SINGLE SNAPSHOT DOA ESTIMATION USING COMPRESSED SENSING

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ABSTRACT

This paper deals with the problem of estimating the Directions of Arrival (DOA) of multiple source signals from a single observation of an array data. In particular, an estimation algorithm based on the emerging theory of Compressed Sensing (CS) is analyzed and its statistical properties are investigated. We show that, unlike the classical Fourier beamformer, a CS-based beamformer (CSB) has some desirable properties typical of the adaptive algorithms (e.g. Capon and MUSIC). Particular attention will be devoted to the super-resolution property. Theoretical arguments and simulation analysis are provided in order to prove that the CSB can achieve a resolution below the classical Rayleigh limit.

***Index Terms*—** DOA estimation, Fourier Beamformer, Compressive Sensing, super-resolution.

1. INTRODUCTION

The problem of estimating the Directions of Arrival (DOA) of a certain number of sources has been an active research area for decades [1], with applications to monostatic and multistatic radar systems ([2], [3], [4], [5], [6]) and remote sensing ([7], [8]). The first approach to carrying out space processing (i.e. DOA estimation) from data sampled by an array of sensors was the well-known Fourier beamformer (FB). However, the main drawbacks of the FB are the high level of the secondary lobes and a poor angular resolution [8]. In fact, the FB suffers from the Rayleigh resolution limit that is independent of the SNR. In order to overcome these limits, adaptive beamformers (such as Capon [9] and MUSIC [10]) has been proposed and their performance investigated, also in the presence of multiplicative noise ([7], [8]) and of array errors [11]. However, most of these adaptive algorithms rely on asymptotic assumptions (e.g. large number of snapshots). In many practical applications, for example in sonar processing, due to physical constraints (e.g. sound speed), only a very small number of snapshots or, in the worst case, a single snapshot is available for DOA estimation ([12], [13]). In the single snapshot scenario, all the adaptive algorithms, which rely on an estimate of the noise covariance matrix (e.g. the Sample Covariance Matrix), cannot be used since it is rank deficient. Recently, new estimation algorithms, based on the emerging field of the Compressed Sensing (CS) theory have been proposed in the array processing literature (see e.g. [14], [15] and [16]). The aim of this paper is then to investigate the statistical properties of these CS-based beamformers (CSBs). Such analysis is carried out in the single snapshot scenario, that is of practical relevance in sonar applications. The multi-snapshot scenario is left to future works. The attention here is paid to two statistical properties: the efficiency and the resolution. It is shown that a CS-based DOA estimator is able to guarantee a some desirable properties, in particular the super-resolution property, typical of the adaptive estimation algorithms. In Section II a brief introduction to the classical FB and to the CSB is provided. In Section III, the DOA estimation problem is described and a comparison between the FB and the CSB is provided. The super-resolution property of the CSB is investigated in Section IV and conclusions are made in Section V.

2. BACKGROUD

2.1. The measurement model

Assume a Uniformly Linear Array (ULA) of N omni-directional sensors spaced by p and a single narrowband source impinging on the array from conic angle $\bar{\theta}$. In the narrow band case, the array vector snapshot can be modeled as [1], [17]:

$$\mathbf{y} = \rho \mathbf{v}(\bar{\nu}_s) + \mathbf{n}, \qquad (1)$$

where $\bar{\nu}_s = p/\lambda_0 \sin\bar{\theta}$ is the spatial frequency, λ_0 is the wavelength of the transmitted signal, $\mathbf{v}(\bar{\nu}_s) = [1, \exp(j2\pi\bar{\nu}_s), \cdots, \exp(j2\pi(N-1)\bar{\nu}_s)]$ is the $N\times 1$ steering vector toward the direction $\bar{\theta}$ and $\mathbf{n}$ is the $N\times 1$ vector of the complex Gaussian sensor noise with zero-mean and diagonal covariance matrix, $\mathbf{n} \in \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I})$. Finally, $\rho \in \mathcal{CN}(0, \sigma_s^2)$ is a complex Gaussian random variable that accounts for the transmitted complex amplitude, the radiation pattern of the array sensors, the two-way path loss, and the sonar or radar cross section of the slowly-fluctuating source. The model in eq. (1) is relative to a single source, but it can be generalized to a multi-source scenario [1], [14].

2.2. The Fourier beamformer (FB)

Under the white noise assumption the noise covariance matrix is diagonal and the maximum likelihood (ML) estimator for $\bar{\nu}_s$ is given by the location of maximum of the data Periodogram [1]:

$$\hat{\nu}_{s,F} = \arg\max_{\nu_s} \left| \sum_{n=0}^{N-1} y_n e^{-j2\pi n \nu_s} \right|^2 = \arg\max_{\nu_s} p(\nu_s) . \quad (2)$$

This estimator is known as the Fourier beamformer (FB).

2.3. The CS-based beamformer (CSB)

As shown in [14], the measurement model in eq. (1) can be recast in a "sparse" representation by defining an overcomplete dictionary of steering vectors evaluated over a set of possible spatial frequencies $\Omega = \{\nu_s^1, \ldots, \nu_s^G\}$. Consequently, an ovecomplete representation matrix can be set up by collecting all the G steering vectors in a matrix

$$\mathbf{A}(\Omega) = [\mathbf{v}(\nu_s^1) | \cdots | \mathbf{v}(\nu_s^G)], \quad (3)$$

where the representation matrix $\mathbf{A}$ do not depend on the actual source spatial frequency $\bar{\nu}_s$. In this framework, the source signal has to be recast as an $G\times 1$ column vector $\mathbf{x}$ where the g^{th} entry is equal to ρ if the source has a spatial frequency equal to ν_s^g and zero otherwise. Since the cardinality G of the dictionary (i.e. the number of grid points used to cover the spatial frequency space) is much larger than the number of possible sources, then the vector $\mathbf{x}$ is sparse. Finally, the measurement model of eq. (1) can be recast in the well-known linear CS measurement model:

$$\mathbf{y} = \mathbf{A}(\Omega)\mathbf{x} + \mathbf{n}. \quad (4)$$

Estimating the vector $\hat{\mathbf{x}}$ from eq. (4), is equivalent to estimate the spatial energy as a function of the set of assumed spatial frequencies Ω. Under the assumptions that the number of sources in the scenario is much smaller than the number of the frequency grid points, the spatial spectrum is sparse. Then, a CS-based estimation of $\bar{\nu}_s$ is given by:

$$\hat{\nu}_{s,CS} = \arg\max_{\Omega} \hat{\mathbf{x}}_{CS}(\Omega), \quad (5)$$

where a sparse estimate of $\mathbf{x}$ is obtained from the measurement vector $\mathbf{y}$ by solving the constrained optimization problem:

$$\hat{\mathbf{x}}_{CS}(\Omega) = \arg\min_{\mathbf{x}\in\mathbb{C}^G} \|\mathbf{x}\|_1 \quad \text{s.t.} \ \|\mathbf{A}(\Omega)\mathbf{x} - \mathbf{y}\|_2 \le \sigma_n . \quad (6)$$

Before proceeding, it is worth noting that in the measurement model of eq. (4), the target frequency to be estimated is handled as a discrete parameter. Recent works deal with the more challenging case of continuous parameter space (see e.g. [18] and [19]).

3. DOA ESTIMATION

In this section, the efficiency of the FB and of the CSB is investigated by comparing their $\text{RMSE} = E\{(\hat{\nu}_s - \bar{\nu}_s)^2\}^{1/2}$ for the DOA estimation with the Cramér-Rao Lower Bound (CRLB) as function of the Signal-to-Noise Ratio (SNR).

3.1. Efficiency

Following [17] and [20], the CRLB on the DOA estimation for the random signal model in (1) can be expressed as:

$$\text{CRLB}(\hat{\nu}_s) = \frac{\sigma_n^2}{2\,\text{Re}\{hl\}}, \quad (7)$$

$h = \mathbf{d}^H[\mathbf{I} - \mathbf{v}\mathbf{v}^H/M]\mathbf{d}$, $l = \sigma_s^4 \mathbf{v}^H \mathbf{R}^{-1}\mathbf{v}$, $\mathbf{d} = d\mathbf{v}(\nu_s)/d\nu_s |_{\nu_s=\bar{\nu}_s}$, $\mathbf{R} = \sigma_s^2 \mathbf{v}\mathbf{v}^H + \sigma_n^2 \mathbf{I}$ where, for notation simplicity, $\mathbf{v} \triangleq \mathbf{v}(\bar{\nu}_s)$. The FB is evaluated using the FFT on 2^{13} point, while the number of grid-points G for the CSB is chosen to be equal to 180. The source spatial frequency is $\bar{\nu}_s = 0.2278$ and the number of independent Monte Carlo trials used to evaluate the RMSE is 10^4. In Fig. 1, we see that the FB and the CSB have similar performance in terms of RMSE. Moreover, it can be noted that, below 30dB of SNR, both the estimators are in the "low SNR" region where the CRLB is not tight. From 30dB to 40dB, the estimators are close to the CRLB. It is important to note that $\bar{\nu}_s$ is chosen to belong to Ω. If this condition is not satisfied, the CSB could present a bias in the estimation of the DOA. This is the so-called 'off-grid' effect. It should be highlighted that the same effect can occur also using the FB if the true DOA $\bar{\nu}_s$ is not on the FFT grid.

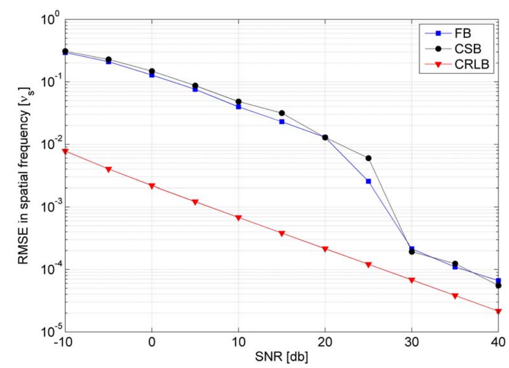

Figure 1 – RMSE of the FB and of the CSB and the CRLB vs SNR.

3.2. Off-grid effects

In order to evaluate the loss in estimation accuracy due to the off-grid effects for both the FB and CSB, in this subsection, the source spatial frequency is generated by sampling a uniform distribution between -0.5 and 0.5, i.e. $\bar{\nu}_s \in \mathcal{U}([-0.5, 0.5))$. The RMSE of both the FB and the CSB is evaluated using 10^3 Monte Carlo trials on the same grid of 2^5, 2^7 and 2^9 points.

As shown in Fig. 2, the RMSE of the FB and of the CSB are very close to each other for a given number of grid-points. Moreover, when the SNR exceeds the value of 40dB, the bias error, due to the finite number of grid-points becomes evident.

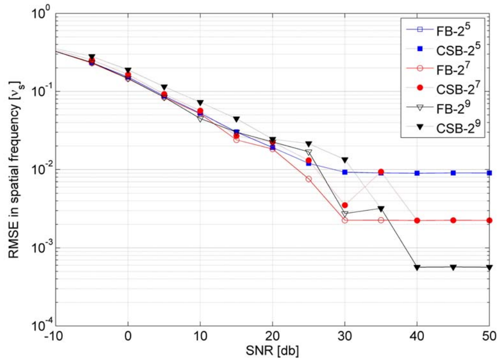

Figure 2 – RMSE for different number of grid-points.

4. SUPER-RESOLUTION PROPERTY

FB suffers from the Rayleigh resolution limit, which is independent of the SNR. Some adaptive methods, e.g. MUSIC and Capon estimators, are able to resolve sources within a Rayleigh cell, but they need a sufficiently high SNR level and a suitable number of snapshots to estimate the disturbance covariance matrix. It would be very useful then to have an algorithm that is able to achieve the super-resolution even if a single snapshot is available. It is well-known that, for an ULA of N array elements, the Rayleigh resolution limit, i.e. the beamwidth in the spatial frequency space defined as the full width of the mainlobe at the half-power level, takes the simple form [1]:

$$\Delta\nu = 0.886/N\,. \tag{8}$$

Then, if two targets are spaced by less than the Rayleigh resolution limit in eq. (8), they cannot be resolved by a classical FB. On the other hand, it can be shown that the CSB is able to resolve targets that are in the same Rayleigh resolution cell [14]. A qualitative proof of this statement is provided in Fig. 3.

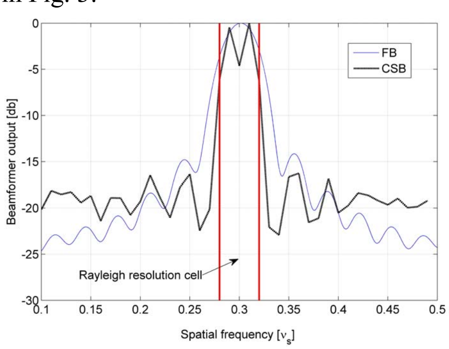

Figure 3 – The super-resolution property: two sources in the same Rayleigh resolution cell (SNR = 10db).

Even if the fact that the CSB can achieve the super-resolution is known in literature (see, e.g. [14]), neither a strong theoretical justification nor a statistical characterization of this property is shown. This paper represents a first attempt to provide a theoretical definition of "CS super-resolution limit" and to give a statistical characterization of this property using classical tools of the decision theory already exploited to characterize the super-resolution property of the classical adaptive beamformers (e.g. MUSIC). The theoretical justification to the fact that the CSB can achieve super-resolution, even using a single snapshot, can be found in a fundamental result of CS theory. Under the sparsity assumption, it can be proved [21] that it is possible to reconstruct (with overwhelming probability) a complex signal **x** from a very low number of its Fourier coefficients. This general result of the CS theory can be easily reformulated in the array processing framework as:

Theorem 1 [21] (Array processing formulation): *Let K and N be the number of targets and the number of array elements, respectively. Let $\Omega = \{\nu_s^1, \dots, \nu_s^G\}$ be the set (of cardinality $|\Omega| = G$) of spatial frequencies in the grid and let $\mathbf{A}(\Omega) = [\mathbf{v}(\nu_s^1) \,|\, \cdots \,|\, \mathbf{v}(\nu_s^G)]$ the overcomplete matrix of steering vectors on Ω. If the sparse energy signal* **x** *(and consequently the source DOA) is recovered from the single (noise free) data snapshot* **y** *by solving the following optimization problem:*

$$\hat{\mathbf{x}}_{CS}(\Omega) = \arg\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{x}\|_1\,, \quad \text{s.t. } \mathbf{y} = \mathbf{A}(\Omega)\mathbf{x} \tag{9}$$

where

$$|\Omega| = G \le e^{\frac{C_d N}{K}}\,, \tag{10}$$

then, with probability at least $\eta = 1 - O(G^{-d})$, the solution of the problem in eq. (9) *is unique and is equal to* **x**.

The value η represents the probability of exact reconstruction $P_{ex} \triangleq \Pr\{\mathbf{x} = \hat{\mathbf{x}}_{CS}\} = \eta$. The value of C_d is explicitly derived in [21] under asymptotic conditions (i.e. valid for $N \le G/4$, $d \ge 2$ and $G \ge 20$) as $C_d = 1/(23(d+1))$. In particular, the value of d in C_d depends on the desired value of P_{ex}. The fundamental link between Theorem 1 and the super-resolution property is now clear: if it is possible to reconstruct **x** on a (discrete) support of cardinality G with probability at least η then, with probability at least η, it is also possible to resolve two targets spaced by

$$\widetilde{\Delta\nu} \ge e^{-\frac{C_d N}{K}}\,. \tag{11}$$

It can be seen, that while the Rayleigh resolution limit in eq. (8) decreases as N^1, the CS super-resolution limit in eq. (11) decreases as $\exp(-C_d N/K)$. In particular, given a fixed number of array elements N, the minimum possible spatial frequency separation between two sources is at least $\Delta\nu = 0.886 N^{-1}$ if the FB is used while, using the CSB, it is of at least $\widetilde{\Delta\nu} = \exp(-C_d N/2)$. It is easy to show that the inequality $\widetilde{\Delta\nu} < \Delta\nu$ holds true for a large set of value of d, but of course, not for all values. In fact, as stated in [21], the asymptotic value of C_d provided by Theorem 1 is a very conservative value. In other word, the same value of P_{ex} can be guaranteed by a smaller value of C_d than the one provided in Theorem 1 and then a finer resolution can be achieved. Additional theoretical investigations are necessary to refine the value of C_d, and this is an active research area among the CS community. Moreover, it must be underlined that Theorem 1 refers to the noise free case. In array processing, however a certain amount of noise is always present, then the optimization problem in eq. (9), should be replaced with the problem in eq. (6). Of course, in the noisy case, one cannot expect exact recovery [16] and Theorem 1 is not valid in the present form. However, in the following, the robustness of the theorem results on CS super-resolution is verified against the measurement noise as function of the SNR. To perform the super-resolution analysis of the CSB in the presence of noise, a resolution criterion is needed. In [22], the following random inequality is used to define a super-resolution event:

$$\gamma\left(\nu_s^1,\nu_s^2\right) \triangleq \frac{1}{2}\left[\hat{\mathbf{x}}_{CS}(\nu_s^1)+\hat{\mathbf{x}}_{CS}(\nu_s^2)\right]-\hat{\mathbf{x}}_{CS}\left(\nu_s^m\right)>0, \quad (12)$$

where:

$$\nu_s^m=(\nu_s^1+\nu_s^2)/2\,. \quad (13)$$

Two sources located at spatial frequencies ν_s^1 and ν_s^2 are said to be resolvable if the inequality in eq. (12) holds true and to be irresolvable otherwise. This problem can then be seen as a binary decision problem, where γ is the decision statistic. The probability of resolution (P_{res}) can then be defined as:

$$P_{res}=\Pr\{\gamma>0\}\,. \quad (14)$$

In the following, P_{res} is evaluated as a function of the SNR, and of the source separation in angular frequency.

4.1. P_{res} vs SNR

Fig. 4 shows the probability of resolution at various SNR's. The following parameters are used in the simulation: $G = 2^7$, $\nu_s^m=0.3$, $\nu_s^1=0.2922$, $\nu_s^1=0.3078$ and the number of independent Monte Carlo trials is 10^4. For the FB, the simulation results merely confirm the theory; in fact, its P_{res} is very low and independent of the SNR. On the other hand, the P_{res} of the CBS is always higher than the one relative to the FB and tends to 1 as the SNR increases.

4.2. P_{res} vs frequency separation

In Fig. 5, the probability of resolution is evaluated as function of the source separation for a SNR = 10db and with $G = 2^9$. The frequency separation, denoted here as $\Delta=i/G$ for i=1,2,…,$L<G$ is defined with reference to $\nu_s^m=0.3$ so that the spatial frequencies of the two sources are given by $\nu_s^1=\nu_s^m-\Delta$ and $\nu_s^2=\nu_s^m+\Delta$. Again, the P_{res} of the CSB is always higher than the one of the FB and tends to 1 right beyond the Rayleigh limit. However, by decreasing the sources separation also the P_{res} of the CSB decreases. This is an expected behavior. In fact, when the source separation goes below the CS super-resolution limit in eq. (11), i.e. $\Delta<\widetilde{\Delta\nu}/2$, the two sources are no longer resolvable.

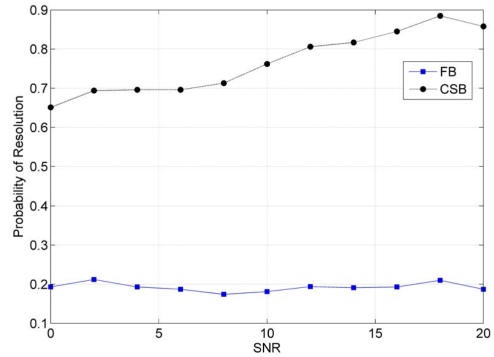

Figure 4 – Probability of Resolution vs SNR (# of grid-points = 2^7).

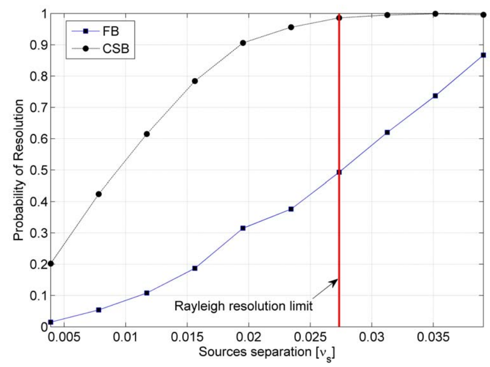

Figure 5 – Probability of resolution vs the frequency separation between two sources (# of grid-points = 2^9).

5. CONLUSIONS

In this paper, statistical efficiency and super-resolution properties of the CSB beamformer have been tested against the classical FB beamformer to estimate the DOA of a target in the single-snapshot scenario. Off-grid effects on the RMSE estimate have been investigated, as well. The efficiency of the CSB estimator is of the same order of magnitude of the FB one and asymptotically reaches the CRLB in the random signal case for a SNR greater than 30dB. The CSB estimator achieves super-resolution beyond the Rayleigh limit with a single pulse, while classical super-resolution algorithms like MUSIC need multiple snapshots. Theoretical arguments have been proposed to link the super-resolution property of the CSB estimator with the CS theory. Off grid effects on the DOA RMSE for the CSB and the FB beamformers are similar, when compared as a function of the SNR and of the number of grid points. Future research efforts will be directed toward the application to sonar real data and additional theoretical insights concerning super-resolution properties.

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