ON THE ROLE OF THE HILBERT TRANSFORM IN BOOSTING THE PERFORMANCE OF THE ANNIHILATING FILTER

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ABSTRACT

We consider the problem of parameter estimation from real-valued multi-tone signals. Such problems arise frequently in spectral estimation. More recently, they have gained new importance in finiterate-of-innovation signal sampling and reconstruction. The annihilating filter is a key tool for parameter estimation in these problems. The standard annihilating filter design has to be modified to result in accurate estimation when dealing with real sinusoids, particularly because the real-valued nature of the sinusoids must be factored into the annihilating filter design. We show that the constraint on the annihilating filter can be relaxed by making use of the Hilbert transform. We refer to this approach as the Hilbert annihilating filter approach. We show that accurate parameter estimation is possible by this approach. In the single-tone case, the mean-square error performance increases by 6 dB for signal-to-noise ratio (SNR) greater than 0 dB. We also present experimental results in the multi-tone case, which show that a significant improvement (about 6 dB) is obtained when the parameters are close to 0 or π . In the mid-frequency range, the improvement is about 2 to 3 dB.

Index Terms— Annihilating filter, discrete Hilbert transform, finite rate of innovation, sampling, spectral estimation.

1. INTRODUCTION

High-resolution spectral estimation (HRSE) is an important problem that arises in applications such as radio detection and ranging (RADAR), sound navigation and ranging (SONAR) and communication systems, etc. [1]. The fundamental goal in HRSE is to accurately estimate the parameters of closely-spaced sinusoids in the presence of noise with as few measurements as possible. Techniques such as the annihilating filter, multiple signal classification (MUSIC) [2], and estimation of signal parameters by rotational invariance (ESPRIT) [3, 4], have been proposed to solve the problem. More recently, the HRSE methods gained importance in the context of a new sampling paradigm, namely, finite-rate-of-innovation (FRI) sampling first proposed by Vetterli et al. [5], [6]. The FRI sampling problem is one of sampling and reconstructing parametric signals (piece-wise polynomial, piece-wise sinusoidal, etc.) from their projections on to a suitably chosen sampling kernel. These signals have a finite number of degrees of freedom over a unit interval and do not fall within Shannon's sampling framework. Typically, the kernel is chosen such that the problem of parameter estimation reduces to one of solving for the parameters of a sum-of-sinusoids signal, which is the standard HRSE problem. The annihilating filter method, MU-SIC, and ESPRIT methods have been deployed to solve the FRI problem. Even though the annihilating filter, in comparison with

ESPRIT and MUSIC, is more susceptible to noise, it requires fewer number of samples for parameter estimation. This is mainly because ESPRIT and MUSIC use the autocorrelation matrix, whereas the annihilating filter works directly on the noisy input samples. To accurately estimate the autocorrelation matrix, more number of samples are required. The standard annihilating filter approach is best suited for complex sinusoids. In case of real sinusoids, the parameters become mutually dependent, which must be taken into account in designing the annihilating filter to obtain optimal performance. Otherwise, there will be a mismatch between the signal model and the annihilating filter design, which would affect noise performance. In view of the simplicity and importance of the annihilating filter in FRI signal sampling, we address the issue of resolving the mismatch by making use of the Hilbert transform. We would like to add here that the Hilbert transform is gaining more importance in the context of FRI problems. Recently, the Hilbert transform was used for pulse modeling in variable pulse width (VPW) FRI problems [7] for electrocardiogram (ECG) signal modeling and compression. Hao and Marziliano [8] proposed a FRI ECG model, where the ECG signals are modeled as sum of bandlimited and nonuniform linear splines. Condat and Hirabayashi [9] proposed a different approach of sampling and reconstruction of FRI signals, where the FRI signals are reconstructed by applying maximum-likelihood estimation method. Kusuma and Goyal [10] derived the Cramér-Rao bounds for parameter estimation of FRI signals by posing the FRI based sampling and reconstruction as powersum-based sampling method. In presence of noise, the performance of parameter estimation of FRI signals from its samples can be improved by applying Cadzow denoising method [11].

1.1. Real versus complex sinusoids

In the standard formulation, the signal model is given as

$$\widetilde{x}[n] = \sum_{k=1}^{K} a_k \, e^{\mathbf{j}\omega_k n} + \epsilon_c[n], \quad 0 \le n \le N - 1, \tag{1}$$

where $\epsilon_c[n]$ is additive white complex Gaussian noise. The problem is one of estimating $\{\omega_k\}_{k=1}^K$ from $\tilde{x}[n]$.

In practical spectral estimation problems, one has to deal with real sinusoids. Also, in case of FRI problems, some practically realizable sampling kernels result in a sum of real sinusoids. Thus, the corresponding real-valued signal model takes the form,

$$x[n] = \sum_{k=1}^{K} a_k \, \cos(\omega_k n) + \epsilon[n], \quad 0 \le n \le N - 1, \qquad (2)$$



Fig. 1. Comparison of bias, variance, and MSE on a single-tone signal for both complex as well as real signals; N = 21 and $\omega_1 = 0.3\pi$. The results are obtained by averaging from 500 independent Monte Carlo realizations.

where $\epsilon[n]$ is white Gaussian noise with zero mean, variance σ^2 and a_k is the amplitude of the k^{th} cosinusoid. In (1), the problem is to estimate K complex sinusoid frequencies, whereas in (2), the goal is to estimate 2K complex sinusoid frequencies. If we deploy the standard annihilating filter machinery to solve the problem in (2), we need to explicitly enforce the constraint that out of the 2K parameters, K are redundant. Otherwise, there will be a difference in the mean-square error (MSE) performance depending on what was input to the annihilating filter. This aspect is illustrated through a numerical simulation in Figure 1. We observe that the MSE performance difference between the real and complex signal cases is nearly 6 dB. Although the bias is nearly identical for SNR greater than 5 dB, the variance and hence the MSE show a significant difference. This example also provides convincing evidence that the complex signal scenario is better for annihilation than the real signal one.

1.2. Prior work in real sinusoid frequency estimation

In the literature, there are broadly two classes of techniques, one based on discrete Fourier transform (DFT) interpolation and the other based on HRSE techniques. Macleod [12] noted that, due to the finite length of the signal there exists spectral leakage from the negative complex exponential onto the positive complex exponential and vice versa. He proposed three-point and five-point interpolation algorithms in the DFT domain to remove the effect of spectral leakage. Mahata and Söderstrom [13] proposed a modification of ESPRIT, referred to as R-ESPRIT for real sinusoids. The authors show that the noise free component of the signal lies in a subspace of dimension K, and thus the signal dimension is reduced from 2Kto K. An FRI based technique was proposed by Bernet et al. [14], to recover the parameters of a piecewise sinusoidal signal from its samples using kernels satisfying Strang-Fix [15] conditions. So et al. [16] proposed iterative methods to estimate the frequencies of multiple sinusoids by constrained weighted least square estimators.

1.3. Organization of the paper

In Section 2, we briefly review the annihilating filter for complex sinusoid parameter estimation. Also in this section, we pose the FRI sampling technique as a problem of solving for the unknown frequencies from a sum of real multi-tone sinusoids. In Section 3, we propose a Hilbert transform based technique to transform a signal of the type given in (2) to that in (1), in order to improve on the estimation accuracy. We refer to the proposed method as the H-annihilating filter. In Section 4, we make a comparative study of performance of the annihilating filter and the H-annihilating filter.

2. THE ANNIHILATING FILTER AND PARTIAL FOURIER SERIES

2.1. Annihilating filter

z[r]

Consider a signal y[n] that is a sum of K complex exponentials,

$$y[n] = \sum_{k=1}^{K} a_k e^{j\omega_k n}, \quad n = 0, 1, \cdots, N-1.$$

Consider a finite causal filter h[n], defined over $n = 0, 1 \cdots M$. The convolution output z[n], for $M \le n \le N - 1$ is,

$$h] = (y * h)[n]$$

$$= \sum_{k=1}^{K} a_k e^{j\omega_k n} \sum_{m=0}^{M} h[m] e^{j\omega_k m}$$

$$= \sum_{k=1}^{K} a_k H(\omega_k) e^{-j\omega_k n}$$

If $H(\omega_k)$ is designed to take a value 0, for $k = \{1, 2, \dots, K\}$, then z[n] = 0 for $M \le n \le N - 1$. The transfer function of the filter h[n] is given by,

$$H(z) = \prod_{k=1}^{K} (1 - e^{j\omega_k} z^{-1}).$$

There are K unknown zeros of h[n], thus M = K. To solve the system of equations,

$$\sum_{m=0}^{K} h[m]y[n-m] = 0, K \le n \le N-1$$

we need at least K equations. This implies that, $N - K \ge K \Rightarrow N \ge 2K$. Thus, we need at least 2K samples of y[n] to estimate $\{\omega_k\}_{k=1}^K$.

2.2. Reconstruction from partial Fourier series

We consider the model proposed by Vetterli et al. [5] and reformulate it differently to obtain a real-valued multi-tone signal. Consider a τ periodic FRI signal having K Dirac impulses in every period,

$$g(t) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{K} a_k \delta(t - t_k - n\tau)$$

where a_k and t_k are the unknown amplitude and time location of the k^{th} Dirac impulse. This periodic signal can be expanded using a trigonometric Fourier series with coefficients c_{ℓ} , b_{ℓ} [17] as,

$$g(t) = c_0 + \sum_{\ell=1}^{\infty} c_\ell \cos(2\pi\ell t/\tau) + \sum_{\ell=1}^{\infty} b_\ell \sin(2\pi\ell t/\tau).$$

Now c_{ℓ} , for $\ell \geq 1$ is given by,

$$c_{\ell} = \frac{2}{\tau} \int_{0}^{\tau} g(t) \cos(2\pi\ell t/\tau) dt$$
$$= \frac{2}{\tau} \int_{0}^{\tau} \sum_{k=1}^{K} a_{k} \delta(t-t_{k}) \cos(2\pi\ell t/\tau) dt$$
$$= \frac{2}{\tau} \sum_{k=1}^{K} a_{k} \cos(2\pi\ell t_{k}/\tau).$$
(3)

Thus, given the coefficient sequence $\{c_{\ell}, 1 \leq \ell \leq 2K\}$, the problem of computing a_k and t_k fits within the framework of (2). Once the parameters t_k are estimated using the annihilating filter approach (more specifically, a suitably modified version of it), the parameters a_k can be estimated using a standard linear least-squares approach. A similar result holds if one were interested in computing the parameters based on coefficients b_{ℓ} . Interestingly, the problem of signal reconstruction from partial Fourier coefficients was also addressed by Eckhoff [18] and more recently by Batenkov and Yomdin [19].

3. REAL SINUSOID PROBLEM

Using Euler's formula, (2) is written as,

$$x[n] = \sum_{k=1}^{K} c_k e^{j\omega_k n} + c_k e^{-j\omega_k n} + \epsilon(n)$$
(4)

where $c_k \triangleq \frac{a_k}{2}$. In general, the annihilating filter solves for 2K complex sinusoids in (4). However, K out of the 2K parameters are redundant, a constraint which must be enforced to enable accurate reconstruction.

3.1. Proposed method

We propose to use the Hilbert transform to go from a signal model of the type given in (2) to that given in (1). The Hilbert transform is a unitary operator that converts cosines to sines and vice versa. The model given in (1) is actually an analytic signal model. Let the discrete Hilbert transform operator be denoted by \mathcal{H} . Then the analytic signal $\tilde{x}[n]$ is given as

$$\widetilde{x}[n] = x[n] + j\mathcal{H}x[n]$$

$$= \sum_{k=1}^{K} a_k e^{j\omega_k n} + \epsilon_c[n]$$
(5)



Fig. 2. H-Annihilating filter technique.

Thus, the real sinusoid problem has been transformed to a complex sinusoid problem. The total number of complex exponentials is reduced by a factor of 2. This has an interesting consequence. Maravic and Vetterli [20] showed that the mean-square error in frequency estimation increases with the number of complex exponentials. By reducing the number of complex exponentials, the mean-square error performance is improved. We show that, by carrying out spectral domain calculations the signal-to-noise ratio remains unchanged in the process of applying the Hilbert transform. Let $X(\omega), \tilde{X}(\omega), \epsilon(\omega)$ denote the Fourier transforms of $x[n], \tilde{x}[n]$, and $\epsilon[n]$, respectively.

$$X(\omega) = \sum_{k=1}^{K} \frac{a_k}{2} \delta(\omega - \omega_k) + \frac{a_k}{2} \delta(\omega + \omega_k) + \epsilon(\omega),$$

$$\widetilde{X}(\omega) = \begin{cases} \sum_{k=1}^{K} a_k \delta(\omega - \omega_k) + 2\epsilon(\omega), & \text{if } \omega > 0, \\ 0, & \text{if } \omega < 0. \end{cases}$$
(6)

The SNR estimated from $X(\omega)$ is

$$SNR_X = \frac{\sum_{k=1}^{K} \left(\frac{a_k}{2}\right)^2 + \left(\frac{a_k}{2}\right)^2}{\int_{-\infty}^{\infty} |\epsilon(\omega)|^2 d\omega}$$
$$= \frac{\sum_{k=1}^{K} \frac{a_k^2}{2}}{\int_{-\infty}^{\infty} |\epsilon(\omega)|^2 d\omega},$$

whereas that estimated from $\widetilde{X}(\omega)$ is

$$SNR_{\tilde{X}} = \frac{\sum_{k=1}^{K} a_k^2}{\int_0^\infty |2\epsilon(\omega)|^2 d\omega}$$
$$= \frac{\sum_{k=1}^{K} a_k^2}{4 * \frac{1}{2} \int_{-\infty}^\infty |\epsilon(\omega)|^2 d\omega}$$
$$\Rightarrow SNR_X = SNR_{\tilde{X}}.$$

3.2. Discrete Hilbert transform

The ideal discrete Hilbert transform has the frequency response $H(\omega) = -j \operatorname{sgn}(\omega), -\pi < \omega \leq \pi$. The corresponding discrete filter impulse response is of infinite duration and is given by $h[n] = \frac{2 \sin^2(n \pi/2)}{n \pi}, n \in \mathbb{Z} - \{0\}$, and h[0] = 0. We designed a (2L + 1)-length Hilbert transform impulse response using the frequency sampling method [17]. Increasing *L* improves the approximation quality to the ideal Hilbert transform behaviour. However, it comes with an increase in computational complexity for performing the filtering operation.

The sequence of operations in the proposed method (which we shall refer to as the H-annihilating filter method) is shown in Figure 2.



Fig. 5. Comparison of MSE of annihilating filter and H-annihilating filter on a cosine signal having three tones $\omega_1 = 0.1\pi$, $\omega_2 = 0.3\pi$, $\omega_3 = 0.9\pi$ and N = 21.



Fig. 3. Comparison of MSE of annihilating filter on a single tone complex exponential signal with $\omega_1 = \{0.02\pi \cdots 0.98\pi\}, N = 21$ and SNR = 15dB.



Fig. 4. Comparison of MSE of H-annihilating filter on a single tone cosine having $\omega_1 = 0.3\pi$ for N = 21, 41, and 61.

4. SIMULATION RESULTS

In this section, we present the numerical simulation results to evaluate the performance of the H-annihilating filter method. All the results were obtained by averaging over 500 independent Monte-Carlo simulations. As a first illustration, we compare in Figure 3, the MSE of the frequency estimated by the annihilating filter on a real and complex sinusoid as the frequency of the sinusoid is changed from 0.02π to 0.98π . The length of the signal, N = 21 and SNR = 15 dB. We observe that the MSE increases for a real sinusoid, when the frequency is close to 0 and π . This is because, the positive and negative exponentials are very close to one another when $\omega_1 < \frac{2\pi}{N}$ or $\omega_1 > (N/2-1)\frac{2\pi}{N}$ [12]. In comparison to this, there is almost no change in MSE of the frequency estimated by the annihilating filter on a complex exponential as the frequency is changed from 0.02π to 0.98π . Thus, we conclude that spectral interference between positive and negative frequencies is responsible for the poor performance of the annihilating filter on real sinusoids. In Figure 5, we have compared the performance of the proposed H-annihilating filter and the standard annihilating filter technique on a real sinusoid containing three frequencies $\omega_1 = 0.1\pi$, $\omega_2 = 0.3\pi$, $\omega_3 = 0.9\pi$ and length of the signal, N = 21. We note that, a relatively higher improvement in MSE performance is observed in the H-annihilating filter method over the annihilating filter method for ω_1 and ω_3 as compared to ω_2 . This is in agreement with observations made from Figure 3. The improvement observed in the H-annihilating filter method for ω_1 and ω_3 is about 6 dB and about 3 dB for ω_2 at a SNR of 15 dB. We see that there is a consistent improvement in performance with the H-annihilating filter method. In Figure 4, we study the effect of increasing the length of the input signal to the H-annihilating filter. There is an improvement of about 6 dB, when the length of the signal is increased from 21 to 61 at an SNR of 15 dB. Thus, the mean-square error performance of the filter improves as the length of the observation sequence is increased.

5. CONCLUSION

We have addressed the issue of frequency estimation of a real-valued multi-tone signal by annihilating filter. We proposed a modified Hannihilating filter, which converts the real signal into a complex one before the annihilating filter is applied. The algorithm is validated using numerical simulations. Experimental results show that there is an improvement of about 6 dB in the H-annihilating filter technique as compared with the annihilating filter technique when the frequency is not close to 0 or π . At frequencies close to 0 or π the annihilating filter suffers from severe spectral interference caused by the closeness of negative and positive complex exponentials. However, the proposed H-annihilating filter method does not suffer from such problems. The performance of the proposed method can be further improved by applying Cadzow's denoising method to the output of H-annihilating filter.

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