2014 IEEE International Conference on Acoustic, Speech and Signal Processing (ICASSP)

ACCURATE RATE-DISTORTION APPROXIMATION FOR SPARSE BERNOULLI-GENERALIZED GAUSSIAN MODELS

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ABSTRACT

The objective of this paper is to study rate-distortion properties of a quantized Bernoulli-Generalized Gaussian source. Such source model has been found to be well-adapted for signals having a sparse representation in a transformed domain. We provide here accurate approximations of the entropy and the distortion functions evaluated through a p-th order error measure. These theoretical results are then validated experimentally. Finally, the benefit that can be drawn from the proposed approximations in bit allocation problems is illustrated for a wavelet-based compression scheme.

Index Terms— Rate-Distortion theory, entropy, quantization, Bernoulli-Generalized Gaussian distribution, Lagrangian approach.

1. INTRODUCTION

Rate-Distortion (R-D) theory has attracted a considerable attention in the information theory literature, especially in the area of data compression [1]. For example, the current image and video coders based on wavelets require the computation of the rate and distortion functions to perform optimal rate allocation among all the wavelet subbands [2]. In a transform coding context, the R-D performance can be usually expressed from the individual rate and distortion functions of the quantized subbands. Therefore, it becomes interesting to efficiently compute the rate and distortion functions of the quantized coefficients.

To this end, a simple empirical approach consists of quantizing the wavelet coefficients and evaluating the resulting entropy and distortion functions for different values of quantization steps. However, this approach may be computationally intensive in the context of bit allocation problems where many R-D points are required [3]. To overcome this drawback, André et al. proposed to select only few R-D points and to interpolate between them using splines [3]. Moreover, numerical approaches were also developed by using different source models and quantizer characteristics. Indeed, the expression of the operational R-D function can be derived in the case of exponential and Laplacian sources with absolute and squared error distortion measures [4], in the case of a uniformly distributed source for both distortion criteria [5], and in the case of a Gaussian source for squared error distortion [6]. Recent studies also considered the Laplace and Generalized Gaussian (GG) probabilistic models to compute the entropy and the distortion resulting from a uniform quantizer [7, 8]. In addition, it should be noticed that the well-known R-D results developed by Gish and Pierce [9], which are only valid at high bit rate (i.e small quantization steps), are useful in the selection mode, in the context of H.264/AVC coding standard for example [10]. Recently, some of us proposed approximations of the entropy as well as *asymptotic* expressions of the distortion for memoryless Generalized Gaussian sources [11] and Bernoulli-Generalized Gaussian ones [12]. Note that these approximations have been developed in [12] under some assumptions about the distribution shape parameter and the deadzone parameter of the quantizer, as discussed at the end of Section 3.

With the objective of designing fast and efficient bit allocation methods, we will make in this paper a detailed study of the R-D functions. More specifically, by considering a uniform scalar quantization (which is used in many embedded coders), we will develop accurate approximations of *both* the entropy and distortion functions. Unlike the results reported in the literature where only the high and/or the low resolution behaviors are often investigated, our results are valid for any given set of quantization parameters. Moreover, instead of using the GG model which was often used to model the wavelet coefficients, we propose to employ a more general one, known as the Bernoulli Generalized Gaussian (BGG) model, which is more appropriate for modelling coefficients of a *sparse* representation.

The remainder of this paper is organized as follows. In Sections 2 and 3, we derive close approximations of the entropy and distortion functions for quantized BGG source. We then show in Section 4 the interest of these approximations in the context of bit allocation problems. Finally, some conclusions are provided in Section 5.

2. ENTROPY OF QUANTIZED BGG SOURCES

Let us first define the source and quantization models. In wavelet-based transform coding applications, the source to be quantized corresponds to the J subbands which will be designated in the following by X_j with $j \in \{1, \ldots, J\}$. An appropriate distribution for modelling the resulting coefficients is the BGG one whose probability density function is given by

$$\forall \xi \in \mathbb{R}, \qquad f_j(\xi) = (1 - \epsilon_j)\delta(\xi) + \epsilon_j \tilde{f}_j(\xi) \tag{1}$$

where $\epsilon_j \in]0, 1]$ denotes the mixture parameter, δ is the Dirac distribution and $\tilde{f}_j$ represents the probability density function for a GG distribution with scale factor $\omega_j \in]0, +\infty[$ and shape parameter $\beta_j \in]0, 2]$:

$$\forall \xi \in \mathbb{R}, \qquad \tilde{f}_j(\xi) = \frac{\beta_j \omega_j^{1/\beta_j}}{2\Gamma(1/\beta_j)} e^{-\omega_j |\xi|^{\beta_j}} \tag{2}$$

where Γ is the gamma function. The differential entropy of such a GG variable is given in [13] by

$$h_{\beta_j}(\omega_j) = -\int_{-\infty}^{\infty} \tilde{f}_j(\xi)\log_2 \tilde{f}_j(\xi)\,d\xi = \log_2\Big(\frac{2\Gamma(1/\beta_j)}{\beta_j \omega_j^{1/\beta_j}}\Big) + \frac{1}{\beta_j}.$$

During the quantization process, we employ a uniform scalar quantizer with a quantization step q_j, having a deadzone of size $(2\tau_j - 1)q_j$ where $\tau_j > 1/2$. Thus, for each input coefficient $X_{j,s}$, the output of the quantizer $\overline{X}_{j,s}$ is expressed as

$$\overline{X}_{j,s} = r_0 = 0, \quad \text{if } |X_{j,s}| < \left(\tau_j - \tfrac{1}{2}\right) q_j, \quad \text{where } \tau_j > 1/2$$

and, for all $i \in \mathbb{Z}$, $\overline{X}_{j,s} = r_{i,j}$,

$$\begin{aligned}&(\text{if } (\tau_j + i - \tfrac{3}{2})q_j \le X_{j,s} < (\tau_j + i - \tfrac{1}{2})q_j \text{ and } i \ge 1)\\ \text{or } &(\text{if } (-\tau_j + i + \tfrac{1}{2})q_j < X_{j,s} \le (-\tau_j + i + \tfrac{3}{2})q_j \text{ and } i \le -1),\end{aligned}$$

where the reconstruction levels are given by

$$\forall i \ge 1, \qquad r_{i,j} = -r_{-i,j} = (\tau_j + i - 1 + \zeta_j)q_j \tag{3}$$

and $\zeta_j \in [-1/2, 1/2]$ is an "offset" parameter used to adjust the values of the reconstruction levels. We will not consider any saturation effect. Note that the quantization rule corresponds often to the case when $\zeta_j = 0$. For example, this rule is used in many encoding techniques, like EBCOT [14], which have been developed in wavelet-based image compression schemes.

Let us now focus on the entropy of the quantized variable $\overline{X}_j$ defined by

$$H_{f_j}(q_j, \epsilon_j) = -\sum_{i=-\infty}^{\infty} \mathsf{p}_{i,j} \log_2 \mathsf{p}_{i,j} \tag{4}$$

where, for every $i \in \mathbb{Z}$, $\mathsf{p}_{i,j} = \mathsf{P}(\overline{X}_{j,s} = r_{i,j})$ is the probability of occurrence of the reconstruction level $r_{i,j}$.

Let Q_a with $a \in \mathbb{R}_+^*$ be the normalized incomplete Gamma function [15] which will be used to compute the approximation of the entropy:

$$\forall \xi \in \mathbb{R}, \qquad Q_a(\xi) = \frac{1}{\Gamma(a)}\int_0^{\xi} \theta^{a-1} e^{-\theta} d\theta. \tag{5}$$

In the following, we first provide an accurate approximation of the entropy and then give the sketch of proof of this result.

Proposition 1. *The entropy of the quantized BGG source is*

$$\widehat{H}_{f_j}(q_j, \epsilon_j) = \Phi(\mathsf{p}_{0,j}, \epsilon_j) + \epsilon_j \widehat{H}_{\tilde{f}_j}(q_j) \tag{6}$$

$$\begin{aligned}\textit{with } \Phi(\mathsf{p}_{0,j}, \epsilon_j) = &-\big(1 - \epsilon_j(1 - \mathsf{p}_{0,j})\big)\log_2\big(1 - \epsilon_j(1 - \mathsf{p}_{0,j})\big)\\ &- \epsilon_j(1 - \mathsf{p}_{0,j})\log_2 \epsilon_j + \epsilon_j \mathsf{p}_{0,j}\log_2 \mathsf{p}_{0,j},\end{aligned}$$

$$\begin{aligned}\textit{and } \widehat{H}_{\tilde{f}_j}(q_j) = &-\mathsf{p}_{0,j}\log_2 \mathsf{p}_{0,j} - 2\mathsf{p}_{1,j}\log_2 \mathsf{p}_{1,j}\\ &+ \big(h_{\beta_j}(\omega_j) - \log_2 q_j\big)\Big(1 - Q_{1/\beta_j}\Big(\omega_j\big(\tau_j + \frac{1}{2}\big)^{\beta_j} q_j^{\beta_j}\Big)\Big)\\ &+ \frac{\omega_j^{1/\beta_j}(\tau_j + \frac{1}{2})q_j}{\Gamma(1/\beta_j)}\, e^{-\omega_j(\tau_j + \frac{1}{2})^{\beta_j} q_j^{\beta_j}}.\end{aligned} \tag{7}$$

The error incurred in this approximation is such that

$$0 \le \widehat{H}_{f_j}(q_j, \epsilon_j) - H_{f_j}(q_j, \epsilon_j) \le 2\epsilon_j q_j C(\beta_j, \tau_j)\tilde{f}_j\big((\tau_j + \frac{1}{2})q_j\big),$$

$$\textit{with } C(\beta_j, \tau_j) = \begin{cases}\left(\frac{2\tau_j + 1}{2\tau_j - 1}\right)^{1-\beta_j} & \textit{if } \beta_j < 1\\ \left(\frac{2\tau_j + 2}{2\tau_j + 1}\right)^{\beta_j - 1} & \textit{if } \beta_j \in [1, 2].\end{cases} \tag{8}$$

Proof. We recall that the entropy of a quantized BGG random variable distributed according to (1) is given by [12]:

$$H_{f_j}(q_j, \epsilon_j) = \Phi(\mathsf{p}_{0,j}, \epsilon_j) + \epsilon_j H_{\tilde{f}_j}(q_j) \tag{9}$$

$$\text{where} \quad H_{\tilde{f}_j}(q_j) = -\mathsf{p}_{0,j}\log_2 \mathsf{p}_{0,j} - 2\sum_{i=1}^{\infty} \mathsf{p}_{i,j}\log_2 \mathsf{p}_{i,j} \tag{10}$$

is the entropy of a quantized GG random variable with probability density function $\widetilde{f}_j$. In order to prove the desired result, it is enough to show that

$$H_{\tilde{f}_j}(q_j) = \widehat{H}_{\tilde{f}_j}(q_j) + \Delta \tag{11}$$

$$\text{where} \quad 0 \le \Delta \le 2q_j C(\beta_j, \tau_j)\tilde{f}_j\big((\tau_j + \frac{1}{2})q_j\big). \tag{12}$$

To this end, depending on the β_j values, two cases can be considered.

- The proof for the case when $\beta_j \in [1, 2]$ can be found in [12].
- When $\beta_j < 1$, it can be checked that

$$\begin{aligned}0 \le -\sum_{i=2}^{+\infty} \mathsf{p}_{i,j}\log_2 \mathsf{p}_{i,j} &+ \int_{(\tau_j + \frac{1}{2})q_j}^{+\infty} \widetilde{f}_j(\xi)\log_2 \widetilde{f}_j(\xi) d\xi\\ &+ \log_2 q_j \int_{(\tau_j + \frac{1}{2})q_j}^{+\infty} \widetilde{f}_j(\xi) d\xi \le I_1\end{aligned} \tag{13}$$

$$\begin{aligned}\text{where} \quad I_1 &= \beta_j \omega_j q_j \int_{(\tau_j + \frac{1}{2})q_j}^{+\infty} (\xi - q_j)^{\beta_j - 1}\widetilde{f}_j(\xi)\, d\xi\\ &\le \frac{\beta_j \omega_j^{1/\beta_j} q_j}{2\Gamma(1/\beta_j)}\Big(\frac{2\tau_j - 1}{2\tau_j + 1}\Big)^{\beta_j - 1} e^{-\omega_j(\tau_j + \frac{1}{2})^{\beta_j} q_j^{\beta_j}}.\end{aligned} \tag{14}$$

Furthermore, it can be noticed that

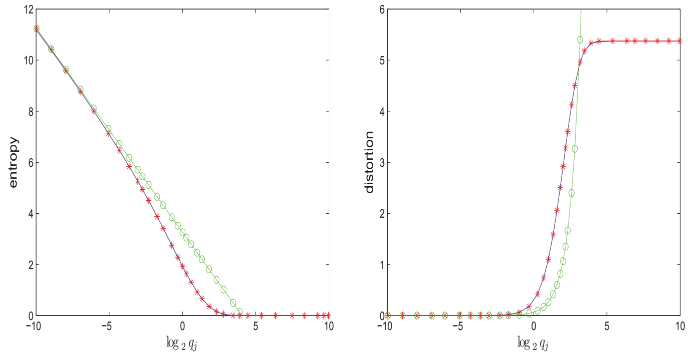
$$\begin{aligned}2\int_{(\tau_j + \frac{1}{2})q_j}^{+\infty} &\widetilde{f}_j(\xi)\log_2 \widetilde{f}_j(\xi) d\xi\\ &= -h_{\beta_j}(\omega_j)\Big(1 - Q_{1/\beta_j}\Big(\omega_j\big(\tau_j + \frac{1}{2}\big)^{\beta_j} q_j^{\beta_j}\Big)\Big)\\ &\quad - \frac{\omega_j^{1/\beta_j}(\tau_j + \frac{1}{2})q_j}{\Gamma(1/\beta_j)}\, e^{-\omega_j(\tau_j + \frac{1}{2})^{\beta_j} q_j^{\beta_j}}.\end{aligned} \tag{15}$$

After noticing that $\mathsf{p}_{i,j}$ can be easily expressed by using the incomplete Gamma functions, combining (13) and (14), and using (10) and (15), we obtain the approximation formula of the entropy of the quantized GG random variable, given by (11)-(12). Finally, the approximation formula for the discrete entropy of the quantized BGG random variable can be easily deduced from (9). □

Taking into account the local behaviour of the incomplete Gamma function around 0 (see [16]), it can be checked that, at high bitrate (i.e. when $q_j \to 0$),

$$\widehat{H}_{f_j}(q_j, \epsilon_j) = H_{\epsilon_j} + \epsilon_j(h_{\beta_j}(\omega_j) - \log_2 q_j) + O(q_j \log_2 q_j) \tag{16}$$

where $H_{\epsilon_j} = -\epsilon_j \log_2 \epsilon_j - (1 - \epsilon_j)\log_2(1 - \epsilon_j)$ is the entropy of a Bernoulli random variable with parameters $(1 - \epsilon_j, \epsilon_j)$. Note that (16) corresponds to the classical high rate approximation of the entropy, known as Bennett's formula [9]. As shown by Fig. 1(a), the latter formula leads to a good approximation of the entropy function H_{f_j} only for small quantization steps, whereas Proposition 1 yields an accurate approximation for any set of quantization parameters.



(a) (b)

Fig. 1. In (a) (resp. (b)): standard high rate approximation plotted in circle green symbol and new approximation developed in Proposition 1 (resp. Proposition 2) plotted with star red symbol, of the entropy H_{f_j} (resp. distortion d_j with $p_j = 2$) plotted in solid blue line versus $\log_2 q_j$. The parameters of the BGG source are $\epsilon_j = 0.8$, $\beta_j = 0.75$ and $\omega_j = 1$.

3. DISTORTION OF QUANTIZED BGG SOURCES

We focus now on the distortion function which can be evaluated through the p_j-th order moment of the quantization error as follows [12]:

$$\begin{aligned} d_j(q_j,\epsilon_j) &= \mathsf{E}[|X_{j,s} - \overline{X}_{j,s}|^{p_j}] \\ &= 2\epsilon_j\Big(\int_0^{(\tau_j-\frac{1}{2})q_j} \xi^{p_j}\widetilde{f}_j(\xi)d\xi \\ &\quad + \sum_{i=1}^{+\infty}\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j} |\xi - r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi\Big), \end{aligned}$$

where $p_j \geq 1$ is a real exponent. Thus, in the usual case when $p_j = 2$, the distortion represents the standard mean square error criterion.

In the following, we first provide in Proposition 2 an accurate approximation of the distortion and then give some details and comment this result.

Proposition 2. *An approximation of the distortion of the quantized BGG random variable is*

$$\begin{aligned} &\widehat{d}_j(q_j,\epsilon_j) \\ &= 2\epsilon_j\Big(\frac{\omega_j^{-p_j/\beta_j}\Gamma((p_j+1)/\beta_j)}{2\Gamma(1/\beta_j)}Q_{(p_j+1)/\beta_j}\big(\omega_j(\tau_j-\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big) \\ &+ \int_{(\tau_j-\frac{1}{2})q_j}^{(\tau_j+\frac{1}{2})q_j}|\xi - r_{1,j}|^{p_j}\widetilde{f}_j(\xi)d\xi \\ &+ \frac{\nu_j q_j^{p_j}}{2(p_j+1)}\big(1 - Q_{1/\beta_j}\big(\omega_j(\tau_j+\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big)\big)\Big) \end{aligned} \quad (17)$$

where the approximation error is such that

$$|d_j(q_j,\epsilon_j) - \widehat{d}_j(q_j,\epsilon_j)| \leq 2\epsilon_j\frac{\nu_j q_j^{p_j+1}}{p_j+1}\widetilde{f}_j\big((\tau_j+\frac{1}{2})q_j\big). \quad (18)$$

Proof. Knowing that

$$\begin{aligned} \int_0^{(\tau_j-\frac{1}{2})q_j}\xi^{p_j}\widetilde{f}_j(\xi)d\xi &= \frac{\omega_j^{-p_j/\beta_j}\Gamma((p_j+1)/\beta_j)}{2\Gamma(1/\beta_j)} \\ &\times Q_{(p_j+1)/\beta_j}\big(\omega_j(\tau_j-\frac{1}{2})^{\beta_j}q_j^{\beta_j}\big), \end{aligned} \quad (19)$$

it can be deduced that the approximation error is given by

$$\begin{aligned} d_j(q_j,\epsilon_j) - \widehat{d}_j(q_j,\epsilon_j) &= 2\epsilon_j\Big(\sum_{i=2}^{+\infty}\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi - r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi \\ &- \frac{\nu_j q_j^{p_j}}{2(p_j+1)}\Big(1 - Q_{1/\beta_j}\Big(\omega_j\big((\tau_j+\frac{1}{2})q_j\big)^{\beta_j}\Big)\Big)\Big). \end{aligned} \quad (20)$$

In addition, we have the following inequalities:

$$\begin{aligned} \forall i \geq 1,\ \widetilde{f}_j\big((\tau_j+i-\frac{1}{2})q_j\big)&\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi - r_{i,j}|^{p_j}d\xi \\ &\leq \int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi - r_{i,j}|^{p_j}\widetilde{f}_j(\xi)d\xi \\ \leq \widetilde{f}_j\big((\tau_j+i-\frac{3}{2})q_j\big)&\int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi - r_{i,j}|^{p_j}d\xi \end{aligned} \quad (21)$$

$$\text{with}\quad \int_{(\tau_j+i-\frac{3}{2})q_j}^{(\tau_j+i-\frac{1}{2})q_j}|\xi - r_{i,j}|^{p_j}d\xi = \frac{\nu_j q_j^{p_j+1}}{p_j+1}. \quad (22)$$

$$\forall i \geq 1,\ \int_{(\tau_j+i-\frac{1}{2})q_j}^{(\tau_j+i+\frac{1}{2})q_j}\widetilde{f}_j(\xi)d\xi \leq q_j\widetilde{f}_j\big((\tau_j+i-\frac{1}{2})q_j\big) \quad (23)$$

$$\forall i \geq 2,\ q_j\widetilde{f}_j\big((\tau_j+i-\frac{3}{2})q_j\big) \leq \int_{(\tau_j+i-\frac{5}{2})q_j}^{(\tau_j+i-\frac{3}{2})q_j}\widetilde{f}_j(\xi)d\xi. \quad (24)$$

By combining (19) and (21)-(24), it can be checked that (17) and (18) hold. □

Two remarks can be made about the developed approximation of the distortion. First, when $q_j \to 0$, we find the well-known high rate approximation of the distortion:

$$\widehat{d}_j(q_j,\epsilon_j) = \epsilon_j\frac{\nu_j}{p_j+1}q_j^{p_j}(1 + O(q_j)) \quad (25)$$

Secondly, when $p_j = 2$, the integral in (17) can be expressed using the incomplete Gamma function. In this case, the approximation formula of the distortion takes the more tractable form:

$$\begin{aligned} \widehat{d}_j(q_j,\epsilon_j) = \epsilon_j\Big(&\omega_j^{-2/\beta_j}\frac{\Gamma(3/\beta_j)}{\Gamma(1/\beta_j)}Q_{3/\beta_j}\big(\omega_j\big((\tau_j+\frac{1}{2})q_j\big)^{\beta_j}\big) \\ &- 2\omega_j^{-1/\beta_j}\frac{\Gamma(2/\beta_j)}{\Gamma(1/\beta_j)}r_{1,j}\Big(Q_{2/\beta_j}\big(\omega_j\big((\tau_j+\frac{1}{2})q_j\big)^{\beta_j}\big) \\ &- Q_{2/\beta_j}\big(\omega_j\big((\tau_j-\frac{1}{2})q_j\big)^{\beta_j}\big)\Big) \\ &+ r_{1,j}^2\Big(Q_{1/\beta_j}\big(\omega_j\big((\tau_j+\frac{1}{2})q_j\big)^{\beta_j}\big) \\ &- Q_{1/\beta_j}\big(\omega_j\big((\tau_j-\frac{1}{2})q_j\big)^{\beta_j}\big)\Big) \\ &+ \frac{\nu_j}{3}q_j^2\Big(1 - Q_{1/\beta_j}\big(\omega_j\big((\tau_j+\frac{1}{2})q_j\big)^{\beta_j}\big)\Big)\Big). \end{aligned} \quad (26)$$

Fig. 1(b) shows that the obtained approximation of the distortion in Proposition 2, leads to good results while the standard formula (25)

is only precise at high bitrate.
It is important to note that Propositions 1 and 2 are useful in practice since they allow to compute fast and accurate approximations of the entropy and distortion functions for any given set of quantization steps. Moreover, compared with the approximation results provided in [12], the main contributions of this paper can be summarized as follows. First, in [12], the β_j exponent is restricted to take its values in $[1, 2]$, whereas it can now have values lower than 1, which is more realistic for modelling sparse sources. Then, the deadzone parameter τ_j was set to 1 in [12] whereas the theoretical expressions are here valid for any value of $\tau_j > \frac{1}{2}$. This can be useful in practice since in JPEG2000, the different subbands can be parameterized to have different deadzone sizes. Finally, only asymptotic expressions of the distortion are given in [12] whereas an accurate approximation of this function is derived in this paper for *any* quantization parameter.

4. APPLICATION TO A BIT ALLOCATION PROBLEM

The developed rate-distortion results can be useful in the context of bit allocation in transform-based coding applications. In this case, the objective is to distribute an available budget of bits among the different subbands resulting from a wavelet decomposition. To this end, the basic idea behind the bit allocation procedure consists of minimizing the distortion subject to a constraint on the global bitrate.
To address this problem, standard Lagrangian optimization techniques have been widely used in the literature [3, 17, 18]. Such techniques are based on two steps. The first one aims at empirically computing the R-D curves of the different subbands, and then resorting to an iterative method to find the optimal bitrate for each subband [19]. As an example, we focus in this section on the improved version of these techniques which has been recently proposed in [3]. Thus, to illustrate the interest of the developed R-D results, we have implemented this method by focusing on the first step and considering the following three cases:
• In the first one (designated in Fig. 3 by "Lagrangian approach-1"), the R-D points are *empirically* computed [3]. It is important to note that the R-D curve resulting from this approach is considered as the reference one which should be very close to that obtained from accurate rate and distortion functions.
• In the second case (designated in Fig. 3 by "Lagrangian approach-2"), the R-D points are computed by using the standard high rate approximations [9] given by Bennett's formula (16) and (25).
• In the third one (designated in Fig. 3 by "Lagrangian approach-3"), the R-D points are evaluated according to the approximations proposed in Propositions 1 and 2.
These results are obtained on the "Einst" image, displayed in Fig. 2, by using a 9/7 wavelet transform carried out over three resolution levels, and setting the deadzone parameter τ_j to 2. As shown by Fig. 3, using only high rate approximations of the entropy and distortion functions affects significantly the standard Lagrangian optimization technique [3] and leads to a large difference between "Lagrangian approach-1" and "Lagrangian approach-2". In turn, the behavior of the Lagrangian optimization technique based on the developed approximation results gets much closer to that based on the empirical procedure.
Fig. 4 illustrates the R-D performance for synthetic wavelet subbands generated according to (1). The parameters of the BGG model of the different subbands are set to those obtained on the "Einst" image after applying the 9/7 wavelet transform. It can be noticed that the two approaches lead to the same optimal rate-distortion points. This observation confirms the fact that the small errors, obtained in Fig. 3 between the two curves produced by "Lagrangian approach-1" and "Lagrangian approach-3", may be accounted for by the modelling error of the wavelet coefficients.
Finally, we should note that using an Intel Core 2 (3.2 GHz) computer with a Matlab 2013 implementation, the generation of the R-D curves for the different subbands takes about 5 seconds (resp. 2 seconds) when the empirical (resp. proposed) computation strategy is applied.

5. CONCLUSION

In this paper, we have proposed accurate approximations of the entropy and distortion functions for a quantized BGG source. Such approximation formulas present two advantages. First, they allow us to efficiently compute these functions for any given value of the quantization parameter. Secondly, these kinds of approximations can be useful to design advanced bit allocation algorithms [20]. The interest of the developed results has been shown in the context of bit allocation for a transform-based coding application.

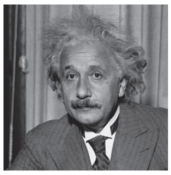


Fig. 2. "Einst" image.

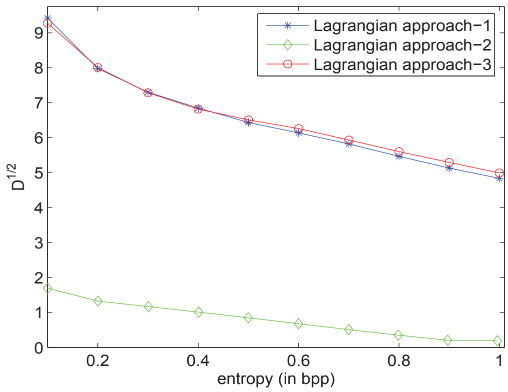


Fig. 3. Distortion versus the entropy (in bpp) for the "Einst" image.

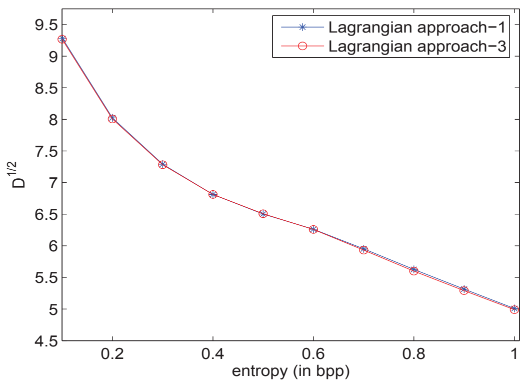


Fig. 4. Distortion versus the entropy (in bpp) for synthetic data generated according to (1) and using the BGG parameters of the wavelet subbands of "Einst".

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