INFORMATION MAXIMIZING DAC NOISE SHAPING

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ABSTRACT

DACs are usually defined as having a constant number of bits over a contiguous bandwidth. Signals in practice, however, frequently have an information or bit content which is not constant with frequency. As DAC power is a function of bits, this creates an opportunity to design a more power efficient DAC. This paper addresses this opportunity by deriving the optimal DAC bit resolution vs frequency shape for maximizing the information content in the channel output given a power constraint. An optimization method that works within the constraints of the delta sigma modulator is also developed as arbitrary noise shaping is not possible with a fixed delta sigma DAC architecture.

Index Terms— DACs, information, noise shaping, delta sigma

1. INTRODUCTION

Digital to analog converters (DACs) sit at the boundary of the digital and analog worlds, converting signals defined at discrete time instants to signals defined at a continuum of time instants. DACs are usually defined as having a constant number of bits over a contiguous bandwidth and consume power which is approximately proportional to 2^{bits} x bandwidth.

Signals in practice, however, frequently have an information or bit content that is not constant with frequency. Multicarrier systems which use bit loading to maximize information transmission through frequency selective channels are an example of this. As such, there is an opportunity to create a more efficient DAC design by shaping the bit resolution as a function of frequency based on the signal being converted.

Section 3 addresses this opportunity by deriving the optimal DAC bit resolution vs frequency shape for maximizing the information content in the channel output given a power constraint. Arbitrary noise shaping is not possible with a fixed delta sigma DAC architecture, so Section 4 develops an optimization method which works within the constraints of the delta sigma modulator. While the philosophy of minimizing the information loss remains the same, the achievable quantization noise shape is now determined by the architecture. An example result is shown in Section 5 and conclusions are provided in Section 6.

2. RELATION TO PRIOR WORK

The derivation of the theoretical optimal DAC noise shape is motivated by [3], which studied optimal quantization noise shaping in ADCs. On the theory side, this paper extends these ideas to DACs.

For practical noise shaping in delta sigma DACs, [4] and [5] optimized the zeros of the NTF to minimize the total integrated noise power. A slightly different approach was taken in [2], which minimized the maximum in band gain of the NTF. This work differs from these works in that it takes non constant information as a function of frequency in the signal into account in optimizing the noise shaping.

3. THEORETICAL NOISE SHAPING

In this section the optimal noise shape given a power constraint is derived based on maximizing the preserved amount of total information when different signal frequencies contain different amounts of information. This can be viewed as maximizing the information in the converted signal at the output of a channel with non constant attenuation and/or noise or as maximizing the information in the DAC output given an input signal with non constant information.

3.1. DAC Noise and Information Loss

Due to power and other constraints, DACs are not infinitely precise. Thus, when using a finite number of bits to approximate an analog signal $x(t)$ which can take on a continuum of levels, an approximation error will exist. This error is referred to as DAC noise $q(t)$ which is modeled as

$$y(t) = x(t) + q(t), \tag{1}$$

where $y(t)$ represents the output of the DAC.

The signal to noise ratio (SNR) of a signal at the output of a linear time invariant channel with additive white Gaussian noise (AWGN) with and without DAC noise is illustrated in

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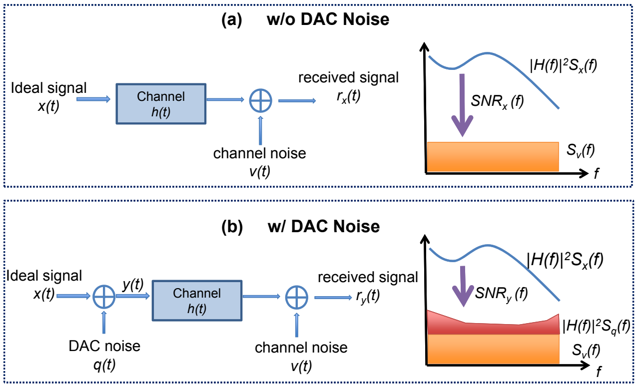

Fig. 1. Illustration of received signal SNR with and without DAC noise.

Fig. 1. A question to address is how much information is lost due to the DAC noise.

First, consider the case where the DAC noise free signal $x(t)$ goes through the linear time invariant channel $h(t)$ with AWGN $v(t)$ at the channel output

$$r_x(t) = x(t) \otimes h(t) + v(t), \tag{2}$$

where $r_x(t)$ denotes the received signal. Transforming (2) to the frequency domain results in

$$R_x(f) = X(f)H(f) + V(f). \tag{3}$$

Denote the power spectral density (PSD) of $x(t)$ as $S_x(f)$ and the PSD of $v(t)$ as S_v. The SNR of the received signal $R_x(f)$ as a function of frequency is

$$\mathrm{SNR}_x(f) = \frac{|H(f)|^2 S_x(f)}{S_v} \tag{4}$$

and the maximum information [6] at frequency f is

$$\begin{aligned} C_x(f) &= \log_2\left(1 + \mathrm{SNR}_x(f)\right) \\ &= \log_2\left(1 + \frac{|H(f)|^2 S_x(f)}{S_v}\right). \end{aligned} \tag{5}$$

Next, consider the case of a signal $y(t)$ with DAC noise which goes through the same channel

$$\begin{aligned} r_y(t) &= y(t) \otimes h(t) + v(t) \\ &= x(t) \otimes h(t) + q(t) \otimes h(t) + v(t), \end{aligned} \tag{6}$$

where $r_y(t)$ denotes the channel output in the time domain and $R_y(f)$ denotes the channel output in the frequency domain

$$R_y(f) = X(f)H(f) + Q(f)H(f) + V(f). \tag{7}$$

The SNR of $R_y(f)$ as a function of frequency is

$$\mathrm{SNR}_y(f) = \frac{|H(f)|^2 S_x(f)}{|H(f)|^2 S_q(f) + S_v(f)} \tag{8}$$

where $S_q(f)$ denotes the PSD of the DAC noise. As before, the maximum information at frequency f can be written as

$$C_y(f) \approx \log_2\left(1 + \frac{|H(f)|^2 S_x(f)}{|H(f)|^2 S_q(f) + S_v}\right), \tag{9}$$

where the approximation is due to the quantization noise not being AWGN.

The loss of information at frequency f due to DAC noise is found by subtracting (9) from (5)

$$C_\Delta(f) \approx C_x(f) - C_y(f) \tag{10}$$

$$= \log_2\left(1 + \frac{|H(f)|^4 S_x(f) S_q(f)}{|H(f)|^2 S_q(f) S_v + S_v^2 + |H(f)|^2 S_x(f) S_v}\right).$$

Assuming that S_v^2 and $|H(f)|^2 S_q(f) S_v$ are $\ll |H(f)|^2 S_x(f) S_v$, (10) simplifies to

$$C_\Delta(f) \approx \log_2\left(1 + \frac{|H(f)|^2 S_q(f)}{S_v}\right). \tag{11}$$

If $x(t)$ is a band limited signal and occupies the frequency range $[f_A, f_B]$, then the total information loss is

$$\begin{aligned} C_L &= \int_{f_A}^{f_B} C_\Delta(f) df \\ &= \int_{f_A}^{f_B} \log_2\left(1 + \frac{|H(f)|^2 S_q(f)}{S_v}\right) df. \end{aligned} \tag{12}$$

As such, information loss not only depends on the shape of the DAC noise PSD, but it is also dependent on the frequency selective channel gain and channel noise power.

3.2. DAC Noise and DAC Power

To find the optimal DAC noise shape given a power constraint, it is necessary to know the relationship between DAC power and DAC noise. DAC power and DAC noise are both related to the bit resolution vs frequency profile $N(f)$ (see Fig. 2 for an illustration).

DAC noise is related to the bit resolution using

$$S_q(f) = 2^{-2N(f)}/12, \tag{13}$$

or after rearranging terms

$$N(f) = -\frac{1}{2}\log_2(12 S_q(f)). \tag{14}$$

DAC power is related to the bit resolution via a dynamic term which is proportional to 2^{bits} x bandwidth and a static term which is proportional to bits x bandwidth [1]. At frequency f this can be written as

$$P(f) = E_1 2^{N(f)} + E_2 N(f), \tag{15}$$

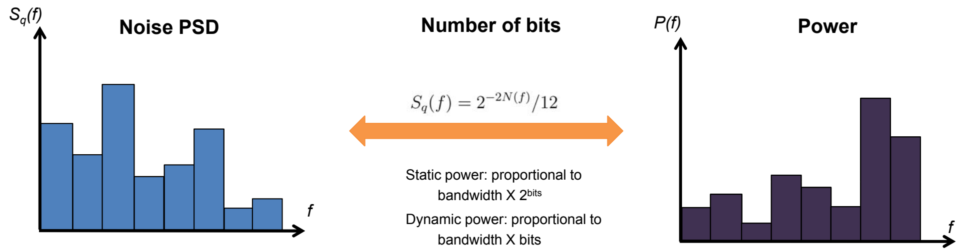

Fig. 2. Relationship between DAC noise, number of bits and DAC power.

where E_1 and E_2 denote the proportionality constants for dynamic power and static power, respectively. Typically, $E_2 \ll E_1$ and the dynamic power term dominates the DAC power. As such, it is the only term considered in this paper.

Substituting (14) into (15), the relationship between DAC power and DAC noise PSD at frequency f is

$$P(f) = E_1 \frac{1}{\sqrt{12}} S_q^{-\frac{1}{2}}(f). \tag{16}$$

Integrating the noise across frequencies yields the total DAC power as a function of DAC noise PSD

$$P_{\text{DAC}} = \int_{f_A}^{f_B} P(f) df = E_1 \frac{1}{\sqrt{12}} \int_{f_A}^{f_B} S_q^{-\frac{1}{2}}(f) df \tag{17}$$

3.3. Optimal DAC Noise Shaping

Given the relationship between capacity loss and DAC noise and the relationship between DAC noise and DAC power, this section derives the optimum PSD of the DAC noise $S_q(f)$ to minimize the information loss in (12) given the power constraint P_{DAC} (17), channel response $H(f)$ and channel noise power S_v.

The approach to the derivation is similar to that in [3]. Integrals are converted into Riemann sums and the Lagrangian is formed as

$$\begin{aligned} J[\lambda, S_q(k)] = {} & \frac{f_B - f_A}{K} \sum_{k=1}^{K} \log_2 \left[1 + \frac{|H(k)|^2 S_q(k)}{S_v} \right] \\ & + \lambda \left(\frac{E_1 (f_B - f_A)}{\sqrt{12} K} \sum_{k=1}^{K} S_q^{-\frac{1}{2}}(k) - P_{\text{DAC}} \right), \end{aligned} \tag{18}$$

where λ is a Lagrange multiplier. Taking partial derivatives with respect to $S_q(k)$ and λ, setting the results to 0

$$\frac{\partial J[\lambda, S_q(k)]}{\partial S_q(k)} = 0, \quad \frac{\partial J[\lambda, S_q(k)]}{\partial \lambda} = 0 \tag{19}$$

and solving the system of equations results in

$$S_q(k) = |H(k)|^{-\frac{4}{3}} \left[\frac{\frac{f_B - f_A}{K} \sum_{k=1}^{K} |H(k)|^{\frac{2}{3}} df}{\sqrt{12} P_{\text{DAC}} / E_1} \right]^2. \tag{20}$$

Letting $K \to \infty$ in (20) to convert back to an integral yields

$$S_q(f) = |H(f)|^{-\frac{4}{3}} \left[\frac{\int_{f_A}^{f_B} |H(f)|^{\frac{2}{3}} df}{\sqrt{12} P_{\text{DAC}} / E_1} \right]^2. \tag{21}$$

As the term in the brackets is a constant, the result is that the optimal DAC noise shape is proportional to $|H(f)|^{-\frac{4}{3}}$.

Conceptually, this result is aligned with intuition and indicates that it's appropriate to make the quantization noise small where the channel is good and the quantization noise large where the channel is bad. The exponent of $-\frac{4}{3}$ indicates that this is done in a manner which is not exactly inversely proportional to the channel magnitude.

4. PRACTICAL NOISE SHAPING

Section 3 derived the optimal DAC noise shape for minimizing information loss given a power constraint. This section determines the optimal DAC noise shape for minimizing information loss given an architectural constraint on noise shaping imposed by using a delta sigma DAC. While the philosophy of minimizing the information loss remains the same, the achievable noise shape is now determined by the architecture.

Delta sigma DACs [5] commonly include an interpolator, a delta sigma modulator, an analog DAC core and an analog filter (see Fig. 3). Typically, a delta sigma DAC converts a many level lower rate signal into a few level higher rate signal while maintaining high in band SNR. Converting to a few level signal allows for analog DAC core designs which are more tolerant of mismatch and nonlinearity errors.

A relatively general delta sigma modulator is shown in Fig. 4. In the z domain, the signal before the quantizer is

$$U(z) = L_0(z) X(z) + L_1(z) Y(z). \tag{22}$$

Modeling the quantizer as an additive noise $W(z)$ independent of the input signal, the delta sigma modulator output is

$$\begin{aligned} Y(z) &= U(z) + W(z) \\ &= L_0(z) X(z) + L_1(z) Y(z) + W(z). \end{aligned} \tag{23}$$

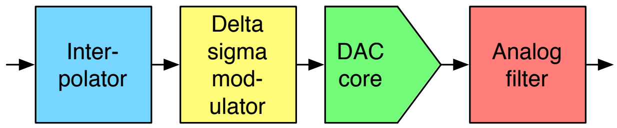

Fig. 3. A typical delta sigma DAC structure.

Rearranging terms and using $L_0(z) = G(z)/F(z)$ and $L_1(z) = 1 - 1/F(z)$ yields

$$\begin{aligned} Y(z) &= \frac{L_0(z)}{1-L_1(z)}X(z) + \frac{1}{1-L_1(z)}W(z) \\ &= G(z)X(z) + F(z)W(z). \end{aligned} \quad (24)$$

$G(z)$ is referred to as the signal transfer function (STF) and $F(z)$ is referred to as the noise transfer function (NTF).

For the simplified quantizer model, the noise signal PSD is modeled as a constant, W_0^2 and the quantization noise PSD is determined by the NTF filter $F(e^{j\omega}) = F(z)\mid_{z=e^{j\omega}}$

$$S_q(e^{j\omega}) = |F(e^{j\omega})|^2 W_0^2. \quad (25)$$

Substituting the expression for quantization noise (25) into the expression for information loss (12), it can be seen that the problem of minimizing the information loss translates to a problem of optimizing the filter coefficients of $F(e^{j\omega})$.

To illustrate the setup for optimizing the NTF, consider the case of a 2nd order filter with it's form chosen for causality in the feed back path [5]

$$F(z) = \frac{b_2 z_2 + b_1 z + b_0}{a_2 z_2 + a_1 z + a_0}. \quad (26)$$

If the denominator is nearly constant in the pass band and the zeros are conjugate symmetric (typical assumptions for delta sigma modulators), then

$$F(z) = F_0(z-\alpha)(z-\alpha^*) \quad (27)$$

where F_0 is a constant, and the optimization problem reduces to finding α which minimizes (12). Numerical methods can be used for this.

5. RESULTS

To illustrate the NTF design, a 4th order delta sigma DAC with 12x oversampling was simulated. The conventional design with zeros optimized to minimize the in band DAC noise [5] was compared to the proposed design with zeroes optimized based on maximizing information. Fig. 5 shows the resulting NTFs and channel response.

The numerical results are in line with the theoretical result. The optimization placed a zero near the frequency where the channel gain is high, which allows more bits to be assigned around those frequencies.

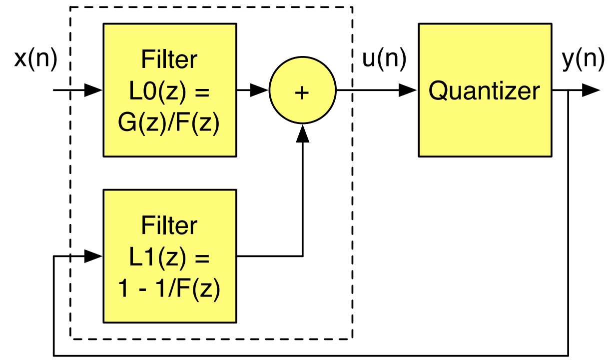

Fig. 4. A general delta sigma modulator.

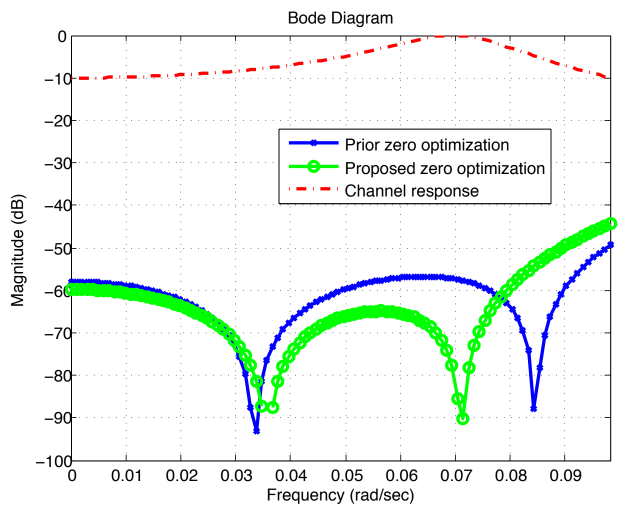

Fig. 5. 4th order NTF with conventional and information maximizing zeros optimization.

6. CONCLUSIONS

This paper derived the optimal noise shape for a DAC given a power constraint based on maximizing information when different signal frequencies contain different amounts of information. A practical method for maximizing information that works within the architectural constraints of a delta sigma DAC was also provided along with a simulation to illustrate the proposed method.

7. REFERENCES

[1] Y. Li, B. Bakkaloglu and C. Chakrabarti, “A system level energy model and energy-quality evaluation for integrated transceiver front-ends,” *IEEE Trans on VLSI Systems*, pp. 90-103, 2007.

[2] M. Nagahara and Y. Yamamoto, “Frequency domain min-max optimization of noise-shaping delta-sigma modulators,” *IEEE Trans on Signal Processing*, pp. 2828-2839, 2012.

[3] A. Redfern and K. Shi, “Quantization noise shaping for information maximizing ADCs,” arXiv, http://arxiv.org/abs/1305.2801, pp. 1-4, 2013.

[4] R. Schreier, “An empirical study of high-order single-bit delta-sigma modulators,” *IEEE Trans on Circuits and Systems II: Analog and Digital Signal Processing*, pp. 461-466, 1993.

[5] R. Schreier and G. Temes, *Understanding delta-sigma data converters*. IEEE Press, Piscataway, NJ, 2005.

[6] C. Shannon, “Communication in the presence of noise,” *Proceedings of the IRE*, pp. 10-21, 1949.