CRAMÉR-RAO BOUND FOR SAMPLING & RECONSTRUCTION OF FRI SIGNALS

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ABSTRACT

This paper analyses the estimation process for Finite Rate of Innovation (FRI) signals. The main contribution is the derivation of the well known Cramér-Rao Bound (CRB) for the estimation of signal parameters for a pulse stream. Other publications consider the estimation of the signal instead of its parameters or omit the effect of sampling. In this contribution both effect are considered and analytical expressions for the Fisher Information Matrix are obtained. Furthermore, for the estimation of parameters of a single pulse analytical expressions for CRB are given and for the case of multiple pulses dependencies of the CRB of the distance between the pulses are illustrated.

Index Terms— Finite Rate of Innovation, FRI, spectral estimation, minimum error variance, Cramér-Rao Bound, CRB, CRLB

1. INTRODUCTION

At the beginning of the last decade a new branch of sampling methods arose denoted as Compressed Sensing. The pioneering works of Donoho [1], Candés, Tao *et al.* [2–5] laid with their theory the foundation for CS. The basic principle is quite easy: if a signal is spare, you do not have to sample at the signal's Nyquist rate.
In parallel, Vetterli *et al.* [6–11] presented their ideas for sampling of signals which are sparse in their parametric description. Such signals can be described by a finite set of parameters per interval and are, therefore, called Finite Rate of Innovation (FRI) signals. For both approaches, CS and FRI signals, a comprehensive overview can be found in [12].
This paper will deal with the second approach. In general, FRI signals are not restricted to a certain bandwidth and sampling according to the Shannon-Nyquist sampling theorem [13] will not be possible. For example, a stream of K diracs per interval is completely described by their amplitudes and positions. Therefore, such a process has an FRI of $2K$ per interval but it cannot be sampled appropriately. After lowpass filtering sampling according to the sampling theorem will be possible. Early publications dealt with Gaussian or ideal lowpass filters [6–11]. However, the Gaussian lowpass leads to numerical problems during the reconstruction of the signal parameters and the ideal lowpass suffers from its infinitely long impulse response. Both problems have been solved by Eldar *et al.* [14,15] introducing the Sum of Sincs (SoS) kernel (as shown in Sec. 2).
Since the theory shows that the reconstruction of signal parameters will be exact, we will analyse the reconstruction performance under noisy conditions. First results for the Cramér-Rao Bound (CRB) in [16] show the minimum error variance for different system setups with continuous or sampled measurements for the reconstruction of the signal. Further results in [17] show the CRB for the parameter estimation but are restricted to a single pulse. In this paper the results are extended for multiple pulses and dependencies between the positions of different pulses will be shown. In the sequel, Sec. 2 will present the system design necessary to derive the CRB in Sec. 3. Finally, we compare the CRB with results obtained by parameter estimation with spectral estimation methods.

2. SYSTEM DESCRIPTION

2.1. FRI Signals & Sampling Scheme

An analogue FRI signal of the form

$$s\left(x\right) = \sum_{k=0}^{K-1} c_k \cdot h\left(x - x_k\right) \tag{1}$$

is assumed where K pulses at positions x_k with amplitudes c_k and a common impulse response $h\left(x\right)$ are superposed. All pulse positions are restricted to a certain interval, $x_k \in [0, \tau)$. Therefore, $s\left(x\right)$ can be interpreted as an FRI signal with a rate of innovation of $\frac{2K}{\tau}$. This model can be extended easily for the 2-D case [18].
Transforming $s\left(x\right)$ into the spectral domain delivers

$$S\left(f\right) = H\left(f\right) \cdot \sum_{k=0}^{K-1} c_k \cdot e^{-j2\pi f x_k} \quad . \tag{2}$$

The bandwidth of the considered FRI signal is assumed to be infinitely high and, therefore, sampling according to the sampling theorem will be infeasible. To avoid aliasing effects by sampling with finite rate, the FRI signal is transformed into a lowpass signal by convolution with a lowpass sampling kernel $g\left(x\right)$.

$$r\left(x\right) = s\left(x\right) * g\left(x\right) \tag{3}$$

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2.2. SoS Sampling Kernel

For lowpass filtering the SoS kernel as proposed in [16] is applied. Equivalent to Orthogonal Frequency Division Multiplexing (OFDM) systems in communications, e.g. in [19], multiple sinc functions are arranged in the spectral domain such that they do not interfere with each other at discrete frequencies $f = k/\tau \,\forall\, k \in \mathbb{Z}$.

$$G(f) = \tau \sum_{l \in \mathbb{L}} \alpha[l] \cdot \mathrm{sinc}(f \cdot \tau - l) \tag{4}$$

The different sinc functions are weighted with non-zero factors $\alpha[l]$ and shifted to positions defined by set $\mathbb{L}$ containing consecutive integers. The total number of sincs is obtained by the cardinality of the set, $|\mathbb{L}|$. Considering the zeros of the different sincs one can see that

$$G(f) = \tau \cdot \begin{cases} \alpha[l] \neq 0 & f \cdot \tau \in \mathbb{L} \\ 0 & f \cdot \tau \in \mathbb{Z} \backslash \mathbb{L} \\ \text{else} & \text{arbitrary} \end{cases} \tag{5}$$

holds. Transforming the kernel into the original domain delivers a superposition of complex exponentials (frequency shift) which are windowed by a rectangular function (obtained by the sincs).

$$g(x) = \mathrm{rect}\left(\frac{x}{\tau}\right) \cdot \sum_{l \in \mathbb{L}} \alpha[l]\, e^{j2\pi l x/\tau} \tag{6}$$

To allow a reconstruction of the unknown signal parameters at least $2K$ spectral coefficients are required. Thus, the set $\mathbb{L}$ should contain at least that many elements, respectively the filter kernel that many shifted sincs. Furthermore, a real valued kernel $g(x)$ is desired. Therefore, kernel $G(f)$ should be symmetric around zero. For this reason we define

$$\mathbb{L} = \left\{ -\frac{|\mathbb{L}| - 1}{2}, ..., \frac{|\mathbb{L}| - 1}{2} \right\} \tag{7}$$

where $|\mathbb{L}|$ has to be an odd integer and the filter coefficients have to be pairwise conjugate complex, $\alpha[l] = \alpha^*[-l]$.

2.3. Sampling, Bounds & Reconstruction

After convolution the signal bandwidth is changed according to $G(f)$ to a lowpass characteristic but still not limited and equidistant sampling at the rate $\frac{M}{\tau}$ with $M \in \{\mathbb{N}\,|M \geq |\mathbb{L}|\}$ leads to shifted copies at integer multiples of the sampling frequency. These copies lead to aliasing expect at discrete frequencies $f \cdot \tau \in \mathbb{L}$ where the shifted copies have their zeros as well (5). Thus, the spectral description at discrete frequencies $f \cdot \tau \in \mathbb{L}$ remains without aliasing after sampling. The samples can be described by

$$r[m] = (s * g)\left(x = \frac{m}{\tau}\right) \quad . \tag{8}$$

Transforming these samples into the spectral domain the Discrete Fourier Transform (DFT) will be exploited where a convolution corresponds to a multiplication in the spectral domain

$$\begin{aligned} R[m] &= G[m] \cdot S[m] \\ &= G[m] \cdot H[m] \cdot \sum_{k=0}^{K-1} c_k \cdot e^{-j2\pi m \frac{x_k}{\tau}} \end{aligned} \tag{9}$$

if either the signal $s(x)$ is periodic, $s(x) = s(x + \tau)$ or if the convolution in (3) is circular, $r(x) = s(x) \circledast g(x)$.
The discrete spectral coefficients can be considered to estimate the unknown positions. On the one hand this can be done via a linear equation system known as the Annihilating Filter method [6–9, 11] (also known as High Order Yule-Walker system or Prony's method). On the other hand more complex spectral estimation algorithms such as the Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) algorithm [20–22] or Unitary ESPRIT [23] can be used. To estimate the positions via the phases of K superposed exponentials (9) at least $2K$ non-zero coefficients are required, $|\mathbb{L}| \geq 2K$. Furthermore, to avoid aliasing sampling has to be done according to the sampling theorem with $M \geq |\mathbb{L}|$. Under the restriction that $|\mathbb{L}|$ has to be an odd integer this leads to

$$M \geq |\mathbb{L}| \geq 2K + 1 \quad . \tag{10}$$

2.4. Noise

Up to now the theory has been presented under ideal conditions neglecting noise at any kind of sensing device, quantisation noise at the sampling stage or errors at the spectral estimation stage due to limited numerical precision. In this paper we consider noisy samples of the form

$$\tilde{r}[m] = r[m] + z[m] \tag{11}$$

where $z[m]$ is assumed to be Additive White Gaussian Noise (AWGN) with variance σ_{Z}^2. Thus, the spectral coefficients considered for the estimation process will suffer from noise as well.

$$\tilde{R}[m] = R[m] + Z[m] \tag{12}$$

For these conditions the reconstruction result depends significantly on the noise variance, the design of the SoS kernel as well as on the sampling rate.

3. CRAMÉR-RAO BOUND

To evaluate whether the unknown signal parameters $\boldsymbol{\theta} = [c_0, ..., c_{K-1}, x_0, ..., x_{K-1}]$ can be estimated reliably from noisy measurements the well known Cramér-Rao Bound (CRB) will be considered. The theory can be found in many textbooks such as [24].

The minimum error variance achievable by an unbiased estimator is given by the diagonal elements of the inverse of the Fisher Information Matrix (FIM) $\boldsymbol{J}(\boldsymbol{\theta})$.

$$\mathrm{MSE}\left(\theta_i, \hat{\theta}_i\right) \geq \left[\boldsymbol{J}(\boldsymbol{\theta})^{-1}\right]_{i,i} \tag{13}$$

Here, subscript i denotes a certain element of a vector or matrix and variables with a hat denote the estimate of the same variable. For the case of AWGN the elements of the FIM can be determined by [24, (3.31)]

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{k,\kappa} = \frac{1}{\sigma_{\mathrm{Z}}^2} \sum_{m=0}^{M-1} \left(\frac{\partial r[m]}{\partial \theta_k}\right)\left(\frac{\partial r[m]}{\partial \theta_\kappa}\right) \tag{14}$$

and, of course, the FIM is symmetric since $[\boldsymbol{J}(\boldsymbol{\theta})]_{k,\kappa} = [\boldsymbol{J}(\boldsymbol{\theta})]_{\kappa,k}$ holds. For the proposed system it is beneficial to determine the partial derivation by the spectral representation of the samples $r[m]$ where we assume $h(x) = \delta(x)$ and, therefore, $H[m] = 1$.

$$r[m] = \sum_{k=0}^{K-1} c_k \sum_{l\in\mathbb{L}} G[l] \cdot e^{j2\pi l \frac{m}{M}} \cdot e^{-j2\pi l \frac{x_k}{\tau}} \tag{15}$$

First, we determine the derivations with respect to any amplitude c_k and position x_k.

$$\frac{\partial r[m]}{\partial c_k} = \sum_{l\in\mathbb{L}} G[l] \cdot e^{j2\pi l\left(\frac{m}{M} - \frac{x_k}{\tau}\right)} \tag{16}$$

$$\frac{\partial r[m]}{\partial x_k} = c_k \sum_{l\in\mathbb{L}} \frac{-j2\pi l}{\tau} G[l] \cdot e^{j2\pi l\left(\frac{m}{M} - \frac{x_k}{\tau}\right)} \tag{17}$$

Second, we determine the elements of the FIM by summing up the products of different derivatives according to (14). In the following the superscripts denote according to which variables the derivatives were calculated.

$$[\boldsymbol{J}(\boldsymbol{\theta})]^{(c_k,c_\kappa)} = \frac{1}{\sigma_{\mathrm{Z}}^2} \cdot \sum_{m=0}^{M-1} \left(\sum_{l\in\mathbb{L}} G[l] \cdot e^{j2\pi l\left(\frac{m}{M} - \frac{x_k}{\tau}\right)}\right) \cdot \left(\sum_{\iota\in\mathbb{L}} G[\iota] \cdot e^{j2\pi \iota\left(\frac{m}{M} - \frac{x_\kappa}{\tau}\right)}\right) \tag{18}$$

By interchanging the sums and combining the exponential function one can obtain the following expression.

$$[\boldsymbol{J}(\boldsymbol{\theta})]^{(c_k,c_\kappa)} = \frac{1}{\sigma_{\mathrm{Z}}^2} \cdot \sum_{l\in\mathbb{L}} \sum_{\iota\in\mathbb{L}} G[l] \cdot G[\iota] \cdot e^{-j2\pi \frac{l x_k + \iota x_\kappa}{\tau}} \cdot \frac{e^{j2\pi(l+\iota)} - 1}{e^{j2\pi(l+\iota)\frac{1}{M}} - 1} \tag{19}$$

The numerator of the last fraction will always be zero since $l, \iota \in \mathbb{L} \subset \mathbb{Z}$. For the case of $l = -\iota$ the denominator becomes zero as well. As long as $M \geq |\mathbb{L}|$ holds no other cases arise where the denominator becomes zero. However, for these cases L'Hôpital's rule show that the last ratio equals M.

$$[\boldsymbol{J}(\boldsymbol{\theta})]^{(c_k,c_\kappa)} = \frac{M}{\sigma_{\mathrm{Z}}^2} \cdot \sum_{l\in\mathbb{L}} G[l] \cdot G[-l] \cdot e^{-j2\pi l \frac{x_k - x_\kappa}{\tau}} \tag{20}$$

Exploiting that the filter kernel is symmetric around zero $G[l] = G^*[-l]$ the complex exponentials can be combined to cosine terms.

$$[\boldsymbol{J}(\boldsymbol{\theta})]^{(c_k,c_\kappa)} = \frac{M}{\sigma_{\mathrm{Z}}^2} \cdot |G[0]|^2 + \frac{M}{\sigma_{\mathrm{Z}}^2} \cdot \sum_{l=1}^{\frac{|\mathbb{L}|-1}{2}} |G[l]|^2 \cdot 2\cos\left(2\pi l \frac{x_k - x_\kappa}{\tau}\right) \tag{21}$$

In the same fashion the remaining elements of the FIM can be determined.

$$[\boldsymbol{J}(\boldsymbol{\theta})]^{(x_k,x_\kappa)} = \frac{4\pi^2 M}{\tau^2} \cdot \frac{c_k c_\kappa}{\sigma_{\mathrm{Z}}^2} \cdot \sum_{l=1}^{\frac{|\mathbb{L}|-1}{2}} l^2 \cdot |G[l]|^2 \cdot 2\cos\left(2\pi l \frac{x_k - x_\kappa}{\tau}\right) \tag{22}$$

$$[\boldsymbol{J}(\boldsymbol{\theta})]^{(x_k,c_\kappa)} = \frac{2\pi M}{\tau} \cdot \frac{c_k}{\sigma_{\mathrm{Z}}^2} \cdot \sum_{l=1}^{\frac{|\mathbb{L}|-1}{2}} l \cdot |G[l]|^2 \cdot 2\sin\left(2\pi l \frac{x_k - x_\kappa}{\tau}\right) \tag{23}$$

Discussion

In general, for all main diagonal elements of the FIM, i.e. $[\boldsymbol{J}(\boldsymbol{\theta})]^{(c_k,c_k)}$ and $[\boldsymbol{J}(\boldsymbol{\theta})]^{(x_k,x_k)}$, the cosine terms equal one. Furthermore, all elements of the form $[\boldsymbol{J}(\boldsymbol{\theta})]^{(x_k,c_k)}$ become zero due to the sine term. The terms of squared amplitudes divided by the noise variance can be interpreted as Signal to Noise Ratio (SNR). In addition, changing the kernels coefficients $\alpha[l]$ leads to a variation of the CRB.

For the single pulse FRI signal, $K = 1$, the FIM becomes a diagonal matrix and the minimum error variance can be determined as simple as shown in [17].

$$\mathrm{MSE}(c_0, \hat{c}_0) \geq [\boldsymbol{J}(\boldsymbol{\theta})]^{-1}_{c_0,c_0} = \frac{\sigma_{\mathrm{Z}}^2}{M \cdot \sum_{l\in\mathbb{L}} |G[l]|^2}$$

$$\mathrm{MSE}(x_0, \hat{x}_0) \geq [\boldsymbol{J}(\boldsymbol{\theta})]^{-1}_{x_0,x_0} = \frac{\tau^2 \sigma_{\mathrm{Z}}^2}{4\pi^2 M c_0^2 \cdot \sum_{l\in\mathbb{L}} l^2 \cdot |G[l]|^2}$$

Here, the minimum error variance is proportional to the SNR and decreases with the number of samples. Increasing the kernel bandwidth the minimum error variance decreases.

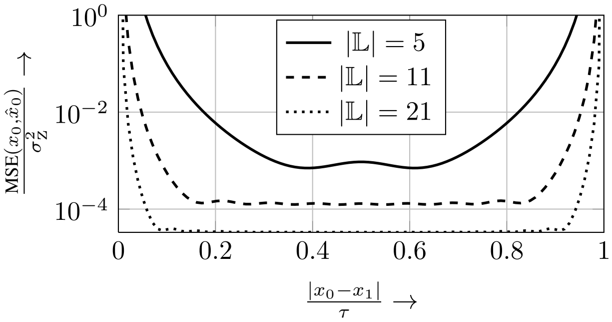

Fig. 1. CRB for two-pulse FRI signal, unit power filter kernel, $M = 21$

For multiple pulse FRI signals the solutions is not that obvious since the inverse FIM has to be determined which is no longer a diagonal matrix but numerical results can be obtained easily. The distance between different pulses has an important role as the sine and cosine terms scale the error variance as shown in Sec. 4. The dependencies of the CRB of M, σ_Z^2 and the kernel remain equal in quality.

4. NUMERICAL EVALUATION

This section numerically evaluates the results obtained for the CRB. For Fig. 1 two pulses, $K = 2$, are assumed with unknown amplitudes $c_0 = c_1 = 1$. The figure shows a strong dependencies of the achievable estimation performance dependent on the filter kernel and the distance between the pulses. Increasing the kernel bandwidth leads to steeper slopes for kernel shape $g(x)$ and closely located pulses can be well separated. Furthermore, if the distance between the two pulses is too small, their position cannot be reliably estimated any more. The illustrated CRB is symmetric around 0.5 due to the periodicity of the problem. In addition, the achievable error floor decreases with $\mathbb{L}$ due to additional spectral coefficients which help to suppress noise. The results for the estimation performance of the amplitudes are not presented here but show a similar dependency of filter bandwidth, sampling rate and distance to each other.

Increasing the number of pulses leads to additional dependencies between the pulses dependent on the distance to each other. The result will be similar: whenever two pulses are closely located they cannot be estimated reliably but increasing the bandwidth, namely parameter $|\mathbb{L}|$, allows smaller distances between the impulses at the same estimation performance. If two out of K pulses are closely located, $K - 2$ pulses can still be estimated reliably when the dimension for the estimation algorithm is changed to $K - 1$. Then two closely located pulses will be estimated as one single pulse.

In Fig. 2 additional results show a comparison between simulation results and the analytically obtained CRB. For these results K pulses have been generated in 10^4 random realisations where the samples are disturbed by AWGN. For each realisation, on the one hand, the CRB has been determined and on the other hand, the positions have been estimated via the ESPRIT algorithm. It can be seen that the results differ significantly as K increases. For many realisations of the FRI signal the illustrated CRB can also be achieved for $K = 5$. However, in average this is not the case since few realisations with very closely located pulses results in a higher MSE than the CRB promises. Fig. 3 illustrates that behaviour for a realisation of four pulses with a fifth pulse changing its position x_4. By comparing the solid lines for ESPRIT and CRB it can be seen that the trends of both curves equal but for closely located pulses, when the FIM is close to singular, the result for ESPRIT is significantly higher than for the CRB. In average, this results in a higher MSE as illustrated by the dotted lines. Performing the same simulation without a fifth pulse and if the distances between the remaining pulses are sufficiently high enough, the CRB can almost be achieved by ESPRIT as the dashed lines show. Summarising, with a certain distance between the pulses ESPRIT can achieve the same results as the CRB for all noise powers.

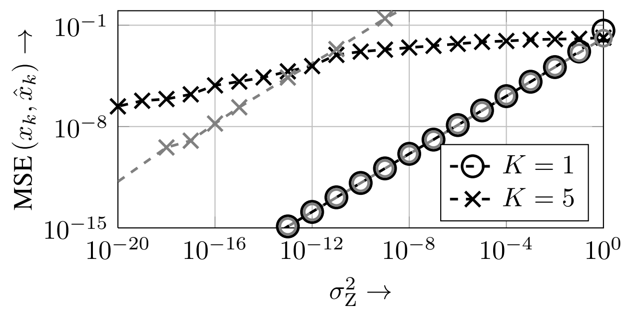

Fig. 2. Comparison of simulation results (via ESPRIT, black) and CRB (grey) for K pulses, $|\mathbb{L}| = 2K + 1$, unit power filter kernel, $M = |\mathbb{L}|$.

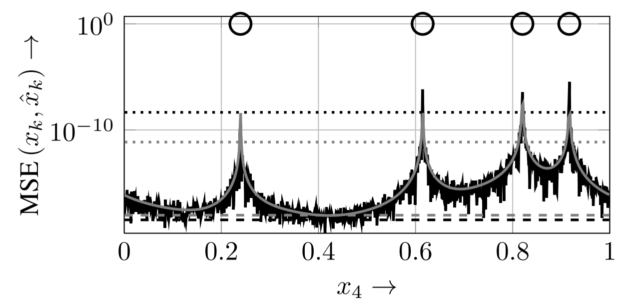

Fig. 3. Comparison of simulation results (via ESPRIT, black) and CRB (grey); $K - 1 = 4$ pulses on fixed positions (circles); x_4 changes on x-axis; solid lines: MSE dependent on x_4; dotted lines: average of MSE for all positions x_4; dashed lines: MSE by omitting x_4; $M = |\mathbb{L}| = 11$; $\sigma_Z^2 = 10^{-15}$

5. CONCLUSION

In this paper it is shown how the CRB can be derived for the estimation of signals parameters of a sampled FRI signal. Furthermore, the dependencies of the CRB of the system design and the FRI signal realisation is shown.

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