CLOCK JITTER IN SAMPLING AND RECONSTRUCTION

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ABSTRACT

Clock jitter plays an important role in A/D and D/A conversions.Various aspects of this problem have been extensively studied. We consider here the effect of clock jitters in the basic system of digital filtering of analog signals. The system consists of an A/D, followed by a digital filter, and then followed by a hold circuit for D/A reconstruction. Thus the readout jitter affects both the leading edge and the trailing edge of the rectangular pulse in the reconstruction. This system includes the familiar sampling of an analog signal and subsequent reconstruction from its samples. Expressions are derived for the mean squared error caused by the sampling clock jitter and readout clock jitters. Expressions are also derived showing how the signal spectra are modified by clock jitters.

Index Terms— Clock jitter, jittered sampling, analog-to-digital conversion, digital-to-analog conversion, sample and hold.

1. INTRODUCTION

Clock jitters play an important role in the performance of A/D and D/A, and has been extensively studied. An incomplete short list of some previous publications is given as references [1]-[7]. Many of the previous works deal separately with A/D and D/A; some assumed specific input or jitter statistics. We consider in this paper the effect of clock jitters in the basic system for digital filtering of analog signals shown in Figure 1. Without the digital filter, the system reduces to the familiar sampling and reconstruction problem. We derive expressions for the mean squared error caused by the sampling and readout clock jitters. Expressions are also derived to describe how the signal spectra are modified by the clock jitters. These results provide quantitative measures of the effects of clock jitters.

The clock jitters are considered as small random errors. The signals affected by the clock jitters are treated as random signals. As we do not consider signal quantization here, the term ‘digital signal’ shall mean sampled signal or discrete-time signal, and the term ‘analog signal’ shall mean continuous-time signal. Subscript ‘a’ denotes analog signals; subscript ‘d’ denotes digital signals.

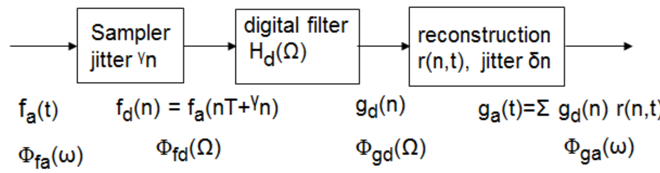

Figure 1 – Digital Processing of Analog Signal

When there is no clock jitter, the input analog signal $f_a(t)$ is sampled every T seconds. The samples form a digital signal

$$f_d(n) = f_a(nT). \quad (1.1)$$

When there is sampling clock jitter, the n-th sample is taken at $t = nT + \gamma_n$ instead of nT. See Figure 2. The actual digital signal is then

$$f_d(n) = f_a(nT + \gamma_n). \quad (1.2)$$

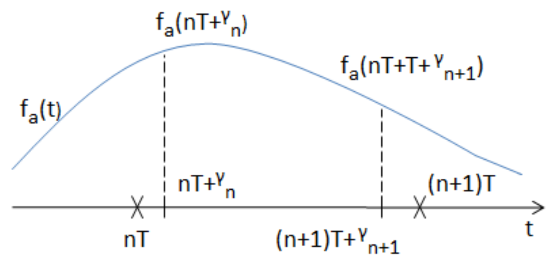

Figure 2 – Sampling with and without clock jitter

The digital signal $f_d(n)$ is processed by a digital filter having frequency response $H_d(\Omega)$. The digital filter output $g_d(n)$ is converted to an analog signal $g_a(t)$ by a hold circuit, where the sample $g_d(n)$ is read out at time $t = nT$ and its value held until the next readout time $t = nT+T$. The next sample $g_d(n+1)$ is then readout. The reconstructed analog signal $g_a(t)$ can be written as

$$g_a(t) = \Sigma_n\, g_d(n)\, r_o(n,t), \quad (1.3)$$

where $r_o(n,t)$ is the rectangular pulse shown in Figure 3. That is, $r_o(n,t)=1$ for $nT<t<nT+T$, and $r_o(n,t) = 0$ for other t. Unless otherwise specified, summations such as Σ_k or Σ_n are from $-\infty$ to ∞.

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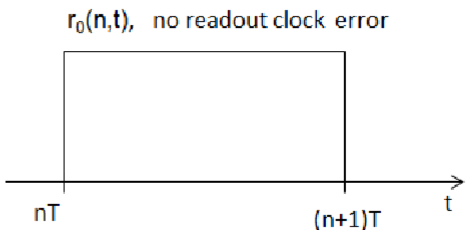

Figure 3 – $r_o(n,t)$ for reconstruction using hold circuit

The subscript 'o' in $r_o(n,t)$ indicates that there is no readout clock jitter. Additional analog anti-image filtering is often applied to $g_a(t)$ to suppress the spectral images beyond the baseband $|\omega| < \pi/T$, and to compensate for the non-ideal lowpass frequency response of the hold circuit.

When there is clock jitter in the reconstruction of $g_a(t)$, the n-th sample $g_d(n)$ is read out at time $nT + \delta_n + \alpha$, instead of nT, where δ_n is the error of the n-th read out time and α is the offset of the read out times from t=nT. [1] The value of $g_d(n)$ is held until the next read-out time $(n+1)T+\delta_{n+1}+\alpha$. Thus

$$g_a(t) = \sum_n g_d(n)\, r(n,t), \tag{1.4}$$

where r(n,t) is a rectangular pulse shown in Figure 4.

$$r(n,t) = 1 \text{ for } nT+\delta_n+\alpha < t < (n+1)T+\delta_{n+1}+\alpha, \text{ and}$$
$$r(n,t) = 0 \text{ for other t.} \tag{1.5}$$

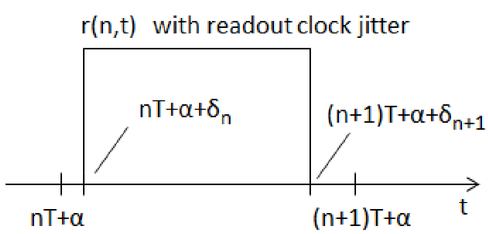

Figure 4 – r(n,t) for reconstruction with clock jitter

This r(n,t) becomes $r_o(n,t)$ of Figure 3 when there is no readout clock jitter. We note that the clock jitter affects both the leading and the trailing edges of the pulse r(n,t).

2. SAMPLING WITH CLOCK JITTER

When there is sampling clock jitter, $f_d(n) = f_a(nT + \gamma_n)$. We assume that the sampling clock jitter $\{\gamma_n\}$ is a stationary random sequence. It is shown in Appendix A that the digital signal $f_d(n)$ is WSS, and its power spectral density $\Phi_{fd}(\Omega)$ is derived there. See (A3).

2.1. Mean squared error due to sampling clock jitter

Denote by $f_{do}(n)$ the signal $f_d(n)$ when there is no clock jitter (γ_n=0). It is shown in Appendix A that the mean squared error (MSE) caused by the sampling jitter is

$$E\{|f_d(n)-f_{do}(n)|^2\}=(1/2\pi) \int \Phi_{fa}(\omega)[2-C(\omega)-C(\omega)^*]d\omega, \tag{2.1}$$

where $C(\omega) = E\{\exp(j\omega\gamma_n)\}$ is the characteristic function of γ_n. Unless otherwise specified, integrations such as $\int d\omega$ are from $-\infty$ to ∞.

For small σ_γ^2 and ω, $C(\omega) \approx 1 - \sigma_\gamma^2\omega^2/2$, and the MSE of (2.1) can be approximated by

$$E\{|f_d(n) - f_{do}(n)|^2\} \approx \sigma_\gamma^2 (1/2\pi) \int \Phi_{fa}(\omega)\omega^2\, d\omega. \tag{2.2}$$

(2.2) can be further simplified for particular inputs, such as sinusoidal input or input having a constant spectrum.

2.2 Signal spectrum $\Phi_{fd}(\Omega)$ for independent clock jitter

Often the sampling clock jitters $\{\gamma_n\}$ are independent and identically distributed having zero mean and small variance σ_γ^2. An expression for $\Phi_{fd}(\Omega)$ is derived in Appendix A.

If $\Phi_{fa}(\omega)$=0 for $|\omega| > \pi/T$, $\Phi_{fd}(\Omega)$ is given by

$$\Phi_{fd}(\Omega) = (1/T)\, \Phi_{fa}(\Omega/T)\, |C(\Omega/T)|^2$$
$$+(1/2\pi)\int_{-\pi/T}^{\pi/T} \Phi_{fa}(\omega)\, [1-|C(\omega)|^2]\, d\omega, \quad |\Omega| < \pi. \tag{2.3}$$

When there is no clock jitter, $C(\Omega/T)$=1. Thus the clock jitter γ_n modifies the signal spectrum $\Phi_{fd}(\Omega)$ from $(1/T)\Phi_{fa}(\Omega/T)$ to $(1/T)\Phi_{fa}(\Omega/T)|C(\Omega/T)|^2$. It also introduces a constant into $\Phi_{fd}(\Omega)$ given by the second term of (2.3), which is a white noise like component. (2.3) can be approximated by

$$\Phi_{fd}(\Omega) = (1/T)\, \Phi_{fa}(\Omega/T) + \sigma_\gamma^2 B(\Omega), \tag{2.4}$$

where

$$B(\Omega)=(1/2\pi)\int_{-\pi/T}^{\pi/T} \Phi_{fa}(\omega)\omega^2 d\omega - \Phi_{fa}(\Omega/T)\Omega^2/T^3. \tag{2.5}$$

Further simplification is possible for specific input spectrum $\Phi_{fa}(\omega)$, such as a sinusoidal test signal.

3. CLOCK JITTER IN RECONSTRUCTION

When there is clock jitter in the reconstruction of $g_a(t)$,

$$g_a(t) = \sum_n g_d(n)\, r(n,t), \tag{3.1}$$

where r(n,t) is given by (1.5). We assume that $\{\delta_n\}$ is a stationary random sequence. It is shown in Appendix B that $g_a(t)$ is wide sense stationary and its power spectral density $\Phi_{ga}(\omega)$ is given by (B6).

3.1. Independent readout clock jitter

Often $\{\delta_n\}$ are independent and identically distributed having zero mean and small variance σ_δ^2. It is shown in

Appendix B that

$$\Phi_{ga}(\omega) = K(\omega)\,|D(\omega)|^2\,\Phi_{gd}(\omega T)+A[1-|D(\omega)|^2]\,/(T\omega^2), \quad (3.2)$$

where $K(\omega) = |(1 - e^{-j\omega T}) / j\omega|^2$ is the frequency response of the hold circuit, $D(\omega) = E\{\exp(j\omega\,\delta_n)\}$ is the characteristic function of δ_n, and A is a constant given by

$$A = (1/\pi)\int_{-\pi}^{\pi} \Phi_{gd}(\Omega)\,[1-\cos(\Omega)]\,d\Omega. \quad (3.3)$$

When there is no readout clock jitter, $D(\omega)=1$, and the first term of (3.2) becomes $K(\omega)\,\Phi_{gd}(\omega T)$. Thus the clock jitter δ_n modifies $\Phi_{ga}(\omega)$ by the factor $|D(\omega)|^2$, and it also introduces an additional "distortion" given by the second term of (3.2).

3.2. MSE due to readout clock jitter

Denoting by $g_{ao}(t)$ the reconstructed analog signal when there is no readout clock jitter, the MSE introduced by the readout clock jitter is $E\{\,|g_a(t) - g_{ao}(t)|^2\}$. Following similar steps used in Appendix A to derive (2.2), it can be shown that

$$E\{|g_a(t)-g_{ao}(t)|^2\} \approx A(1/2\pi)\int [1-|D(\omega)|^2]/(T\omega^2)\,d\omega. \quad (3.4)$$

The integrand of (3.4) is $\approx \sigma_\delta^2$ for small ω, and is dominated by $1/\omega^2$ for large ω. Note, however, the approximation of $|D(\omega)|^2$ by $1 - \omega^2\sigma_\delta^2$ is not appropriate here because of the range of integration.

4. RELATING OUTPUT SPECTRUM $\Phi_{ga}(\omega)$ TO INPUT SPECTRUM $\Phi_{fa}(\omega)$

Suppose $\Phi_{fa}(\omega) = 0$ for $|\omega| > \pi/T$. If the analog anti-image filtering following the hold circuit suppresses $\Phi_{ga}(\omega)$ beyond $|\omega| > \pi/T$, then $\Phi_{ga}(\omega) \sim 0$ outside the band $-\pi/T < \omega < \pi/T$. Following some tedious but rather straightforward derivation, it can be shown that (3.2) simplifies to

$$\Phi_{ga}(\omega)=K(\omega)\,|H_d(\omega T)|^2\Phi_{fa}(\omega)/T+\sigma_\gamma^2 U(\omega)+\sigma_\delta^2 V(\omega), \quad (4.1)$$

where

$$U(\omega) = B(\omega T)\,K(\omega)\,T\,|H_d(\omega T)|^2, \quad (4.2)$$

$$B(\omega T) = (1/2\pi)\int \Phi_{fa}(\omega)\,\omega^2\,d\omega - (1/T)\,\Phi_{fa}(\omega)\,\omega^2, \quad (4.3)$$

$$V(\omega) = (1/2\pi T^2)\int_{-\pi}^{\pi} \Phi_{fa}(\Omega/T)|H_d(\Omega)|^2[2-2\cos(\Omega)]d\Omega - \omega^2 K(\omega)\Phi_{fa}(\omega)|\,H_d(\omega T)\,|^2. \quad (4.4)$$

The first term in (4.1) is the jitter free spectrum. The second and third terms are due to clock jitters. $U(\omega)$ and $V(\omega)$ can be evaluated for specific input signal spectrum.

For example, if the input is a sinusoidal signal of frequency ω_o, then $\Phi_{fa}(\omega) = \pi\,[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]$, and the first term of (4.1) is $[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]\,K(\omega_o)\,|\,H_d(\omega_o T)\,|^2$. (4.2), (4.3) and (4.4) can be similiarly calculated.

5. SAMPLING AND RECONSTRUCTION WITHOUT DIGITAL FILTER

The sampling of a properly band limited analog signal $f_a(t)$ and using the samples $f_d(n)$ to reconstruct an analog signal $g_a(t)$ by a hold circuit is a special case of Figure 1 with $H_d(\Omega) = 1$. Denote by $g_a(t)$ the reconstructed analog signal with clock jitters in sampling and in reconstruction, and by $g_{ao}(t)$ the reconstructed signal when there is no clock jitters. It can be shown that the MSE is given by

$$E\{|g_a(t)-g_{ao}(t)|^2\}= \sigma_\gamma^2(1/2\pi)\int_{-\pi/T}^{\pi/T} \Phi_{fa}(\omega)\omega^2 d\omega + (A/2\pi)\int [1-|D(\omega)|^2]\,/\,(T\omega^2)\,d\omega, \quad (5.1)$$

where A is given by (3.3). The remark after (3.4) also applies to the second term.

Appendix A – derivation of results in Section 2

The autocorrelation function of $f_d(n)$ is

$$E\{f_d(n)\,f_d(n-k)^*\} = E\{R_{fa}(kT+\gamma_n - \gamma_{n-k})\}, \quad (A1)$$

where $R_{fa}(\,.\,)$ is the autocorrelation function of $f_a(\,.\,)$. The expectation is to be taken with respect to γ_n. By writing $R_{fa}(kT + \gamma_n - \gamma_{n-k})$ as the inverse Fourier transform of $\Phi_{fa}(\omega)$, (A1) becomes

$$E\{f_d(n)\,f_d(n-k)^*\}= (1/2\pi)\int \Phi_{fa}(\omega)\,e^{jkT\omega}\,M_\gamma(\omega,k)\,d\omega, \quad (A2)$$

where $M_\gamma(\omega,k) = E\{\exp(j\omega\gamma_n - j\omega\gamma_{n-k})\}$.

Because γ_n is stationary, $M_\gamma(\omega,k)$ does not depend on n and (A2) is not a function of n. So $f_d(n)$ is WSS and we have $R_{fd}(k) = (1/2\pi)\int \Phi_{fa}(\omega)\,e^{jkT\omega}\,M_\gamma(\omega,k)\,d\omega$.

Taking the Fourier transform of $R_{fd}(k)$ gives the power spectral density

$$\Phi_{fd}(\Omega)=(1/2\pi)\int \Phi_{fa}(\omega)\sum_k M_\gamma(\omega,k)e^{jk(T\omega-\Omega)}d\omega. \quad (A3)$$

A.1. Independent sampling clock jitter

If $\{\gamma_n\}$ is independent and identically distributed with zero mean and small variance σ_γ^2, $M_\gamma(\omega,k) = |\,C(\omega)\,|^2$ for $k \neq 0$ and $M_\gamma(\omega,0) = 1$. (A3) then becomes

$$\Phi_{fd}(\Omega)=(1/T)\sum_k \Phi_{fa}(\Omega/T-2\pi k/T)\,|C(\Omega/T-2\pi k/T)|^2 +(1/2\pi)\int \Phi_{fa}(\omega)\,[1-|C(\omega)|^2]\,d\omega. \quad (A4)$$

For small σ_γ^2 and ω, $C(\omega) \approx 1 - \sigma_\gamma^2 \omega^2/2$. If $\Phi_{fa}(\omega) = 0$ for $|\omega| > \pi/T$, the k-th term in the summation in (A4) is 0 outside the interval $|\Omega - 2\pi k| < \pi$. (A4) simplifies to (2.4).

A.2. Derivation of MSE.

The MSE introduced by the sampling clock jitter is $E\{| f_d(n) - f_{do}(n) |^2\}$, where $f_{do}(n)$ is $f_d(n)$ in the absence of sampling clock jitter (γ_n=0). This MSE is

$$E\{ | f_d(n) |^2 + | f_{do}(n) |^2 - f_d(n) f_{do}(n)^* - f_d(n)^* f_{do}(n) \}. \quad (A5)$$

The first terms in (A5) is given by (A2) with k=0. The second term of (A5) is also given by (A2), but with γ_n=0 and k=0. Both reduce to $(1/2\pi)\int \Phi_{fa}(\omega)\, d\omega$. The third term is $-(1/2\pi) \int \Phi_{fa}(\omega)C(\omega)d\omega$, where $C(\omega)=E\{\exp(j\omega \gamma_n)\}$ is the characteristic function of γ_n. The fourth term of (A5) is the complex conjugate of the third term. With these simplifications, the MSE (A5) becomes (2.1).

If $\{ \gamma_n \}$ is independent and identically distributed with zero mean and small mean square error σ_γ^2, $\Phi_{fd}(\Omega)$ is given by (A4) above. If $\Phi_{fa}(\omega) = 0$ for $|\omega| > \pi/T$, the k-th term in the Summation in (A4) is 0 outside $|\Omega - 2\pi k| < \pi$. (A4) then simplifies to (2.3)

Appendix B – derivation of results in Section 3

The autocorrelation function of $g_a(t)$ is

$$E\{g_a(t)g_a(t-\tau)^*\}=\Sigma_k \Sigma_n R_{gd}(k)E\{r(n,t)\, r(n-k,t-\tau)^*\}, \quad (B1)$$

where $R_{gd}(k) = E\{ g_d(n)\, g_d(n-k)^* \}$ is the autocorrelation function of $g_d(n)$. $r(n,t)$ and $r(n-k,t-\tau)^*$ can be expressed as inverse Fourier transform. See Figures 3 and 4.

$$r(n,t) = (1/2\pi) \int d\omega\, e^{j\omega t}\, e^{-j\omega nT - j\omega\alpha} \times [\exp(-j\omega\delta_n) - \exp(-j\omega T - j\omega\delta_{n+1})] / j\omega, \quad (B2)$$

$$r(n-k,t-\tau)^* = (1/2\pi) \int d\lambda\, e^{-j\lambda(t-\tau)}\, e^{j\lambda(nT-kT+\alpha)} \times [\exp(j\lambda\delta_{n-k}) - \exp(j\lambda T + j\lambda\delta_{n-k+1})]/(-j\lambda). \quad (B3)$$

Insert these expressions in (B1), and consider the expectation with respect to the δ's:

$$E\{ [\exp(-j\omega\delta_n) - \exp(-j\omega T - j\omega\delta_{n+1})] \times [\exp(j\lambda\delta_{n-k}) - \exp(j\lambda T + j\lambda\delta_{n-k+1})]\}.$$

This expectation does not depend on n. So it can be taken out of the summation Σ_n.

Taking the expectation with respect to α involves the integration $\int_0^T (1/T)\, d\alpha$. Letting $x = t - nT - \alpha$ and summing over n gives $\int_{-\infty}^{\infty} (1/T)\, dx$. We now collect all terms involving x and performing this integration. The result is $\int_{-\infty}^{\infty} e^{jx(\omega-\lambda)}\, dx = 2\pi\, \delta(\omega-\lambda)$, where $\delta(\,.\,)$ is the unit impulse function. Integrating with λ replaces λ by ω. (B1) then becomes

$$R_{ga}(\tau)=(1/2\pi)\int e^{j\omega\tau}\{ \Sigma_k R_{gd}(k)M_\delta(\omega,k)e^{-jkT\omega} / (T\omega^2) \}d\omega,$$

where (B4)

$$M_\delta(\omega,k) = E\{ [\exp(-j\omega\delta_n) - \exp(-j\omega T - j\omega\delta_{n+1})] \times [\exp(j\omega\delta_{n-k}) - \exp(j\omega T + j\omega\delta_{n-k+1})] \}. \quad (B5)$$

It is seen that $g_a(t)$ is WSS. $\Phi_{ga}(\omega)$ can be identified from (B4) as the Fourier transform of $R_{ga}(\tau)$. So

$$\Phi_{ga}(\omega) = \Sigma_k\, e^{-jkT\omega} R_{gd}(k)\, M_\delta(\omega,k) / (T\omega^2). \quad (B6)$$

When the readout timing errors $\{ \delta_n \}$ are independent and identically distributed having zero mean and variance σ_δ^2. $M_\delta(\omega,k)$ of (B6) can be explicitly evaluated to facilitate the summation of (B6). After some simplifications, (B6) reduces to (3.2).

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