SSIM PERFORMANCE LIMITATION OF LINEAR EQUALIZERS

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ABSTRACT

The performance limitation of linear equalizers is studied for the structural similarity (SSIM) criteria. Given a blurring filter and an image with zero mean, the closed form formula is obtained to compute the maximal SSIM index and the corresponding optimal linear equalizer. The formula shows that the equalizer with maximal SSIM index is equal to the one with minimal mean square error (MSE) multiplied by a positive real number. Numerical examples are given to demonstrate the theoretical results.

Index Terms— Structure similarity; performance limitation; image restoration; linear equalizer

1. INTRODUCTION

A quality-assessment measure, called structural similarity (SSIM) index, has been drawing much attention in the field of image/video processing [1]-[3], and has found a variety of applications, ranging from image coding, restoration and fusion, to watermarking and biometrics [3].

The SSIM index represents the quality of a distorted image by comparing the correlations in luminance, contrast, and structure, between the reference and distorted images [2]. In most existing works, SSIM has been used for quality evaluation and algorithm comparison purposes only. There have been efforts to use SSIM as optimization criteria in order to improve perceived image/video quality in a number of image processing problems [3]-[8]. In particular, an algorithm was proposed in [8] for designing the optimal linear filter that maximizes the Stat-SSIM index, which is a statistical version of the SSIM index. It has been shown that the non-convex optimization problem can be transformed into a quasi-convex problem, which has a near closed-form solution and can be efficiently solved by using a bisection procedure [8].

In this paper, we address the SSIM performance limitation that can be achieved by linear equalizers. Both the maximal SSIM index and the corresponding optimal linear equalizer can be computed by closed-form formulas, which provides more insights into the SSIM based equalization problem than the iterative algorithm in [8]. We adopt the same assumption as [8] that the blurring filter and the power spectral density of the additive noise component is known at the receiver, under which the advanced technique can be used to estimate power spectral density of the original image [9, 10].

The main contributions of this paper are as follows. (i) The closed-form formulas of the optimal linear equalizer and the optimal SSIM performance are provided for the source with zero mean. (ii) It is shown that the optimal equalizer designed by SSIM criterion and that by mean square error (MSE) criterion differ only from a scalar factor, which turns out to be the inverse of the achievable SSIM index. (iii) The solution establishes the relation of the SSIM performance to the property of the blurring filters and the length of the equalization filters.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this section, we provide some preliminaries about the equalization and the SSIM index. Please refer to [2, 8] for more details.

The system model is shown in Fig. 1, where $H(z)$ and $G(z)$ denote the blurring filter and the equalization filter respectively, $x[n]$ and $\hat{x}[n]$ denote the original image and the reconstructed image respectively, $y[n]$ is the image generated

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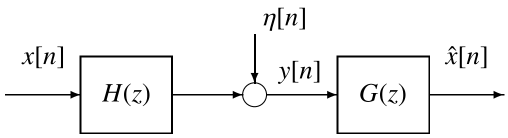

Fig. 1 The system diagram

by the blurring filter and $\eta[n]$ is the noise into the channel. Throughout the paper, we use the same assumption as [8] that the blurring filter and the power spectral density of the additive noise component are known at the receiver. Let $H(z)$ and $G(z)$ be written as

$$H(z) = \sum_{i=0}^{M-1} h_i z^{-i} \text{ and } G(z) = \sum_{j=0}^{N-1} g_j z^{-j},$$

where $h[i]$ and $g[j]$ are the impulse coefficients, M and N are the length of the corresponding filters. Our goal is to design the equalizer $G(z)$ such that the SSIM index between $\hat{x}[n]$ and $x[n]$ is maximized. Note that

$$\begin{aligned} y[n] &= \sum_{i=0}^{M-1} h[i]x[n-i] + \eta[n] \\ \hat{x}[n] &= \sum_{j=0}^{N-1} g[j]y[n-j]. \end{aligned}$$

In this paper, we use the simplified form of the SSIM index as defined by [4, 8]:

$$\text{StatSSIM}(x, \hat{x}) = \frac{2\mu_x\mu_{\hat{x}} + c_1}{\mu_x^2 + \mu_{\hat{x}}^2 + c_1} \frac{2\sigma_{x\hat{x}} + c_2}{\sigma_x^2 + \sigma_{\hat{x}}^2 + c_2} \tag{1}$$

where x and $\hat{x}$ are the source and the reconstructed image respectively.

The problem can be stated as follows: Given the blurring M-tap filter $\mathbf{H}(z) = \sum_{i=0}^{M-1} h[i]z^{-i}$, the noise $\eta[n]$ with known PSD, and the observed image $y[n]$, to design the N-tap linear equalizer $\mathbf{G}(z) = \sum_{j=0}^{N-1} g[j]z^{-j}$ such that the StatSSIM index between $x[n]$ and $\hat{x}[n]$ is maximized. To compute the StatSSIM index, we need more notations. Denote

$$\begin{aligned} \mathbf{g} &= \begin{bmatrix} g[0] & g[1] & \cdots & g[N-1] \end{bmatrix} \\ \mathbf{h} &= \begin{bmatrix} h[0] & h[1] & \cdots & h[M-1] \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{y}[n] &= \begin{bmatrix} y[n] & y[n-1] & \cdots & y[n-N+1] \end{bmatrix}^T \\ \underline{\eta}[n] &= \begin{bmatrix} \eta[n] & \eta[n-1] & \cdots & \eta[n-N+1] \end{bmatrix}^T \end{aligned}$$

$$\mathbf{x}(n:n-j) = \begin{bmatrix} x[n] & x[n-1] & \cdots & x[n-j] \end{bmatrix}^T.$$

Then we have

$$\begin{aligned} \mathbf{y}[n] &= \mathbf{H}(N,M)\mathbf{x}(n:n-N-M+2) + \underline{\eta}[n], \\ \hat{x}[n] &= \mathbf{g}\mathbf{y}[n] \\ &= \mathbf{g}\mathbf{H}(N,M)\mathbf{x}(n:n-N-M+2) + \mathbf{g}\underline{\eta}[n], \end{aligned}$$

where $\mathbf{H}(N,M) \in \mathbb{R}^{N\times(N+M-1)}$ is given by

$$\mathbf{H}(N,M) = \begin{bmatrix} h[0] & h[1] & & \cdots & & 0 \\ 0 & h[0] & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & & h[0] & \cdots & h[M-1] \end{bmatrix}. \tag{2}$$

Since $x[n]$ is WSS, we can define

$$C_{xx}(i) = \mathcal{E}\left((x[n-i] - \mu_x)(x[n] - \mu_x)\right),$$

for $i = 1, 2, \ldots$. Note that $C_{xx}(0) = \sigma_x^2$. Denote

$$\Sigma_{\mathbf{x}} = \begin{bmatrix} C_{xx}(0) & C_{xx}(1) & \cdots & C_{xx}(N+M-2) \\ C_{xx}(1) & C_{xx}(0) & \cdots & C_{xx}(N+M-3) \\ \vdots & & \ddots & \vdots \\ C_{xx}(N+M-2) & \cdots & \cdots & C_{xx}(0) \end{bmatrix}$$

$$\Sigma_\eta = \mathcal{E}\left(\underline{\eta}[n]\underline{\eta}^T[n]\right) \in \mathbb{R}^{N\times N}$$

By direct computation, we have

$$\begin{aligned} \mu_{\hat{x}} &= \mathcal{E}(\hat{x}[n]) = \mathbf{g}\mathbf{H}(N,M)\mathbf{e}(N+M-1)\mu_x \\ &= \mathbf{g}\mathbf{e}(NK)\left(\sum_{i=0}^{M-1} h[i]\right)\mu_x \\ &= \left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=0}^{M-1} h[i]\right)\mu_x \end{aligned}$$

$$\begin{aligned} \sigma_{x\hat{x}} &= \mathcal{E}((x[n]-\mu_x)(\hat{x}[n]-\mu_{\hat{x}})) \\ &= \mathcal{E}((\hat{x}[n]-\mu_{\hat{x}})(x[n]-\mu_x)) \\ &= \mathcal{E}\left((\mathbf{g}\mathbf{H}(N,M)\mathbf{x}(n, N+M-1) - \mu_{\hat{x}})(x[n]-\mu_x)\right) \\ &= \mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) \end{aligned} \tag{3}$$

$$\begin{aligned} \sigma_{\hat{x}}^2 &= \mathcal{E}\left((\hat{x}[n]-\mu_{\hat{x}})^2\right) \\ &= \mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}\mathbf{H}^T(N,M)\mathbf{g}^T + \mathbf{g}\Sigma_\eta\mathbf{g}^T \\ &= \mathbf{g}\mathbf{Q}\mathbf{g}^T \end{aligned} \tag{4}$$

where

$$\mathbf{Q} = \mathbf{H}(N,M)\Sigma_{\mathbf{x}}\mathbf{H}^T(N,M) + \Sigma_\eta. \tag{5}$$

It follows from the above derivations that the StatSSIM in (1) can be written as

$$\text{StatSSIM}(x, \hat{x}) = S_1(x, \hat{x})S_2(x, \hat{x}) \tag{6}$$

where

$$S_1(x,\hat{x}) = \frac{2\mu_x^2\left(\sum_{j=0}^{N-1} g[j]\right)\left(\sum_{i=1}^{M-1} h[i]\right) + c_1}{\mu_x^2\left(1 + \left(\sum_{j=0}^{N-1} g[j]\right)^2\left(\sum_{i=1}^{M-1} h[i]\right)^2\right) + c_1} \tag{7}$$

and

$$S_2(x,\hat{x}) = \frac{2\mathbf{g}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \mathbf{g}\mathbf{Q}\mathbf{g}^T + c_2}. \tag{8}$$

Note that $S_1(x,\hat{x}) \leq 1$ and the equality holds when $\mu_x = 0$. Hence $\mathrm{StatSSIM}(x,\hat{x}) \leq S_2(x,\hat{x})$. Throughout the paper, assume that the source image is zero mean. Under this assumption, we have $\mathrm{StatSSIM}(x,\hat{x}) = S_2(x,\hat{x})$. The performance limitation problem becomes to find $\mathbf{g}$ so that $S_2(x,\hat{x})$ is maximized.

3. MAIN RESULTS

In this section, we shall show that the optimal equalizer that maximizes the SSIM index is equal to the one that minimizes the MSE criterion multiplied by a scalar factor, of which the analytical formula is derived. We need some notations. Define

$$b = \Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1), \tag{9}$$

where $\mathbf{H}(N,M)$ and $\mathbf{Q}$ are given by (2) and (5) respectively, $\Sigma_{\mathbf{x}}(1)$ denotes the first column of $\Sigma_{\mathbf{x}}$.

Theorem 1 *Assume that $x[n]$ and the noise $\eta[n]$ are two zero mean WSS random processes and that they are not correlated with each other. Given the blurring filter $\mathbf{H}(z)$, let b be defined by (9). Then we have*

(i) The optimal SSIM index is given by $StatSSIM_{\max}(x,\hat{x}) = \frac{1}{\beta}$, where

$$\beta = \frac{\sqrt{c_2^2 + 4b(\sigma_x^2 + c_2)} - c_2}{2b}. \tag{10}$$

(ii)The equalization filter $\mathbf{G}(z) = \sum_{j=0}^{N-1} g[j]z^{-j}$ that maximizes $S_2(x,\hat{x})$ in (8) is unique and given by

$$\mathbf{g}_{opt} = \beta\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}. \tag{11}$$

Proof. Note that $\mathbf{Q}$ is positive definite, it follows from singular value decomposition (SVD) [11] that there exists unitary $\mathbf{U}$ and diagonal matrix $\Sigma_{\mathbf{Q}} > 0$ such that $\mathbf{Q} = \mathbf{U}\Sigma_{\mathbf{Q}}\mathbf{U}^*$. Denote

$$\mathbf{q} = \mathbf{g}\mathbf{U}\Sigma_{\mathbf{Q}}^{\frac{1}{2}}. \tag{12}$$

Then the equation (6) can be written as

$$\begin{aligned}\mathrm{StatSSIM}(x,\hat{x}) &= S_2(x,\hat{x}) \\ &= \frac{2\mathbf{q}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^*\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \mathbf{q}\mathbf{q}^* + c_2}.\end{aligned} \tag{13}$$

Define $\bar{\mathbf{q}} = \frac{1}{\alpha}\mathbf{q}$, where $\alpha^2 = \mathbf{q}\mathbf{q}^*$. The optimization problem can be written as

$$\max_{\alpha,\bar{\mathbf{q}}} S_2(x,\hat{x}) = \max_{\alpha,\bar{\mathbf{q}}} \frac{2\alpha\bar{\mathbf{q}}\Sigma_Q^{-\frac{1}{2}}\mathbf{U}^*\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) + c_2}{\sigma_x^2 + \alpha^2 + c_2} \tag{14}$$

subject to $\bar{\mathbf{q}}\bar{\mathbf{q}}^* = 1$. Note that for any α, the optimal argument $\bar{\mathbf{q}}$ that maximizes $2\alpha\bar{\mathbf{q}}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^*\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1)$ is given by

$$\bar{\mathbf{q}}_{opt} = \frac{1}{\sqrt{b}}\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{U}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}, \tag{15}$$

where b is defined by (9). Moreover, the resulting maximum is

$$\frac{1}{\sqrt{b}}\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}\mathbf{H}(N,M)\Sigma_{\mathbf{x}}(1) = 2\alpha\sqrt{b}$$

Then the optimization problem (14) is equivalent to

$$\max_{\alpha} S_2(x,\hat{x}) = \max_{\alpha} \frac{2\sqrt{b}\alpha + c_2}{\alpha^2 + \sigma_x^2 + c_2}, \tag{16}$$

and $\bar{\mathbf{q}}_{opt}$ is given by (15). By using basic calculus rules, the optimal parameter α can be computed directly

$$\alpha_{opt} = \frac{\sqrt{c_2^2 + 4b(\sigma_x^2 + c_2)} - c_2}{2\sqrt{b}}, \tag{17}$$

and the resulting maximum is given by

$$S_{2\max}(x,\hat{x}) = \frac{2b}{\sqrt{c_2^2 + 4b(\sigma_x^2 + c_2)} - c_2} = \frac{1}{\beta},$$

where β is defined by (10). It follows from (12) and (15) that the optimal $\mathbf{g}_{opt}$ is given by

$$\begin{aligned}\mathbf{g}_{opt} &= \mathbf{q}_{opt}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^* = a_{opt}\bar{\mathbf{q}}_{opt}\Sigma_{\mathbf{Q}}^{-\frac{1}{2}}\mathbf{U}^* \\ &= \frac{a_{opt}}{\sqrt{b}}\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{U}\Sigma_{\mathbf{Q}}^{-1}\mathbf{U}^* \\ &= \beta\Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}.\end{aligned}$$

This completes the proof. □

The MSE performance limitation has been extensively studied in signal processing and control [12, 13]. The SSIM performance limitation can be computed directly by using Theorem 1.

As stated in Theorem 1 that the optimal equalizer is unique. Therefore, the value $1/\beta$ should be the same as γ defined in [8] for single channel systems, of which an iterative algorithm is given. Theorem 1 tells us that β can be computed explicitly by (10).

It is well-known that $\mathbf{g}_{MSE} = \Sigma_{\mathbf{x}}^T(1)\mathbf{H}^T(N,M)\mathbf{Q}^{-1}$ is the optimal Wiener solution, which minimizes the MSE. In [3], the authors show the advantage of SSIM index compared with MSE by various applications in image processing. However,

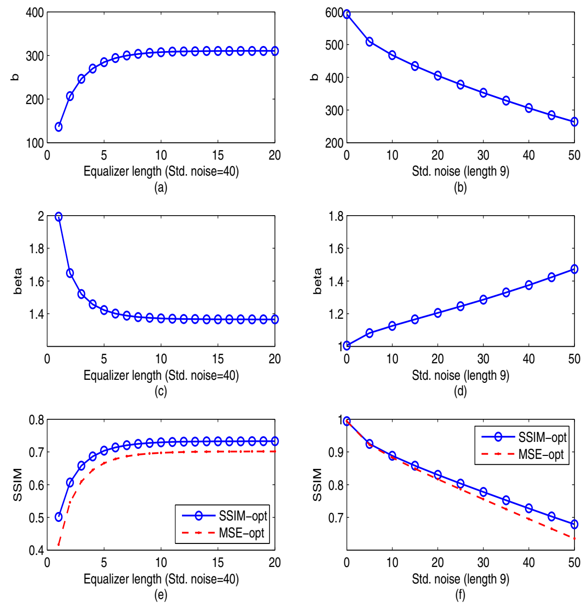

Fig. 2 b, β and the maximal SSIM index.

it follows from Theorem 1 that there is a direct connection between the equalizer with maximal SSIM index and the one with minimal MSE. Indeed, the only difference between $\mathbf{g}_{opt}$ in (11) and $\mathbf{g}_{MSE}$ is a scalar β, which turns out to be the inverse of the optimal SSIM index.

Theorem 1 provides not only the optimal SSIM solution, but also some deep insights into the design process. For instance, we know intuitively that the longer the length of the equalization filters, the better performance of SSIM can be achieved for fixed $H(z), \Sigma_{\mathbf{x}}$ and Σ_η. But what is the performance limitation as N tends to infinity? What is a good trade-off between the N and the achievable SSIM index. These problems can be partly solved by drawing the relation of β and N by using (9) and (10).

4. NUMERICAL EXAMPLES

In this section, we will present two numerical examples to illustrate the usage of the proposed theoretical results. To avoid the influence of the estimation error of $C_{\mathbf{xx}}$, we use a first-order AR model with correlation $\rho = 0.95$ as the model of the source image in the examples. The first-order AR model is a good representation of many image. In both examples, we set $\sigma_x^2 = 600$ and $c_2 = 58.5225$.

Example 1. This example demonstrates how the value of b, β and the optimal SSIM vary with respect to the equalizer length and the noise strength. The blurring filter is the Gaussian filter with length 5 and standard variance 2. The noise is white Gaussian with the standard variance $\sigma_n = 40$. The subplots in the left column of Fig. 2 show the relation with the equalizer length N. We can see that the performance improves as the length N of the equalizer increases. However, the improvement is not significant for $N \geq 9$ as shown by Fig. 2e. This suggests a good choice of the equalizer length between the restoration performance and the implementation complexity. The subplots in the right column of Fig.2 show the relation with the noise strength. Generally the SSIM index decreases linearly as the standard variance of the noise increases. The SSIM performance of the optimal MSE equalizer is also drawn in Fig.2e-2f, both of which show that the optimal SSIM equalizer does give a better performance. However, the performance difference between SSIM index and MSE is small when the noise standard variance small as shown by Fig. 2f.

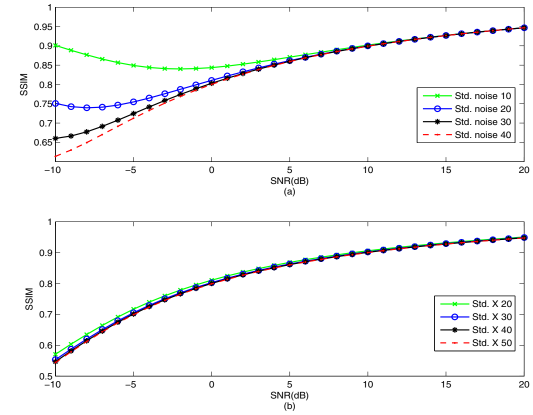

Fig. 3 The SSIM index w.r.t SNR.

Example 2. The relation between the optimal SSIM index and the SNR is shown in Fig. 3. In Fig. 3a, the SNR increases caused by increasing σ_x for fixed σ_n, while in Fig. 3b, the SNR increases caused by reducing σ_n for fixed σ_x. For fixed blurring filters, the SSIM performance is determined by the SNR uniquely when the SNR is high.

5. CONCLUDING REMARKS

For a given blurring filter, the SSIM performance limitation of linear equalizers has been revealed. Closed-form formulas have been presented to compute the maximal SSIM index and the linear equalizer achieving the optimal performance. The results show that the equalizer with maximal SSIM index is the one with minimal MSE multiplied by a scalar β, which turns out to be the inverse of the achievable maximum of the SSIM index. Two examples have been given to illustrate the validity of the theoretical results. Finally we would like to remark that Theorem 1 can be extended to multi-channel systems directly.

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