BAYESIAN QUICKEST DETECTION WITH STOCHASTIC ENERGY CONSTRAINT

Jun Geng and Lifeng Lai

Department of Electrical and Computer Engineering Worcester Polytechnic Institute, Worcester, MA, 01605 Email:{jgeng,llai}@wpi.edu

ABSTRACT

In this paper, Bayesian quickest change-point detection problem with a stochastic energy constraint is considered. This work is motivated by applications of renewable energy powered wireless sensor networks. In particular, a renewable energy powered wireless sensor is deployed to detect the change in the probability distribution of the observation sequence. The energy in the sensor is consumed by taking observations and is replenished randomly. The sensor cannot store extra energy if its battery is full and cannot take observations if it has no energy left. Hence, the sensor needs to use its energy efficiently. Our goal is to design a power allocation scheme and a detection strategy to minimize the average detection delay while keeping a low false alarm probability. We show that this problem can be written into a set of iteratively defined functions and then solved by the tools from the optimal stopping theory. It turns out that the optimal solution has a very complex structure. For practical applications, we propose a low complexity algorithm, in which the sensor adopts a greedy power allocation scheme with a threshold detection rule. We show that this algorithm is first order asymptotically optimal as the false alarm probability goes to zero.

Index Terms— Bayesian quickest change detection; energy harvested sensor; stochastic energy constraint; sequential detection.

1. INTRODUCTION

Quickest change-point detection problem aims to detect the abrupt change in probability distribution of a random sequence as quickly and reliably as possible [1, 2, 3, 4]. This technique has found a lot of applications in wireless sensor networks [5, 6, 7, 8, 9, 10] for network intrusion detection [11], seismic sensing, structural health monitoring, etc. The sensor networks powered by renewable energy have attracted considerable interests in recent years. In such networks, each sensor can harvest energy from the ambient environment hence it has unlimited life span. However, the stochastic nature of the energy replenishing process also brings power management challenges. In this paper, we focus on the design

of the optimal power allocation strategy for the renewable energy powered sensor network when the detection delay is of interest.

In particular, we extend the classic Bayesian quickest change detection problem, which was first studied by A. Shiryaev [1, 2], by imposing a stochastic energy constraint. In the classic setup, there is no energy constraint and the sensor can take observations at every time slot. In this paper, we extend this problem to sensors that are powered by renewable energy. In this case, the energy stored in the sensor is replenished by a random process and consumed by taking observations. The sensor cannot store extra energy if the battery is full, and the sensor cannot take observations if there is no energy left. Hence, the sensor cannot take observation at every time instant anymore. Since the energy collected by the harvester in each time instant is not a constant but a random variable, this brings new optimization challenges.

There have been some existing works on the quickest change-point detection problem that take the observation cost into consideration. [12] considers the Bayesian quickest change-point detection problem with sample right constraints in the continuous time scenario. [13] considers the Bayesian quickest detection with observation cost, which assumes that each observation is worth either 1 if it is observed or 0 if it is skipped. Compared with [12, 13], our paper focuses on the discrete time case, and provides a low complexity asymptotically optimal solution as well as the optimal solution. [14] considers the non-Bayesian quickest detection with a stochastic sampling right constraint. [11] considers the design of detection strategy that strikes a balance between the detection delay, false alarm probability and the number of sensors being active for a multiple sensor network. [15] and [16] take the average number of observations taken before the change-point into consideration, and they provide the optimal solutions along with low-complexity but asymptotically optimal rules for Bayesian setup and non-Bayesian setup respectively. [17] is a recent survey about the quickest change-point detection problem.

The remainder of the paper is organized as follows. The mathematical model is given in Section 2. Section 3 and Section 4 present the optimal solution and the asymptotically optimal solution, respectively. Numerical examples are given in Section 5. Section 6 offers concluding remarks. Due to space

The work of J. Geng and L. Lai was supported by the National Science Foundation under grant DMS-12-65663.

limitations, we present only main ideas and conclusions. Details of proofs can be found in [18].

2. PROBLEM FORMULATION

We consider a random sequence $\{X_k, k = 1, 2, ...\}$ with a geometrically distributed change-point τ such that $X_1, X_2, ..., X_{\tau-1}$ are independent and identically distributed (i.i.d.) with probability density function (pdf) f_0 while $X_{\tau}, X_{\tau+1}, ...$ are i.i.d. with pdf f_1 . The distribution of the change-point τ is given as

$$P(\tau = t) = \begin{cases} \pi & \text{if } t = 0\\ (1 - \pi)(1 - \rho)^{t - 1}\rho & \text{if } t = 1, 2, \dots \end{cases}$$
(1)

We use P_{π} to denote the probability measure under which τ has above geometric distribution, and use \mathbb{E}_{π} to denote the expectation with respect to P_{π} .

At each time slot, the energy arrives randomly to the energy harvested wireless sensor. Let $\nu = \{\nu_1, \nu_2, \ldots, \nu_k, \ldots\}$ denote the energy arriving process, where ν_k is the amount of energy arrived at time slot k. Specially, $\nu_k \in \mathcal{V} = \{0, 1, 2, \ldots\}$, in which $\{\nu_k = 0\}$ means that the energy harvester collects nothing at time slot k and $\{\nu_k = i\}$ means that the energy harvester collects i units of energy at time slot k. We use $p_i = P^{\nu}(\nu_k = i)$ to denote its probability mass function (pmf). ν_k 's are i.i.d. over k.

The wireless sensor can decide how to allocate these collected energies. Let $\mu = \{\mu_1, \mu_2, \dots, \mu_k, \dots\}$ denote the energy utility process with $\mu_k \in \{0, 1\}$. $\{\mu_k = 1\}$ means that the wireless sensor spends a unit of energy on taking observation at time slot k, while $\{\mu_k = 0\}$ means that no energy is spent at time slot k and hence no observation is taken.

We assume that the wireless sensor has a battery with finite capacity C. Denote E_k as the amount of energy left in the battery at the end of time slot k. Then, E_k evolves according to

$$E_k = \min\{C, E_{k-1} + \nu_k - \mu_k\}, \quad k = 1, 2, \dots,$$
 (2)

and $E_0 = E$ is the energy initially stored in the battery. The energy utility process must obey the causality constraint: *the energy cannot be used before it is harvested.* The energy causality constraint can be represented as

$$E_k \ge 0, \quad k = 1, 2, \dots \tag{3}$$

Let \mathcal{U} be the admissible strategy set, which contains all the strategies satisfied with (3).

Let $\{Z_k, k = 1, 2, ...\}$ denote the observation sequence obtained by the sensor, in which

$$Z_k = \begin{cases} X_k & \text{if } \mu_k = 1\\ \phi & \text{if } \mu_k = 0 \end{cases}$$
(4)

The observation sequence $\{Z_k\}$ generates the filtration $\{\mathcal{F}_k\}$ with

$$\mathcal{F}_k = \sigma(\{\tau = 0\}, Z_1, \cdots, Z_k), \quad k = 1, 2, \dots,$$

and \mathcal{F}_0 contains the sample space Ω and $\{\tau = 0\}$.

We notice that the distribution of Z_k is related to both X_k and μ_k . Unlike the classic Bayesian setup which only considers the probability measure P_{π} , we should take both P_{π} and P^{ν} into consideration since both of them affect the distribution of Z_k . Hence, in our problem setup, we use the superscript ν over the probability measure and the expectation, i.e. P_{π}^{ν} and \mathbb{E}_{π}^{ν} , to emphasize that we are working with a probability measure taken the distribution of the process ν into consideration.

The sensor aims to detect the change as soon as it occurs. Let \mathcal{T} be the set of all finite stopping times with respect to $\{\mathcal{F}_k\}$. A stopping time $T \in \mathcal{T}$ will decide when the sensor should stop taking observations and declare that the change has occurred. A false alarm occurs if $T < \tau$. Our goal is to minimize the average detection delay (ADD) subjected to a false alarm constraint. Specially, we want to solve the following optimization problem:

$$\min_{\mu \in \mathcal{U}, T \in \mathcal{T}} \mathbb{E}_{\pi}^{\nu}[(T-\tau)^{+}] \text{ subject to } P_{\pi}^{\nu}(T < \tau) \le \alpha, \quad (5)$$

where α is a constant characterizing the probability of false alarm (PFA). By Lagrangian multiplier, for any given $\alpha \in (0, 1)$, we can define a cost function

$$L(\pi, E, T, \mu) = \mathbb{E}_{\pi}^{\nu} [\mathbf{1}_{\{T < \tau\}} + c(T - \tau)^{+}]$$
(6)

for some proper chosen c such that the optimization problem

$$J(\pi, E) = \inf_{\mu \in \mathcal{U}, T \in \mathcal{T}} L(\pi, E, T, \mu)$$
(7)

is equivalent to (5).

3. OPTIMAL SOLUTION

In this section, we study the optimal solution for the proposed problem. Let π_k be the posterior probability that a change has occurred at the k^{th} time instant, namely,

$$\pi_k = P^{\nu}_{\pi} (\tau \le k | \mathcal{F}_k), \quad k = 0, 1, \dots$$
(8)

It is easy to see $\pi_0 = \pi$. Using the Bayes' rule, π_k can be shown to satisfy the following recursive formula

$$\pi_k = \begin{cases} \Phi_0(\pi_{k-1}) & \text{if } \mu_k = 0\\ \Phi_1(Z_k, \pi_{k-1}) & \text{if } \mu_k = 1 \end{cases},$$
(9)

in which

$$\Phi_0(\pi_{k-1}) = \pi_{k-1} + (1 - \pi_{k-1})\rho$$

and

$$\Phi_1(Z_k, \pi_{k-1}) = \frac{\Phi_0(\pi_{k-1})f_1(Z_k)}{\Phi_0(\pi_{k-1})f_1(Z_k) + [1 - \Phi_0(\pi_{k-1})]f_0(Z_k)}$$

The cost function (6) can be converted into an expression in terms of π_k .

Proposition 3.1. *For any arbitrary power allocation* $\mu \in U$ *,*

$$L(\pi, E, T, \mu) = \mathbb{E}_{\pi}^{\nu} \left[1 - \pi_T + c \sum_{k=1}^{T-1} \pi_k \right].$$
 (10)

This problem can be solved by backward induction. In particular, we first consider a finite horizon problem with a horizon N, that is, we consider the case that the sensor must claim a stop at a time no later than N. We define

$$J_{k}^{N}(\pi_{k}, E_{k}) = \inf_{\substack{\mu_{k+1}^{N} \in \mathcal{U}_{k+1}^{N}, T \in \mathcal{T}_{k}^{N}}} L(\pi_{k}, E_{k}, T, \mu_{k+1}^{N})$$

with

$$L(\pi_k, E_k, T, \mu_{k+1}^N) = \mathbb{E}_{\pi_k}^{\nu} \left[1 - \pi_T + c \sum_{i=k}^{T-1} \pi_i \right],$$

in which $\mu_k^N = \{\mu_k, \mu_{k+1}, \dots, \mu_N\}$ is the power allocation strategy adopted by the sensor from k to N, $\mathcal{U}_k^N = \{\mu_k^N : E_i \ge 0, i = k, \dots, N\}$, and $\mathcal{T}_k^N = \{T \in \mathcal{T} : k \le T \le N\}$. By setting $k = 0, J_0^N(\pi, E)$ is the cost function for the finite horizon problem with a horizon N.

We introduce a set of iteratively defined functions. Let

$$V_N^N(\pi_N, E_N) = 1 - \pi_N,$$

and for $k = N - 1, N - 2, \ldots, 0$, we define

$$W_{k+1}^{N}(\pi_{k}, E_{k}, \nu_{k+1})$$

$$= \min \left\{ \mathbb{E}_{\pi_{k}}^{\nu} [V_{k+1}^{N}(\pi_{k+1}, E_{k+1}) | \nu_{k+1}, \mu_{k+1} = 0], \\ \mathbb{E}_{\pi_{k}}^{\nu} [V_{k+1}^{N}(\pi_{k+1}, N_{k+1}) | \nu_{k+1}, \mu_{k+1} = 1] \right\},$$

$$V_{k}^{N}(\pi_{k}, E_{k})$$

$$= \min \{ 1 - \pi_{k}, c\pi_{k} + \mathbb{E}^{\nu} [W_{k+1}^{N}(\pi_{k}, E_{k}, \nu_{k+1})] \}.$$

This set of functions convert the original problem into a Markov stopping problem:

Lemma 3.2. For all k = 0, 1, ..., N, we have

$$J_k^N(\pi_k, E_k) = V_k^N(\pi_k, E_k).$$

Furthermore, the optimal sampling strategy is given as

$$\mu_k^* = \operatorname*{argmin}_{\mu_k \in \{0,1\}} \mathbb{E}_{\pi_{k-1}}^{\nu} [V_k^N(\pi_k, E_k) | \nu_k, \mu_k].$$

The optimal stopping rule is given as

$$T^* = \inf \left\{ 0 \le k \le N : 1 - \pi_k \le c\pi_k + \mathbb{E}^{\nu}[W_{k+1}^N(\pi_k, E_k, \nu_{k+1})] \right\}$$

Remark 3.3. We give a heuristic explanation of the iterative functions W_{k+1}^N , V_k^N and Lemma 3.2. In each time slot, the sensor needs to make two decisions: the sampling decision μ_k and the terminal decision δ_k . Both decisions affect the

cost function, however these two decisions are based on different information. In particular, the sensor decides whether to take an observation or not at time slot k after knowing how much energy has been collected at time slot k. Hence, μ_k is a function of ν_k , π_{k-1} and E_{k-1} . When μ_k is decided, the sensor could determine the way that π_k and E_k evolve, and hence the decision δ_k is a function of π_k and E_k . Actually, the iterative function V_k^N is the cost function associated with δ_k , and W_k^N is that associated with μ_k . At the end of time slot k, the sensor could choose either to stop, which costs $1 - \pi_k$, or to continue. Since μ_{k+1} is the next decision after δ_k , the future cost in V_k^N is $\mathbb{E}^{\nu}[W_{k+1}^N]$. On the other hand, since δ_{k+1} is the decision after μ_{k+1} , hence the sensor chooses μ_{k+1} based on the rule that the future cost is minimized, that is the conditional expectation of V_{k+1}^N is minimized, which leads the expression of W_{k+1}^N . Since the decision δ_k is made at the end of time slot k, V_k^N and J_k^N coincide.

In the following, we use a limit argument to extend the above conclusion to the infinite horizon problem. Since $V_k^N(\pi_k, E_k) \ge 0$ and

$$V_k^{N+1}(\pi_k, E_k) \le V_k^N(\pi_k, E_k),$$

which is true due to the fact that all strategies admissible for horizon N are also admissible for horizon N + 1, the limit of $V_k^N(\pi_k, E_k)$ as $N \to \infty$ exists. Furthermore, as π_k and E_k are homogenous Markov chains, the form of the limit function is the same for different values of k, which we define as

$$V(\pi_k, E_k) = \lim_{N \to \infty} V_k^N(\pi_k, E_k).$$

Similarly, we have

$$W(\pi_k, E_k, \nu_{k+1}) = \lim_{N \to \infty} W_{k+1}^N(\pi_k, E_k, \nu_{k+1}).$$

Hence, we have the following conclusion for the infinite horizon problem.

Theorem 3.4. The optimal sampling strategy is given as

$$\mu_k^* = \underset{\mu_k \in \{0,1\}}{\operatorname{argmin}} \mathbb{E}_{\pi_{k-1}}^{\nu} [V(\pi_k, N_k) | \nu_k, \mu_k].$$
(11)

The optimal stopping rule is given as

$$T^* = \inf \{k \ge 0 : 1 - \pi_k \le c\pi_k + \mathbb{E}^{\nu}[W(\pi_k, N_k, \nu_{k+1})]\}.$$
 (12)

4. ASYMPTOTICALLY OPTIMAL SOLUTION

The optimal solution derived in Section 3 has a very complex structure. To facilitate practical applications, we propose a low complexity algorithm, and we show it is asymptotically optimal when the PFA goes to zero. The proposed algorithm is given as

$$\tilde{\mu}_k = \begin{cases} 1 & \text{if } E_{k-1} + \nu_k \ge 1 \\ 0 & \text{if } E_{k-1} + \nu_k = 0 \end{cases},$$
(13)

and

$$\tilde{T} = \inf\{k \ge 0 | \pi_k \ge 1 - \alpha\}.$$
(14)

That is, the sensor adopts a greedy energy utility strategy, in which the sensor spends energy on taking observations as long as the battery is not empty, and a threshold detection rule. We simply denote this strategy as $(\tilde{\mu}, \tilde{T})$.

The asymptotic optimality of $(\tilde{\mu}, \tilde{T})$ is revealed in the following two theorems. We first derive a lower bound on the ADD for any power allocation scheme and detection strategy:

Theorem 4.1. As $\alpha \rightarrow 0$,

$$\inf_{\mu \in \mathcal{U}, T \in \mathcal{T}} \mathbb{E}_{\pi}^{\nu}[(T-\tau)^{+}] \ge \frac{|\log \alpha|}{\tilde{p}D(f_{1}||f_{0}) + |\log(1-\rho)|}(1+o(1)).$$

where $D(f_1||f_0)$ is the Kullback-Leibler (KL) divergence of f_1 and f_0 , and $\tilde{p} \triangleq \mathbb{E}^{\nu}[\tilde{\mu}]$.

This lower bound can be obtained by $(\tilde{\mu}, \tilde{T})$:

Theorem 4.2. $(\tilde{\mu}, \tilde{T})$ is asymptotically optimal as $\alpha \to 0$. Specifically,

$$\mathbb{E}_{\pi}^{\nu}[(\tilde{T}-\tau)^{+}] = \frac{|\log \alpha|}{\tilde{p}D(f_{1}||f_{0}) + |\log(1-\rho)|}(1+o(1)).$$

5. NUMERICAL SIMULATION

In this section, we give two numerical examples to illustrate the results obtained in our paper. In these numerical examples, we assume that the pre-change distribution f_0 is $\mathcal{N}(0, \sigma^2)$ and the post-change distribution f_1 is $\mathcal{N}(0, P + \sigma^2)$. The signal-to-noise ratio is defined as $SNR = 10 \log P/\sigma^2$. The geometrically distributed change-point has parameters $\pi_0 = 0.15$ and $\rho = 0.01$.

In the first scenario, we illustrate the relationship between the ADD and the PFA with respect to different \tilde{p} . In this simulation, we set $\mathcal{V} = \{0, 1\}$ and $E_0 = 0$. Then $\tilde{\mu}$ is simplified into an immediate power allocation, i.e., the sensor spends the energy on taking observation immediately when it obtains an energy from the environment. Hence we have $\tilde{p} = p_1$ in this case. The simulation result for SNR = 0dB is shown in Figure 1. In this figure, the blue line with circles is the simulation result for $p_1 = 0.2$, the green line with stars and the red line with squares are the results for $p_1 = 0.5$ and $p_1 = 0.8$, respectively. The black dash line is the performance of the classic Bayesian problem, which is served as a lower bound because the sensor can take observation at every time slot. As we can see, for a given α , the detection delay is in inverse proportion to \tilde{p} . The larger \tilde{p} is, the closer is the performance to the lower bound.

In the second simulation, we examine the asymptotic optimality of $(\tilde{\mu}, \tilde{T})$. In the simulation, we set C = 3, $E_0 = 2$ and we assume that the amount of arrived energy is taken from the set $\mathcal{V} = \{0, 1, \dots, 4\}$. We set $p_0 = 0.85$, $p_1 = 0.1$,



Fig. 1. ADD vs. PFA under different \tilde{p}

 $p_2 = 0.03$, $p_3 = 0.01$, $p_4 = 0.01$, and one can find $\tilde{p} = 0.3610$ under this setting. Furthermore, we set $\sigma^2 = 1$ and SNR = 5dB. The simulation result is shown in Figure 2. In this figure the red line with squares is the performance of the proposed strategy $(\tilde{\mu}, \tilde{T})$, and the black dash line is calculated by $|\log \alpha|/(\tilde{p}D(f_1||f_0) + |\log(1-\rho)|)$. As we can see, along all the scales, these two curves are parallel to each other, which confirms that the proposed strategy, $(\tilde{\mu}, \tilde{T})$, is asymptotically optimal as $\alpha \to 0$ since the constant difference can be ignored when the detection delay goes to infinity.



Fig. 2. ADD vs. PFA under strategy $(\tilde{\mu}, \tilde{T})$

6. CONCLUSION

In this paper, we have studied the Bayesian quickest detection problem with a casual energy constraint. We have characterized the optimal solution, which unfortunately has a very high complexity. For practical applications, we have proposed a low complexity algorithm, in which the sensor adopts an greedy power allocation with a threshold detection rule. We have shown that this simple algorithm is first order asymptotically optimal as the PFA goes to zero.

7. REFERENCES

- A. N. Shiryaev, "The problem of the most rapid detection of a disturbance in a stationary process," *Soviet Math. Dokl.*, no. 2, pp. 795–799, 1961, (translation from Dokl. Akad. Nauk SSSR vol. 138, pp. 1039-1042, 1961).
- [2] A. N. Shiryaev, "On optimal methods in quickest detection problems," *Theory of Probability and Its Applications*, vol. 8, pp. 22–46, 1963.
- [3] G. Lorden, "Procedures for reacting to a change in distribution," *Annals of Mathematical Statistics*, vol. 42, no. 6, pp. 1897–1908, 1971.
- [4] M. Pollak, "Optimal detection of a change in distribution," *Annals of Statistics*, vol. 13, no. 1, pp. 206–227, Mar. 1985.
- [5] Y. Mei, "Information bounds and quickest change detection in decentralized decision systems," *IEEE Trans. Inform. Theory*, vol. 51, pp. 2669–2681, Jul. 2005.
- [6] G. V. Moustakides, "Decentralized cusum change detection," in *Proc. Intl. Conf. on Information Fusion*, Florence, Italy, Jul. 2006, pp. 1–6.
- [7] A. G. Tartakovsky and V. V. Veeravalli, "Asymptotically optimal quickest change detection in distributed sensor systems," *Sequential Analysis: Design Methods and Application*, vol. 27, no. 4, pp. 441–475, 2008.
- [8] A. G. Tartakovsky and H. Kim, "Performance of certain decentralized distributed change detection procedures," in *Proc. Intl. Conf. on Information Fusion*, Florence, Italy, Jul. 2006, pp. 1–8.
- [9] G. Fellouris and G. V. Moustakides, "Bandwidth and energy efficient decentralized sequential change detection," *Bernoulli*, 2013, submitted.
- [10] O. Hadjiliadis, T. Schaefer, and H. V. Poor, "Quickest detection in coupled systems," in *Proc. IEEE Conf. on Decision and Control*, Shanghai, China, 2009, pp. 4723–4728.
- [11] K. Premkumar and A. Kumar, "Optimal sleep-wake scheduling for quickest intrusion detection using sensor network," in *Proc. IEEE Conf. Computer Communications (Infocom)*, Phoenix, AZ, USA, Apr. 2008, pp. 1400–1408.
- [12] Erhan Bayraktar and Ross Kravitz, "Quickest detection with discretely controlled observations," 2012, available at [arXiv:1212.4717].
- [13] L. G. Dzhamburia, "On one generalization of the quickest change-point detection problem," *Bulletin of the Academy of Sciences of the Georgian SSR*, vol. 110, no. 1, pp. 17–19, 1983.

- [14] J. Geng and L. Lai, "Non-Bayesian quickest detection with stochastic sample right constraints," *IEEE Trans. Signal Processing*, vol. 61, pp. 5090–5102, Oct. 2013.
- [15] T. Banerjee and V. V. Veeravalli, "Data-efficient quickest change detection with on-off observation control," *Sequential Analysis: Design Methods and Application*, vol. 31, no. 1, pp. 40–77, 2012.
- [16] T. Banerjee and V. V. Veeravalli, "Data-efficient quickest change detection in minimax settings," *IEEE Trans. Inform. Theory*, vol. 59, no. 10, Oct. 2013.
- [17] A. Polunchenko and A. G. Tarkakovsky, "State-of-theart in sequential change-point detection," *Methodology and Computing in Applied Probability*, vol. 14, no. 3, pp. 649–684, 2012.
- [18] J. Geng, E. Bayraktar, and L. Lai, "Bayesian quickest change point detection with sampling right constraints," *IEEE Trans. Inform. Theory*, Jan. 2014, accepted with minor revision. available at [arXiv:1309.5396].