lambda) = h_k(2-\lambda)$. Then (13) becomes

$$\sum_{k=0}^{M/2-1} g_k(\lambda)h_k(\lambda) + g_k(2-\lambda)h_k(2-\lambda) = 2. \tag{15}$$

As a result, the product filter $p_k(\lambda)$ must satisfy the following condition:

$$\sum_{k=0}^{M/2-1} p_k(\lambda) + p_k(2-\lambda) = 2. \tag{16}$$

4. DESIGN METHOD OF M-CHANNEL OVERSAMPLED GRAPH FILTER BANKS

First we consider the case of $M = 4$. Let us define $q(\lambda) = p_0(\lambda) + p_1(\lambda)$. (12) can be rewritten as

$$q(\lambda) + q(2-\lambda) = 2, \tag{17}$$

This equation is the same as that of a two-channel biorthogonal graph filter bank [3]. Therefore, the design problem boils down to separating the critically-sampled product filter $q(\lambda)$ into lowpass and bandpass (Fig. 3) filters $p_0(\lambda)$ and $p_1(\lambda)$ such that the *sum* of filters is $q(\lambda)$.

Let us assume that a lowpass product filter $p_0(\lambda)$ is arbitrarily chosen so that $h_0(\lambda)$ and $g_0(\lambda)$ are "good" lowpass filters with the cutoff frequency $\lambda = 0.5$. By changing the variable of $\lambda = 1 + l$ [3], $p_0(1+l)$ can be expressed as

$$p_0(1+l) = (1+l)^K \left(\frac{1}{2} + \sum_{m=1}^{K-1} \alpha_m l^m \right), \tag{18}$$

where α_m is an arbitrary parameter. From the halfband condition [3], the even degree of $q(\lambda) = p_0(\lambda) + p_1(\lambda)$ must be zero.[2] To *compensate* the even powers in $p_0(1+l)$, $p_1(1+l)$ is defined as follows:

$$p_1(1+l) = (1+l)^K \left(\frac{1}{2} + \sum_{m=1}^{K-1} \beta_m l^m \right), \tag{19}$$

where β_m is determined in such a way as to cancel the even powers in $p_0(1+l)$. Clearly $q(1+l)$ becomes

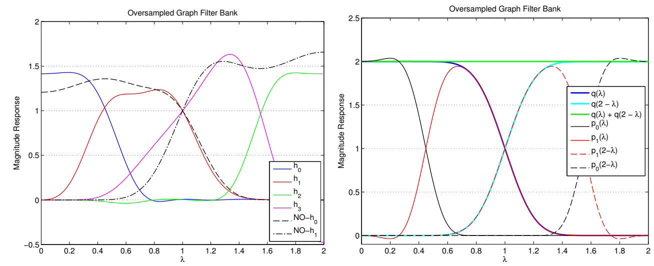
$$q(1+l) = (1+l)^K \left(1 + \sum_{m=1}^{K-1} (\alpha_m + \beta_m) l^m \right) \tag{20}$$

and its degree is $2K-1$, thus $q(\lambda)$ is a *maximally-flat* filter.

A similar derivation is possible for the general M-channel graph filter banks. In that case, parameters for $(M-2)/2$ product filters can be freely chosen, and the last product filter can be designed so that the entire product filter $q(\lambda)$ is a maximally-flat halfband filter.

5. DESIGN EXAMPLES AND EXPERIMENTAL RESULTS

In this section, we show the design methodology of M-channel oversampled graph filter banks and a few design examples.



(a) (b)

Fig. 4. Four-channel oversampled graph filter banks with $(k_0, k_1) = (4, 4)$. (a) analysis filter bank. Black lines indicate graphBior(6,6) [3]. (b): halfband filters.

Table 1. Denoised Results of Minnesota Traffic Graph: SNR (dB)

σ	1/2	1/4	1/8	1/16	1/32
noisy	6.01	12.08	17.99	24.04	30.04
sym8 (1 level)	6.17	11.92	18.58	24.28	30.01
sym8 (5 levels)	5.71	10.97	17.98	24.06	30.09
graphBior(6, 6)	8.42	14.25	20.00	25.57	31.25
OSGFB	**10.15**	**15.36**	**21.73**	**28.77**	**34.55**

[2] Although *Proposition 1* in [3] has a restriction that $q(\lambda)$ is a *product* of two kernels, it is also applicable for a *sum* of two kernels assumed in this paper.

5.1. Design Methodology

As mentioned above, we can use arbitrary parameters to design filters. In what follows, we will use a sequential design method to obtain good filter banks:

1. Design $h_k(1-l)$ and $g_k(1-l)$ ($k = 0, \dots, M/2-2$) with k_0 and k_1 zeros at $l = 1$ ($\lambda = 2$). They are represented as follows:

$$h_k(1-l) = (1-l)^{k_0} \sum_{j=0}^{J_h-1} s_{h,k,j} l^j \tag{21}$$

$$g_k(1-l) = (1-l)^{k_1} \sum_{j=0}^{J_g-1} s_{g,k,j} l^j \tag{22}$$

   where $s_{h,k,j}$ and $s_{g,k,j}$ are filter coefficients. The numbers of arbitrary parameters in $h_k(1-l)$ and $g_k(1-l)$ are J_h and J_g, respectively. The filters are optimized by using the cost function of the stopband attenuation shown below:

$$\begin{aligned} C(h_k) =& w_0 \int_{l\in\omega_p} (\sqrt{2} - h_k(1-l))^2 dl \\ &+ w_1 \int_{l\in\omega_s} h_k^2(1-l) dl, \end{aligned} \tag{23}$$

   where w_0 and w_1 are weights and ω_p and ω_s are defined as the passband and stopband ($-1 \le \omega_p, \omega_s \le 1$), respectively.
2. Calculate two-channel halfband filter pair $q(1-l) = q(\lambda)$ and $q(1+l) = q(2-\lambda)$ with $K = k_0 + k_1$ zeros at $l = 1$ so that the pair satisfies (17).
3. Calculate the bandpass product filter

$$\begin{aligned} p_{\frac{M}{2}-1}(1-l) &= q(1-l) - \sum_{k=0}^{M/2-2} p_k(1-l) \\ &= (1-l)^K \tilde{p}_{\frac{M}{2}-1}(1-l), \end{aligned} \tag{24}$$

   where $p_k(1-l) = g_k(1-l)h_k(1-l)$.
4. Factorize $p_{\frac{M}{2}-1}(1-l)$ into two bandpass filters $h_{\frac{M}{2}-1}(1-l)$ and $g_{\frac{M}{2}-1}(1-l)$. Test all combinations of roots as long as both bandpass filters have real-valued coefficients, and then the best combination, i.e., the filters minimizing $C(h_{\frac{M}{2}-1}) + C(g_{\frac{M}{2}-1})$, are selected.

Fig. 4 shows an example of oversampled graph filter banks. The arbitrary lowpass filters $h_0(\lambda)$ and $g_0(\lambda)$ are designed to have the degree 10 and 11, respectively. For comparison, the frequency responses of the critically-sampled graphBior(6, 6) [3] are also plotted. They have 13-taps for the lowpass filter and 12-taps for the highpass filter. It is clear that our oversampled lowpass filter has a shaper transition band and a more uniform response in the passband than the critically-sampled graph filter banks.

5.2. Denoising of Graph Signal

Here we show the potential ability of using oversampled graph filter banks to remove additive white Gaussian noise from graph signals. The oversampled graph filter bank is compared with graphBior(6, 6) [3] and regular one-dimensional wavelet *sym8*, which can be found in the Wavelet Toolbox in MATLAB. For the regular wavelet transform, the input signal $\boldsymbol{f}$ is treated as a vector, and one-level and five-level dyadic decompositions are performed. Only one level transform is used for the graph filter banks. All methods retain the lowest-frequency subband and the remaining high-frequency subbands are hard-thresholded with $T = 3\sigma$, where σ is the standard deviation of noise.

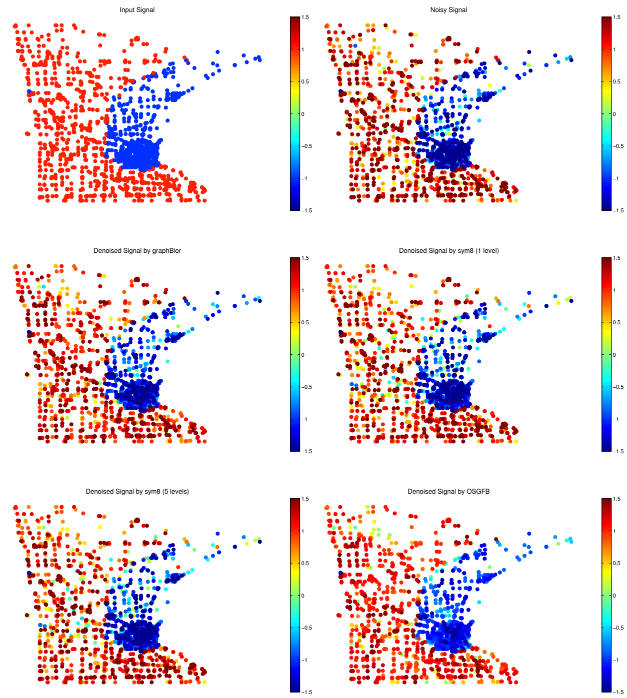


Fig. 5. Denoising results. From left to right, top to bottom: Original signal, noisy signal ($\sigma = 1/2$), denoised signal by graphBior(6,6) [3], denoised signal by sym8 (1 level), denoised signal by sym8 (5 levels), and denoised signal by oversampled graph filter bank.

Fig. 5(a) shows the original signal of the *Minnesota Traffic Graph*, where the signal value is $\{-1, 1\}$, and Table 1 summarizes the denoising performances. As expected, graph filter banks perform much better than the regular wavelet transform. Furthermore, our proposed oversampled graph filter bank outperforms graphBior by 1–3 dB in SNR.

The denoised signals for $\sigma = 1/2$ are shown in Fig. 5. Since the regular wavelet transform does not consider the structure of signals, the signals are over-smoothened across the boundary of the center and surrounding areas; many blue points appear in the surrounding area. In contrast, graph filter banks preserve the solid boundary. It is clear that the proposed filter bank performs better than the critically-sampled one.

6. CONCLUSIONS

We presented the design method of M-channel oversampled filter banks for graph signals. It satisfies the perfect reconstruction condition, and allows us to use arbitrary parameters, unlike critically-sampled graph filter banks. it was shown to outperform other transforms, including regular wavelet transforms, in a graph signal denoising experiment.

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