DOA ESTIMATION PERFORMANCES OF MULTI-PARAMETRIC MUSIC IN PRESENCE OF MODELING ERRORS - CASE OF COHERENT MULTI-PATHS

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ABSTRACT

The purpose of this paper is to give a closed form expression of the RMS (Root Mean Square) error of the estimated DOA (Direction Of Arrival) for multi-parametric MUSIC in presence of a modeling error. The multi-parametric MUSIC approach, [1] firstly introduced by [2] in polarization diversity context, estimates with a subspace approach the sources DOAs jointly to the nuisance parameters such as the polarization vector. The results are based on a second order approximation of the multi-parametric criterion with respect to modeling errors. DOA estimation errors is then an Hermitian form of multi-variate complex random variables. Theoretical results are validated by simulations in the context of coherent multi-paths in polarizations diversity.

***Index Terms*—** DOA estimation, polarization , coherent paths, performances, modeling error

1. INTRODUCTION

The subspace-based estimation of direction of arrival of radio-electric sources using an array of spatially distributed antennas has been intensively studied these last decades with MUSIC algorithm [3]. The paper focuses on the performances of such algorithms in multi-parameters context [1][11] where the DOAs have to be jointly estimated with some nuisance parameters. Indeed in some applications, the steering vector can be factorized with respect to DOA and nuisance parameters allowing a concentration of the MUSIC criterion with respect to the DOA. This approach based on an optimization of a quadratic form [4] has been first introduced with array of antennas in diverse polarization [2]. Other distortions of the wave-front have also been considered such as coherent local scattering on DOA [5] and self calibration [6][1] of mutual coupling coefficients [7]. The approach can also be used also in the context of coherent multi-paths [8][9] where the nuisance parameters are the multi-paths amplitudes and where their directions are jointly estimated. Thus, the algorithm [1] is able to mix these different distortions in a single nuisance vector that can be composed by mutual coupling coefficients, polarization coefficients or amplitudes of coherent multi-paths.

The theoretical performances of [1] in presence of modeling error have been proposed in [10] in a context of non coherent multi-paths with a two components nuisance vector. The numerical results have been given for self calibration where a single mutual coupling coefficient is estimated jointly to source direction. The theoretical approach in [10] can be extended to a nuisance vector with more than two components. However, it will conduct to inextricable calculations and expressions. Thus, the purpose of this paper is to give these performances with an other approach in order to obtain tractable results independently of the length of the nuisance vector. In addition, these theoretical results are extended to coherent multi-paths context where the directions of a set of coherent multi-paths must been jointly estimated. In order to calculate the theoretical performances of the multi-parametric MUSIC [1], a local Taylor expansion of the associated criterion is first proposed to obtain a closed form expression of the DOAs error. In a second step, the second order approximation of the MUSIC noise projector with respect to modeling errors of [12] is used. The DOAs estimation error can then be written as an Hermitian form of multi-variate complex random variables where the associated statistics are given by [12].

2. SIGNAL MODELING AND PROBLEM FORMULATION

According to the modeling of [8], the signal at the output of an array of antenna is a noisy mixture of R uncoherent sources associated to a set of coherent multi-paths. The associated observation vectors, $\mathbf{x}(t)$, whose components $x_n(t)$ $(1 \le n \le N)$ are the complex envelopes of the signals at the output of the antennas, is thus given by

$$\mathbf{x}(t) = \sum_{i=1}^{R} \tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i)\, s_i(t) + \mathbf{n}(t) = \tilde{\mathbf{B}}\,\mathbf{s}(t) + \mathbf{n}(t) \quad (1)$$

where $\mathbf{s}(t) = [s_1(t) \cdots s_R(t)]^T$ (^{T} denotes the transpose operator), $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_R]$ with $\tilde{\mathbf{b}}_i = \tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i)$, $s_i(t)$ is the complex envelope of the i^{th} source, $\mathbf{n}(t)$ is a spatially white noise vector and $\tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i)$ is the steering vector of the i^{th} coherent set, such that

$$\tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i) = \sum_{p=1}^{P_i} \alpha_{pi} \tilde{\mathbf{a}}(\theta_{pi}, \boldsymbol{\varphi}_{pi})$$

where $\Theta_i = [\theta_{1i} \cdots \theta_{P_i i}]^T$, $\boldsymbol{\eta}_i = [\alpha_{1i} \times \boldsymbol{\varphi}_{1i}^T \cdots \alpha_{P_i i} \times \boldsymbol{\varphi}_{P_i i}^T]^T$ and θ_{pi} is the direction of the p^{th} multi-paths, α_{pi} the attenuation and $\boldsymbol{\varphi}_{pi}$ the associated nuisance parameters that can be the path polarization. The estimation problem under consideration is to jointly estimate the directions θ_{pi} for $1 \leq p \leq P_i$ of each coherent paths set with the multi-parametric MUSIC algorithm [1] in presence of modeling error. The multi-parametric steering vector of a set of P coherent paths is

$$\mathbf{b}(\Theta, \boldsymbol{\eta}) = \sum_{i=1}^{P} \alpha_i \mathbf{a}(\theta_i, \boldsymbol{\varphi}_i) = \mathbf{V}(\Theta)\, \boldsymbol{\eta} \tag{2}$$
$$\mathbf{V}(\Theta) = \left[\begin{array}{ccc} \mathbf{U}(\theta_1) & \cdots & \mathbf{U}(\theta_P) \end{array}\right]$$
$$\boldsymbol{\eta} = \left[\begin{array}{ccc} \alpha_1 (\boldsymbol{\varphi}_1)^T & \cdots & \alpha_P (\boldsymbol{\varphi}_P)^T \end{array}\right]^T$$

where $\mathbf{a}(\theta_i, \boldsymbol{\varphi}_i) = \mathbf{U}(\theta_i)\, \boldsymbol{\varphi}_i$ is the steering vector of a single path such that $\|\mathbf{a}(\theta_i, \boldsymbol{\varphi}_i)\|^2 = \mathbf{a}(\theta_i, \boldsymbol{\varphi}_i)^H \mathbf{a}(\theta_i, \boldsymbol{\varphi}_i) = N$ (^{H} denotes the transpose and conjugate). If the polarization is the only nuisance, the parameters vector $\boldsymbol{\varphi}_i$ is the 2×1 polarization vector. The steering vector modeling errors $\mathbf{e}_i$ of the i^{th} set of coherent sources is then

$$\mathbf{e}_i = \tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i) - \mathbf{b}(\Theta_i, \boldsymbol{\eta}_i) = \sum_{p=1}^{P_i} \alpha_{pi} \mathbf{e}_{pi} \tag{3}$$

where $\mathbf{e}_{pi}$ is the modeling error of the steering vector $\mathbf{a}(\theta_i, \boldsymbol{\eta}_i)$ of a single path such that $\mathbf{e}_{pi} = \tilde{\mathbf{a}}(\theta_{pi}, \boldsymbol{\varphi}_{pi}) - \mathbf{a}(\theta_{pi}, \boldsymbol{\varphi}_{pi})$. In these conditions the matrix $\tilde{\mathbf{B}}$ only depends on the modeling error vectors $\mathbf{e}_i$ for $1 \leq i \leq R$ such that

$$\tilde{\mathbf{B}} = \mathbf{B} + \mathbf{E} \text{ with } \mathbf{E} = \left[\begin{array}{ccc} \mathbf{e}_1 & \cdots & \mathbf{e}_R \end{array}\right]$$

where $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_R]$ with $\mathbf{b}_i = \mathbf{b}(\Theta_i, \boldsymbol{\eta}_i)$. The covariance matrix $\mathbf{R}_x(\mathbf{E})$ of the observation $\mathbf{x}(t)$ is exact and the dependence with $\mathbf{E}$ is clearly stated such that

$$\mathbf{R}_x(\mathbf{E}) = \mathbb{E}\left[\mathbf{x}(t)\, \mathbf{x}(t)^H\right] = \tilde{\mathbf{B}} \mathbf{R}_s \tilde{\mathbf{B}}^H + \sigma^2 \mathbf{I}_N$$

where $\mathbf{R}_s = \mathbb{E}[\mathbf{s}(t)\, \mathbf{s}(t)^H]$ is full rank, $\mathbb{E}[\mathbf{n}(t)\, \mathbf{n}(t)]^H = \sigma^2 \mathbf{I}_N$ and $\mathbf{I}_N$ denotes the $N \times N$ identity matrix and $\mathbb{E}[.]$ the mathematical mean. The eigenvalue decomposition of the covariance matrix is then $\mathbf{R}_x(\mathbf{E}) = \sum_{i=1}^{N} \lambda_i \mathbf{w}_i (\mathbf{w}_i)^H$ where $\lambda_1 \geq \cdots \geq \lambda_N$ are the eigenvalues associated to the eigenvectors $\mathbf{w}_i$. The vectors $\mathbf{w}_i$ for $1 \leq i \leq R$ span the signal subspace of the columns of $\tilde{\mathbf{B}}$ and the vectors $\mathbf{w}_{i+R}$ span the noise subspace such that $\mathbf{w}_{i+R}$ and the steering vectors $\tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i)$ are orthogonals. The noise projector is then

$$\boldsymbol{\Pi}(\mathbf{E}) = \sum_{i=R+1}^{N} \mathbf{w}_i (\mathbf{w}_i)^H = \mathbf{I}_N - \tilde{\mathbf{B}} \left(\tilde{\mathbf{B}}^H \tilde{\mathbf{B}}\right)^{-1} \tilde{\mathbf{B}}^H \tag{4}$$

where $\tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i)^H \boldsymbol{\Pi}(\mathbf{E})\, \tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i) = 0$. Then, the multi-parametric MUSIC algorithm minimizes the projection of the steering vector $\mathbf{b}(\Theta, \boldsymbol{\eta})$ on the noise projector $\boldsymbol{\Pi}(\mathbf{E})$ in order to estimate $(\hat{\Theta}_i, \hat{\boldsymbol{\eta}}_i)$ for $1 \leq i \leq R$. According to (2) and [4][1], the previous optimization can be concentrated in Θ as

$$\hat{\Theta}_i = \min_{\Theta_i} (J_{\mathbf{E}}(\Theta)) \tag{5}$$

$$J_{\mathbf{E}}(\Theta) = \frac{\mathbf{k}^H \mathbf{Q}_1(\Theta) \mathbf{k}}{\mathbf{k}^H \mathbf{Q}_2(\Theta) \mathbf{k}} \tag{6}$$

$$\mathbf{Q}_1(\Theta)\, \mathbf{k} = J_{\mathbf{E}}(\Theta)\, \mathbf{Q}_2(\Theta)\, \mathbf{k} \tag{7}$$

$$\mathbf{Q}_1(\Theta) = \mathbf{V}(\Theta)^H \boldsymbol{\Pi}(\mathbf{E})\, \mathbf{V}(\Theta) \tag{8}$$
$$\mathbf{Q}_2(\Theta) = \mathbf{V}(\Theta)^H \mathbf{V}(\Theta)$$

where $J_{\mathbf{E}}(\Theta)$ is the minimum eigenvalue of $(\mathbf{Q}_2(\Theta))^{-1} \mathbf{Q}_1(\Theta)$ and $\mathbf{k}$ is the associated eigenvector. The eigenvector $\mathbf{k}$ is an estimate of $\boldsymbol{\eta}_i$ for $\Theta = \hat{\Theta}_i$. Without modeling error (where $\tilde{\mathbf{b}}(\Theta_i, \boldsymbol{\eta}_i)$=$\mathbf{b}(\Theta_i, \boldsymbol{\eta}_i)$), the DOA error $\Delta\Theta_i = \hat{\Theta}_i - \Theta_i$ is null, because the projection of the steering vector $\mathbf{b}(\Theta_i, \boldsymbol{\eta}_i)$ to the noise projector $\boldsymbol{\Pi}(\mathbf{E} = \mathbf{0})$ is null and then the criterion $J_{\mathbf{E}}(\Theta_i)$ is null. In presence of modeling error the DOA error $\Delta\Theta_i \neq 0$ is not null. Thus, the purpose of this paper is to evaluate the statistics of the vector $\Delta\Theta_i = [\Delta\theta_{1i} \cdots \Delta\theta_{P_i i}]^T$ such as the RMS error $\sqrt{\mathbb{E}[(\Delta\theta_{pi})^2]}$ of the p^{th} path of the i^{th} coherent set where $\Delta\theta_{pi} = \hat{\theta}_{pi} - \theta_{pi}$.

The first step is to propose a local Taylor expansion of the quadratic criterion $J_{\mathbf{E}}(\Theta, \boldsymbol{\eta}_{\min})$ in section 3 in order to obtain a link between $\Delta\Theta_i$ and the noise projector $\boldsymbol{\Pi}(\mathbf{E})$. According to the results of [12], a link between $\Delta\Theta_i$ and the modeling error $\mathbf{E}$ is established in section 4 in order to deduce the statistics of the DOA error.

3. TAYLOR EXPANSION OF THE MULTI-PARAMETRIC MUSIC CRITERION

The Taylor expansion of the criterion $J_{\mathbf{E}}(\Theta)$ needs a closed form expression of its differentials. At the second order, the criterion $J_{\mathbf{E}}(\Theta)$ is

$$J_{\mathbf{E}}(\Theta_i + \partial\Theta) = J_{\mathbf{E}}(\Theta_i) + \partial J_{\mathbf{E}}(\Theta_i) + \frac{1}{2}\partial^2 J_{\mathbf{E}}(\Theta_i) + o\left(\|\partial\Theta\|^2\right) \tag{9}$$

where $\partial J_{\mathbf{E}}(\Theta_i)$ and $\partial^2 J_{\mathbf{E}}(\Theta_i)$ are the first and second order differential of $J_{\mathbf{E}}(\Theta)$. Assuming that the eigenvector $\mathbf{k}$ verify $\mathbf{k}^H \mathbf{k} = 1$, we deduce that $\partial \mathbf{k}^H \mathbf{k} + \mathbf{k}^H \partial \mathbf{k} = 0$ noting that $\partial \mathbf{k}$ is the differential of $\mathbf{k}$. Using in addition the relation $\mathbf{Q}_1(\Theta)\, \mathbf{k} = J_{\mathbf{E}}(\Theta)\, (\mathbf{Q}_2(\Theta_i)\, \mathbf{k})$, the first order differential $\partial J_{\mathbf{E}}(\Theta)$ is then

$$\partial J_{\mathbf{E}}(\Theta) = \frac{\dot{J}_1(\Theta) - J_{\mathbf{E}}(\Theta)\, \dot{J}_2(\Theta)}{J_2(\Theta)} \tag{10}$$

$$J_p(\Theta) = \mathbf{k}^H \mathbf{Q}_p(\Theta)\, \mathbf{k} \quad \dot{J}_p(\Theta) = \mathbf{k}^H \partial \mathbf{Q}_{p,\Theta} \mathbf{k} \tag{11}$$

where $\partial \mathbf{Q}_{p,\Theta}$ is the first order differential of $\mathbf{Q}_p(\Theta)$. The criterion $J_p(\Theta)$ and $\dot{J}_p(\Theta)$ can be rewritten as

$$\begin{cases} J_1(\Theta_i) = \mathbf{b}_i^H \mathbf{\Pi}(\mathbf{E})\, \mathbf{b}_i \\ J_2(\Theta_i) = \mathbf{b}_i^H \mathbf{b}_i \\ \dot{J}_1(\Theta_i) = 2\Re\left(\mathbf{b}_i^H \mathbf{\Pi}(\mathbf{E})\, \partial \mathbf{b}_i\right) \\ \dot{J}_2(\Theta_i) = 2\Re\left(\mathbf{b}_i^H \partial \mathbf{b}_i\right) \end{cases} \tag{12}$$

$$\begin{cases} \mathbf{b}_i = \mathbf{b}(\Theta_i, \boldsymbol{\eta}_i) \\ \partial \mathbf{b}_i = \dot{\mathbf{V}}_i\, \partial \Theta \end{cases} \tag{13}$$

$$\begin{cases} \dot{\mathbf{V}}_i = \left[\begin{array}{ccc} \dot{\mathbf{a}}_{1i} & \cdots & \dot{\mathbf{a}}_{P_i i} \end{array}\right] \\ \dot{\mathbf{a}}_{pi} = \alpha_{pi} \frac{\partial(\mathbf{a}(\theta_i, \boldsymbol{\varphi}_i))}{\partial \theta_{pi}} = \alpha_{pi} \dot{\mathbf{U}}\ (\theta_{pi})\, \boldsymbol{\varphi}_{pi} \end{cases} \tag{14}$$

where $\Re(.)$ denotes the real part, $\dot{\mathbf{U}}\ (\theta)$ is the first derivative of $\mathbf{U}\ (\theta)$ and $\partial\Theta = [\partial\theta_{1i} \cdots \partial\theta_{P_i i}]^T$. According to (12)(13)(14) and noting that $J_{\mathbf{E}}(\Theta_i) = J_1(\Theta_i)/J_2(\Theta_i)$, the first differential $\partial J_{\mathbf{E}}(\Theta)$ only depends on $\partial\Theta$ as

$$\begin{aligned} \partial J_{\mathbf{E}}(\Theta_i) &= \tfrac{(\nabla_i(\mathbf{E}))^T \partial\Theta}{J_2(\Theta_i)} && (15) \\ (\nabla_i(\mathbf{E}))^T &= 2\Re\left(\mathbf{b}_i^H \mathbf{\Pi}(\mathbf{E}) \left(\mathbf{I}_N - \tfrac{\mathbf{b}_i \mathbf{b}_i^H}{\mathbf{b}_i^H \mathbf{b}_i}\right) \dot{\mathbf{V}}_i\ \right) && (16) \end{aligned}$$

where the gradient $\nabla_i(\mathbf{E})$ depends on the modeling error.

The second order differential of the criterion $J_{\mathbf{E}}(\Theta)$ is deduced from (10) and (7) by using the differential of the relation (7). Finally, after some algebraic manipulations, the second order differential of $J_{\mathbf{E}}(\Theta)$ is

$$\begin{aligned} \partial^2 J_{\mathbf{E}}(\Theta) &= \tfrac{\ddot{J}_{\mathbf{E}}(\Theta) + 2\partial\mathbf{k}^H\ \mathbf{H}_{\mathbf{k}}(\Theta)\ \partial\mathbf{k}}{J_2(\Theta)} && (17) \\ \ddot{J}_{\mathbf{E}}(\Theta) &= \ddot{J}_1(\Theta) - J_{\mathbf{E}}(\Theta)\ \ddot{J}_2(\Theta) - 2\partial J_{\mathbf{E}}(\Theta)\ \dot{J}_2(\Theta) \\ \mathbf{H}_{\mathbf{k}}(\Theta) &= J_{\mathbf{E}}(\Theta)\, \mathbf{Q}_2(\Theta) - \mathbf{Q}_1(\Theta) \\ \ddot{J}_i(\Theta) &= \mathbf{k}^H \partial^2 \mathbf{Q}_{i,\Theta} \mathbf{k} && (18) \end{aligned}$$

where $\partial^2\mathbf{Q}_{p,\Theta}$ is the second order differential of $\mathbf{Q}_p(\Theta)$. As $\mathbf{\Pi}(\mathbf{E})\, \mathbf{b}_i$ is closed to zero for small modeling error $\mathbf{E}$, thus the assumption $|\Re\left(\partial^2\mathbf{b}_i^H \mathbf{\Pi}(\mathbf{E})\, \mathbf{b}_i\right)| << \partial\mathbf{b}_i^H \mathbf{\Pi}(\mathbf{E})\, \partial\mathbf{b}_i$ is verify where $\partial^2\mathbf{b}_i$ is the second order differential of $\mathbf{b}_i$. Under the previous approximation, the criterion $\ddot{J}_p(\Theta)$ becomes

$$\begin{aligned} \ddot{J}_1(\Theta_i) &\approx 2\Re\left(\partial\mathbf{b}_i^H \mathbf{\Pi}(\mathbf{E})\, \partial\mathbf{b}_i\right) \\ \ddot{J}_2(\Theta_i) &\approx 2\Re\left(\partial\mathbf{b}_i^H \partial\mathbf{b}_i\right) \end{aligned} \tag{19}$$

According to (14) and (19), the second order differential $\partial^2 J_{\mathbf{E}}(\Theta)$ only depends on $\partial\Theta$ as

$$\begin{aligned} \partial^2 J_{\mathbf{E}}(\Theta_i) &= \tfrac{\partial\Theta^H \mathbf{H}_i(\mathbf{E}) \partial\Theta + 2\partial\mathbf{k}^H\ \mathbf{H}_{\mathbf{k}}(\Theta)\ \partial\mathbf{k}}{J_2(\Theta_i)} && (20) \\ \mathbf{H}_i(\mathbf{E}) &= 2\Re\left(\begin{array}{c} \dot{\mathbf{V}}_i^H \mathbf{\Pi}(\mathbf{E})\, \dot{\mathbf{V}}_i - J_{\mathbf{E}}(\Theta_i)\, \dot{\mathbf{V}}_i^H \dot{\mathbf{V}}_i \\ - \frac{\left(\nabla_i(\mathbf{E}) \Re\left(\mathbf{b}_i^H \dot{\mathbf{V}}_i\right)\right)}{\mathbf{b}_i^H \mathbf{b}_i} \end{array} \right) && (21) \end{aligned}$$

where $\mathbf{H}_i(\mathbf{E})$ is the Hessian with respect to the vector Θ_i associated to the directions of the i^{th} set of coherent multipaths. The Hessian $\mathbf{H}_i(\mathbf{E})$ depends on the modeling error. According to (9)(16) and (20), the DOA error is

$$\Delta\Theta_i \approx \partial\Theta = -\left(\mathbf{H}_i(\mathbf{E})\right)^{-1} \nabla_i(\mathbf{E}) \tag{22}$$

where $\partial\Theta$ is close to $\Delta\Theta_i = \hat{\Theta}_i - \Theta_i$. As the steering vector $\mathbf{b}_i$ verify $\mathbf{\Pi}(\mathbf{E})\, \mathbf{b}_i = 0$ without modeling error for $\mathbf{E} = \mathbf{0}$, the expression (16) shows that $\nabla_i(\mathbf{E} = \mathbf{0}) = 0$. The purpose is now to obtain a link between $\Delta\Theta_i$ and the modeling error matrix $\mathbf{E}$.

4. PERFORMANCES IN PRESENCE OF MODELING ERRORS

According to (22) the link between the DOA error $\Delta\Theta_i$ and the modeling error $\mathbf{E}$ is not linear and need a Taylor expansion. The second order approximation of $\Delta\Theta_i$ with respect to $\mathbf{E}$ is

$$\Delta\Theta_i \approx -\left(\mathbf{H}_{i0}\right)^{-1} \nabla_i^{(2)}(\mathbf{E}) + \Delta\Theta_i^{(b)} \tag{23}$$

$$\Delta\Theta_i^{(b)} = 2\left(\mathbf{H}_{i0}\right)^{-1} \Delta\mathbf{H}_i\ \left(\mathbf{H}_{i0}\right)^{-1} \nabla_i^{(1)} \tag{24}$$

where $\mathbf{H}_{i0} = \mathbf{H}_i(\mathbf{E} = \mathbf{0})$, $\Delta\mathbf{H}_i = \mathbf{H}_i^{(1)} - \mathbf{H}_{i0}$, $\nabla_i^{(r)}$ and $\mathbf{H}_i^{(r)}$ are the r^{th} order approximation of $\nabla_i(\mathbf{E})$ and $\mathbf{H}_i(\mathbf{E})$ respectively. According to [12], the first order approximation of $\mathbf{\Pi}(\mathbf{E})$ is $\mathbf{\Pi}^{(1)}(\mathbf{E}) = \mathbf{\Pi}_0 + \Delta\mathbf{\Pi}$ where $\Delta\mathbf{\Pi} = \Delta\mathbf{U} + \Delta\mathbf{U}^H$, $\Delta\mathbf{U} = -\mathbf{\Pi}_0 \mathbf{B}^{\#}\mathbf{E}$, $\mathbf{B}^{\#} = (\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H$ and $\mathbf{\Pi}_0 = \mathbf{\Pi}(\mathbf{E} = \mathbf{0})$. Thus, at the first order the criterion $J_{\mathbf{E}}(\Theta)$ is null and according to (21),

$$\begin{aligned} \mathbf{H}_{i0} &= 2\Re\left(\dot{\mathbf{V}}_i^H \mathbf{\Pi}_0 \dot{\mathbf{V}}_i\right) \text{ where } \mathbf{\Pi}_0 = \mathbf{\Pi}(\mathbf{E} = \mathbf{0}) \\ \Delta\mathbf{H}_i &= 2\Re\left(\dot{\mathbf{V}}_i^H \Delta\mathbf{\Pi}\ \dot{\mathbf{V}}_i - \frac{\left(\nabla_i^{(1)} \Re\left(\mathbf{b}_i^H \dot{\mathbf{V}}_i\right)\right)}{\mathbf{b}_i^H \mathbf{b}_i}\right) \end{aligned}$$

The DOA error $\Delta\theta_{pi}$ of the p^{th} path is associated to the p^{th} component of $\Delta\Theta_i$. According to (23), $\Delta\theta_{pi}$ is

$$\begin{aligned} \Delta\theta_{pi} &\approx -\sum_{j=1}^{P_i} \left(\mathbf{H}_{i0}^{-1}\right)_{[p,j]} \left(\nabla_i^{(2)}\right)_{[j]} + 2\Delta\theta_{pi}^{(b)} && (25) \\ \Delta\theta_{pi}^{(b)} &= \sum_{m,n,o=1}^{P_i} \left(\mathbf{H}_{i0}^{-1}\right)_{[p,n]} \left(\mathbf{H}_{i0}^{-1}\right)_{[m,o]} \left(\Delta\mathbf{H}_i\right)_{[n,m]} \left(\nabla_i^{(1)}\right)_{[o]} \end{aligned}$$

where $(\mathbf{H})_{[i,j]}$ is the ij^{th} component of the matrix $\mathbf{H}$ and $(\mathbf{h})_{[i]}$ is the i^{th} component of the vector $\mathbf{h}$. According to (14)(16) the components of $\nabla_i^{(r)}$ and $\Delta\mathbf{H}_i$ are

$$\begin{aligned} \left(\nabla_i^{(r)}\right)_{[j]} &= 2\Re\left(\left(\dot{\mathbf{b}}_{ji}\right)^H\ \mathbf{\Pi}^{(r)}(\mathbf{E})\, \mathbf{b}_i\ \right) && (26) \\ \left(\Delta\mathbf{H}_i\right)_{[n,m]} &= 2\Re\left(\dot{\mathbf{a}}_{ni}^H\ \Delta\mathbf{\Pi}\ \dot{\mathbf{a}}_{mi}\right) - 4\Re\left(\dot{\mathbf{b}}_{ni}^H\ \Delta\mathbf{\Pi} \dot{\mathbf{c}}_{mi}\ \right) \\ \dot{\mathbf{b}}_{ji} &= \left(\mathbf{I}_N - \tfrac{\mathbf{b}_i \mathbf{b}_i^H}{\mathbf{b}_i^H \mathbf{b}_i}\right) \dot{\mathbf{a}}_{ji} \quad \dot{\mathbf{c}}_{mi} = \tfrac{\left(\mathbf{b}_i \Re\left(\mathbf{b}_i^H \dot{\mathbf{a}}_{mi}\right)\right)}{\mathbf{b}_i^H \mathbf{b}_i} \end{aligned}$$

where $\mathbf{\Pi}^{(r)}(\mathbf{E})$ is the r^{th} order approximation of the noise projector $\mathbf{\Pi}(\mathbf{E})$ such that $\mathbf{\Pi}^{(1)}(\mathbf{E})\, \mathbf{b}_i = \Delta\mathbf{\Pi}\mathbf{b}_i$. The results of [12] show that

$$\mathbf{v}^H\mathbf{\Pi}^{(2)}\left(\mathbf{E}\right)\mathbf{u}{=}\boldsymbol{\varepsilon}^H\mathbf{Q}(\mathbf{u},\mathbf{v})\,\boldsymbol{\varepsilon} \tag{27}$$
$$\mathbf{Q}(\mathbf{u},\mathbf{v}){=}\begin{bmatrix} q & -\mathbf{q}_{12}^H & \mathbf{0}^T \\ -\mathbf{q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{0} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{bmatrix} \text{ and } \boldsymbol{\varepsilon}{=}\begin{bmatrix} 1 \\ \mathbf{e} \\ \mathbf{e}^* \end{bmatrix}$$

where $\boldsymbol{\varepsilon} = [1\ \mathbf{e}^T\mathbf{e}^H]^T$ $\mathbf{e}{=}[\mathbf{e}_1^T...\mathbf{e}_R^T]^T$, $q{=}\ \mathbf{v}^H\ \mathbf{\Pi}_0\ \mathbf{u}$, $\mathbf{q}_{12}{=}\ \Phi(\mathbf{u},\mathbf{v})$, $\mathbf{q}_{21}{=}\ \Phi(\mathbf{v},\mathbf{u})$, $\Phi(\mathbf{x}{,}\mathbf{y}){=}\ \left((\mathbf{B}^{\#}\mathbf{x})^*\otimes(\mathbf{\Pi}_0\mathbf{y})\right)$, $\mathbf{Q}_{22}{=}\Psi(\mathbf{B}^{\#},\mathbf{B}^{\#},\mathbf{\Pi}_0)$, $\mathbf{Q}_{23}{=}\Psi(\mathbf{B}^{\#},\mathbf{\Pi}_0,(\mathbf{B}^{\#})^H)\,\mathbf{P}$, $\mathbf{Q}_{32}{=}\ \mathbf{P}^H\ \Psi(\mathbf{\Pi}_0,\mathbf{B}^{\#}{,}\mathbf{B}^{\#})$, $\mathbf{Q}_{33}{=}\ \mathbf{P}^H\ \Psi(\mathbf{\Pi}_0,\mathbf{\Pi}_0,\mathbf{B}^{\#}\mathbf{B}^{\#H})\ \mathbf{P}$, $\Psi(\mathbf{X}{,}\mathbf{Y}{,}\mathbf{Z}){=}\left((\mathbf{X}\mathbf{v})^*(\mathbf{Y}\mathbf{u})^T\right)\otimes\mathbf{Z}$, $\otimes$ is the Kronecker product and $\mathbf{P}$ the permutation matrix such that: $vec(\mathbf{E}^T){=}\mathbf{P}vec(\mathbf{E})$. According to (27), the product $\left(\mathbf{v}^H\Delta\mathbf{\Pi}\mathbf{u}\right)\left(\tilde{\mathbf{v}}^H\Delta\mathbf{\Pi}\tilde{\mathbf{u}}\right)$ is

$$4\Re\left(\mathbf{v}^H\Delta\mathbf{\Pi}\mathbf{u}\right)\Re\left(\tilde{\mathbf{v}}^H\Delta\mathbf{\Pi}\tilde{\mathbf{u}}\right) = \boldsymbol{\varepsilon}^H\Delta\mathbf{Q}_{(\mathbf{u},\mathbf{v})}^{(\tilde{\mathbf{u}},\tilde{\mathbf{v}})}\,\boldsymbol{\varepsilon} \tag{28}$$
$$\Delta\mathbf{Q}_{(\mathbf{u},\mathbf{v})}^{(\tilde{\mathbf{u}},\tilde{\mathbf{v}})} = \mathbf{Q}_{(\mathbf{u},\mathbf{v})}^{(\tilde{\mathbf{u}},\tilde{\mathbf{v}})} + \mathbf{Q}_{(\mathbf{u},\mathbf{v})}^{(\tilde{\mathbf{v}},\tilde{\mathbf{u}})} + \mathbf{Q}_{(\mathbf{v},\mathbf{u})}^{(\tilde{\mathbf{u}},\tilde{\mathbf{v}})} + \mathbf{Q}_{(\mathbf{v},\mathbf{u})}^{(\tilde{\mathbf{v}},\tilde{\mathbf{u}})}$$
$$\mathbf{Q}_{(\mathbf{u},\mathbf{v})}^{(\tilde{\mathbf{u}},\tilde{\mathbf{v}})}{=}\begin{bmatrix} 0 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{q}_{21}\tilde{\mathbf{q}}_{12}^H & \mathbf{q}_{21}\tilde{\mathbf{q}}_{21}^T \\ \mathbf{0} & (\mathbf{q}_{21})^*\,\tilde{\mathbf{q}}_{12}^H & (\mathbf{q}_{21})^*\,\tilde{\mathbf{q}}_{21}^T \end{bmatrix}$$

where $\tilde{\mathbf{q}}_{12}{=}\ \Phi(\tilde{\mathbf{u}},\tilde{\mathbf{v}})$, $\tilde{\mathbf{q}}_{21}{=}\ \Phi(\tilde{\mathbf{v}},\tilde{\mathbf{u}})$. According to (26)(28)

$$(\Delta\mathbf{H}_i)_{[n,m]}\left(\nabla_i^{(1)}\right)_{[o]} = \boldsymbol{\varepsilon}^H\Delta\mathbf{Q}_{nmo}^i\boldsymbol{\varepsilon}$$
$$\Delta\mathbf{Q}_{nmo}^i = \Delta\mathbf{Q}_{(\dot{\mathbf{a}}_{ni},\dot{\mathbf{a}}_{mi})}^{(\dot{\mathbf{b}}_{oi},\mathbf{b}_i)} - 2\Delta\mathbf{Q}_{(\dot{\mathbf{b}}_{ni},\dot{\mathbf{c}}_{mi})}^{(\dot{\mathbf{b}}_{oi},\mathbf{b}_i)}$$

According to (25), the DOA error $\Delta\theta_{pi}$ can be rewritten as

$$\Delta\theta_{pi} = \boldsymbol{\varepsilon}^H\mathbf{Q}_{pi}\,\boldsymbol{\varepsilon} \text{ with } \mathbf{Q}_{pi} = -\mathbf{Q}_{pi}^{(a)} + 2\mathbf{Q}_{pi}^{(b)} \tag{29}$$
$$\mathbf{Q}_{pi}^{(a)} = \sum_{j=1}^{P_i}(\mathbf{H}_{i0})_{[p][j]}^{-1}\left(\mathbf{Q}\left(\dot{\mathbf{b}}_{ji},\mathbf{b}_i\right) + \mathbf{Q}\left(\mathbf{b}_i,\dot{\mathbf{b}}_{ji}\right)\right) \tag{30}$$
$$\mathbf{Q}_{pi}^{(b)} = \sum_{m,n,o=1}^{P_i}\left(\mathbf{H}_{i0}^{-1}\right)_{[p,n]}\left(\mathbf{H}_{i0}^{-1}\right)_{[m,o]}\Delta\mathbf{Q}_{nmo}^i \tag{31}$$

According to [12], the RMS $RMS_{pj} = \sqrt{\mathbb{E}[(\Delta\theta_{pi})^2]}$ of the DOA estimation error $\Delta\theta_{pi}$ can be written as $(RMS_{pj})^2 \approx trace(\mathbf{Q}_{pi}^{\otimes 2}\mathbf{R}_{\boldsymbol{\varepsilon}}^{(4)})$ where $\mathbf{R}_{\boldsymbol{\varepsilon}}^{(4)}{=}\mathbb{E}\left[\boldsymbol{\varepsilon}^{\otimes 2}\boldsymbol{\varepsilon}^{\otimes 2H}\right]$, $\mathbf{v}^{\otimes 2} = \mathbf{v}\otimes\mathbf{v}$ and $trace(.)$ is the trace. In the case of a circular Gaussian distribution of $\boldsymbol{\varepsilon}$, the RMS of $\Delta\theta_{pi}$ only depends on $\mathbf{R}_{\boldsymbol{\varepsilon}} = \mathbb{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^H\right]$ as

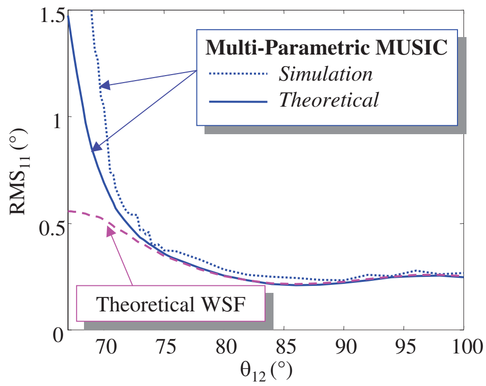
$$RMS_{pj} \approx \sqrt{trace\left((\mathbf{Q}_{pi}\mathbf{R}_{\boldsymbol{\varepsilon}})^2\right) - trace\left(\mathbf{Q}_{pi}\mathbf{R}_{\boldsymbol{\varepsilon}}\right)^2} \tag{32}$$

where $\mathbf{R}_{\boldsymbol{\varepsilon}} = \mathbb{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^H\right]$ and $(\mathbf{R}_{\boldsymbol{\varepsilon}})_{[1,1]} = 1$. In presence of multi-paths with zero mean and independent modeling error such that $\mathbb{E}\left[\mathbf{e}_{pi}\mathbf{e}_{pi}^H\right] = (\sigma_e)^2\,\mathbf{I}_N$ where $\left\|\mathbf{a}(\theta_{pi},\varphi_{pi})\right\|^2 = N$ and $\sum_i(\alpha_{pi})^2$, the covariance is $\mathbb{E}\left[\mathbf{e}_i\mathbf{e}_i^H\right] = (\sigma_e)^2\,\mathbf{I}_N$ according to (3) and the matrix $\mathbf{R}_{\boldsymbol{\varepsilon}}$ verify $(\mathbf{R}_{\boldsymbol{\varepsilon}})_{[i,j\neq i]} = 0$ and $(\mathbf{R}_{\boldsymbol{\varepsilon}})_{[i,i]} = (\sigma_e)^2$ for $1 < i \leq 2NR+1$.

5. SIMULATIONS

A uniform circular array with $N/2 = 7$ antennas of radius λ is used in simulation. Each antenna is composed of two colocated orthogonal loops of responses $\cos(\theta+\alpha_i)$ for $1 \leq i \leq 2$ for a first polarization ($\alpha_1 = 0°$ and $\alpha_2 = 90°$) and $\sin(\theta+\alpha_i)$ for a second polarization. The first source is the combination of two coherent multi-paths of directions $\theta_{11} = 60°$ and $\theta_{21} = 150°$ with $\alpha_{21} = \alpha_{11}/10$ in presence of a second source of direction θ_{12}. The modeling error is $\sigma_e = 0.12$ and the polarization verifies $\varphi_{pi[1]} = \varphi_{pi[2]}$. In Figure.1, the root mean square error RMS_{11} of the first path of the first source is represented with respect to the direction θ_{12} of the second source. The theoretical (32) and simulated performances of the multi-parametric MUSIC with unknown polarization are compared to the theoretical performances of (Weighting subspace fitting) WSF [13] given by [14] where the polarization is assumed to be known. The results show in one hand that the performances of the multi-parametric MUSIC are correctly predicted and in a second hand it gives a tool to compare the performances with other algorithms such as WSF. The simulations show that the performances of multi-parametric MUSIC are correctly predicted for coherent multi-paths in polarization diversity and that the WSF algorithm is more accurate than the multi-parametric MUSIC because the polarization is assumed to be known.

Fig. 1. RMS_{11} of the path of direction $\theta_{11} = 60°$ in presence of a coherent path and a 2^{nd} source of directions $\theta_{21} = 150°$ and θ_{12} respectively. ($\sigma_e = 0.12$)



6. CONCLUSIONS

Theoretical performances of multi-parametric MUSIC algorithm in presence of modeling errors [1] is proposed independently to the length of the nuisance vector. In addition the context of coherent multi-paths is considered and allows a comparison with others algorithms such as WSF [13] or coherent MUSIC [8][9] where the nuisance vector is composed only by the amplitudes of coherent multi-paths.

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