# IMPROPER GAUSSIAN SIGNALING FOR THE Z-INTERFERENCE CHANNEL

*Sandra Lagen, Adrian Agustin, Josep Vidal*

Universitat Politècnica de Catalunya (UPC), Barcelona, Spain
email: {sandra.lagen, adrian.agustin, josep.vidal}@upc.edu

## ABSTRACT

The optimal transmission scheme for a multiple-input multiple-output point-to-point channel (MIMO-P2P) is dependent on the type of received interference, which can be either *proper* or *improper*. We will show that the presence of improper interference is beneficial in terms of improving the achievable rate compared to a case with proper interference. These benefits are due to using widely linear precoder and receiver. Additionally, we investigate a cellular scenario where two types of users are coexisting, ones that are receiving proper interference while others which might be receiving improper interference, trying to devise the best transmission scheme to be adopted. We propose an improper Gaussian signaling-based scheme that allows controlling system performance and fairness in terms of bitrate through a single parameter.

***Index Terms***— improper Gaussian signaling, MIMO-P2P channel, Z-channel, improper interference, precoding design.

## 1. INTRODUCTION

The MIMO-P2P channel models a communication system in which a transmitter (TX) equipped with multiple antennas wish to send information to a receiver (RX) also equipped with multiple antennas. The performance of such communication link is usually degraded due to noise and interference [1]. The noise is commonly modelled as a proper (i.e. circularly symmetric complex) Gaussian random vector (RV), while the received interference can be modelled either as a proper or improper (i.e. circularly asymmetric complex) Gaussian RV depending on the transmission strategy of the interfering TX. The noise-plus-interference RV characteristics influences the transmission scheme obtained from the minimization of any Schur-concave function of the MSE-matrix [2], as minimization of the mean squared error (MSE) or maximization of the rate.

The second-order characterization of the noise-plus-interference RV is given by the covariance and pseudo-covariance matrices [3][4], being the pseudo-covariance matrix of a proper Gaussian RV null. In this latter case, as it is well known in the literature [1], the optimal transmission scheme is given by proper Gaussian signaling. In contrast, if the received noise-plus-interference is improper, the optimum scheme is given by improper Gaussian signaling and widely linear receiver [3], which can be derived by using the equivalent double-sized real-valued MIMO channel matrix [1] whereby real and imaginary parts of the channel are separated.

In this work we analyze improper transmission strategies in a two-user interfering scenario where one of receivers just observes proper noise-plus-interference while the other might observe improper noise-plus-interference caused by improper signaling. This kind of scenario is described by the Z-channel (also known as the Z-interference channel [5]) and is present in certain situations in cellular networks, as it is shown in Fig. 1. Two MIMO-P2P links are considered: RX1 is receiving noise and interference due to the active transmission towards RX2 on the same time/frequency resource, and RX2 is receiving mainly noise. The received interference at RX1 can be either proper or improper depending on the transmission scheme adopted by interfering TX2.

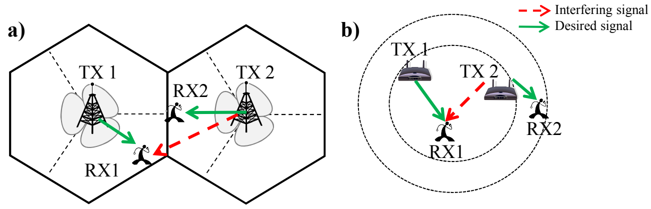


**Fig. 1**. Examples of the Z-channel scenario as observed in: a) macrocell-based deployment and b) smallcell-based deployment.

Previous works tackle the interference channel, which is difficult to analyze due to non-convexity of the problem and hence performance gains are observed in terms of degrees of freedom at high signal-to-noise ratios [6][7] or in specific MIMO configurations [8][9][10]. In the present study, we use the equivalent double-sized real-valued MIMO matrix [1] and the majorization theory [2] to gain new insights of both the reception of improper interference and the use of improper Gaussian signaling at transmission on the MIMO-P2P channel and Z-channel. Our main contributions are:

- We demonstrate that, for a given noise-plus-interference covariance matrix, the use of improper Gaussian signaling when received interference is improper outperforms (in terms of achievable rate) the transmission with a proper Gaussian signaling scheme when interference is proper.
- Although an improper Gaussian signaling-based transmission scheme leads to a loss in terms of throughput when the user is receiving a proper noise-plus-interference (as RX2 in Fig. 1), we propose a simple scheme that controls such bitrate loss with a single parameter and allows improving system performance and fairness in terms of bitrate without requiring knowledge of the interfering channel neither at RX nor TX.

## 2. IMPROPER SIGNALING FOR MIMO-P2P CHANNELS

Consider a MIMO-P2P channel between one transmitter equipped with $N_t$ antennas and one receiver with $N_r$ receive antennas. The MIMO channel is described by a matrix $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ containing complex-valued channel gains of the different antenna-pairs. Hence, assuming narrow-band transmissions, the equivalent baseband signal observed at the user can be expressed as

$$\mathbf{y} = \mathbf{Hx} + \mathbf{s}, \quad \mathbf{s} = \mathbf{n} + \mathbf{i} \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^{N_t \times 1}$ is the complex-valued transmitted vector and $\mathbf{s} \in \mathbb{C}^{N_r \times 1}$ denotes the interference-plus-noise vector at the receiver, containing circularly symmetric white Gaussian noise $\mathbf{n} \sim CN(\mathbf{0}, \sigma_n^2 \mathbf{I})$ and an interference component $\mathbf{i}$ that can be either proper or improper. Different from the conventional transmission setup where proper Gaussian signaling is assumed ($\mathbf{x} \sim CN(\mathbf{0}, \mathbf{C_x})$),

This work has been done in the framework of the projects TEC2010-19171/TCM, CONSOLIDER INGENIO CSD2008-00010 COMONSENS, 2009SGR1236 (AGAUR) of the Catalan Administration, and the project TROPIC FP7 ICT-2011-8-318784 funded by the European Commission.

978-1-4799-2893-4/14/$31.00 ©2014 IEEE

in this paper the more general improper Gaussian signaling is adopted for $\mathbf{x}$ [3]. An efficient way to map a proper information-bearing signal $\mathbf{b}$ into an improper transmitted signal $\mathbf{x}$ is by means of using *widely linear precoding* [9]. This way, the improper transmitted signal is given by

$$\mathbf{x} = \mathbf{W}_1\mathbf{b} + \mathbf{W}_2\mathbf{b}^* \tag{2}$$

where matrices $\mathbf{W}_1$ and $\mathbf{W}_2 \in \mathbb{C}^{N_t\times m}$ are the transmit linear precoders for the information-bearing signal $\mathbf{b} \in \mathbb{C}^{m\times 1}$ and its complex conjugate $\mathbf{b}^*$, respectively, and $m$=min($N_r$,$N_t$) denotes the maximum number of active transmission modes. It is assumed that $\mathbf{b}$ is a proper RV, such that $E\{\mathbf{bb}^H\}=\mathbf{I}$ and $E\{\mathbf{bb}^T\}=\mathbf{0}$. The covariance matrix $\mathbf{C}_\mathbf{x}=E\{\mathbf{xx}^H\}$ and the pseudo-covariance matrix $\tilde{\mathbf{C}}_\mathbf{x}=E\{\mathbf{xx}^T\}$ of the improper transmitted signal $\mathbf{x}$ in (2) are [3]

$$\mathbf{C}_\mathbf{x} = \mathbf{W}_1\mathbf{W}_1^H + \mathbf{W}_2\mathbf{W}_2^H, \quad \tilde{\mathbf{C}}_\mathbf{x} = \mathbf{W}_1\mathbf{W}_2^T + \mathbf{W}_2\mathbf{W}_1^T \tag{3}$$

Similarly, $\mathbf{C}_\mathbf{s}$ and $\tilde{\mathbf{C}}_\mathbf{s}$ denote the covariance and pseudo-covariance matrices of the interference-plus-noise RV $\mathbf{s}$.

Using the double-sized real-valued decomposition, the input-output relation in (1) can be equivalently written as follows [1]

$$\bar{\mathbf{y}} = \begin{bmatrix}\Re\{\mathbf{y}\}\\ \Im\{\mathbf{y}\}\end{bmatrix} = \bar{\mathbf{H}}\bar{\mathbf{x}} + \bar{\mathbf{s}}, \quad \bar{\mathbf{H}} = \begin{bmatrix}\Re\{\mathbf{H}\} & -\Im\{\mathbf{H}\}\\ \Im\{\mathbf{H}\} & \Re\{\mathbf{H}\}\end{bmatrix} \tag{4}$$

where $\Re\{.\}$ and $\Im\{.\}$ denote the real and imaginary operators, $\bar{\mathbf{x}} = [\Re\{\mathbf{x}^T\}\ \ \Im\{\mathbf{x}^T\}]^T$ and $\bar{\mathbf{s}} = [\Re\{\mathbf{s}^T\}\ \ \Im\{\mathbf{s}^T\}]^T$. The double-sized real-valued transmitted signal $\bar{\mathbf{x}}$ in (4) can be expressed as

$$\bar{\mathbf{x}} = \bar{\mathbf{W}}\bar{\mathbf{b}}, \quad \bar{\mathbf{W}} = \begin{bmatrix}\Re\{\mathbf{W}_1+\mathbf{W}_2\} & -\Im\{\mathbf{W}_1-\mathbf{W}_2\}\\ \Im\{\mathbf{W}_1+\mathbf{W}_2\} & \Re\{\mathbf{W}_1-\mathbf{W}_2\}\end{bmatrix} \tag{5}$$

where $\bar{\mathbf{b}} = [\Re\{\mathbf{b}^T\}\ \ \Im\{\mathbf{b}^T\}]^T$. Furthermore, the covariance matrix of $\bar{\mathbf{x}}$ in (5) is related with $\mathbf{C}_\mathbf{x}$ and $\tilde{\mathbf{C}}_\mathbf{x}$ in (3) as

$$\mathbf{C}_{\bar{\mathbf{x}}} = E\{\bar{\mathbf{x}}\bar{\mathbf{x}}^H\} = \tfrac{1}{2}\bar{\mathbf{W}}\bar{\mathbf{W}}^H = \tfrac{1}{2}\begin{bmatrix}\Re\{\mathbf{C}_\mathbf{x}+\tilde{\mathbf{C}}_\mathbf{x}\} & -\Im\{\mathbf{C}_\mathbf{x}-\tilde{\mathbf{C}}_\mathbf{x}\}\\ \Im\{\mathbf{C}_\mathbf{x}+\tilde{\mathbf{C}}_\mathbf{x}\} & \Re\{\mathbf{C}_\mathbf{x}-\tilde{\mathbf{C}}_\mathbf{x}\}\end{bmatrix} \tag{6}$$

Existing optimization algorithms in the literature can be applied to optimally design $\bar{\mathbf{W}}$ based on different criteria. In the most general MSE-based designs (including the minimum MSE and the maximum rate designs) [2], the transmit precoder $\bar{\mathbf{W}}$ is obtained from the minimization of a Schur-concave function of the MSE-matrix of the transmitted symbols $\mathbf{E} = E\{(\mathbf{b}-\hat{\mathbf{b}})(\mathbf{b}-\hat{\mathbf{b}})^H\}$ [2]. Equivalently, the MSE-matrix in the double-sized form is

$$\bar{\mathbf{E}} = E\{(\bar{\mathbf{b}}-\hat{\bar{\mathbf{b}}})(\bar{\mathbf{b}}-\hat{\bar{\mathbf{b}}})^H\} = \tfrac{1}{2}\left(\mathbf{I} + \tfrac{1}{2}\bar{\mathbf{W}}^H\bar{\mathbf{H}}^H\mathbf{C}_{\bar{\mathbf{s}}}^{-1}\bar{\mathbf{H}}\bar{\mathbf{W}}\right)^{-1} \tag{7}$$

where $\hat{\bar{\mathbf{b}}}$ denotes the double-sized real-valued vector of the estimated symbol stream $\hat{\mathbf{b}}$ at the receiver when using the *widely linear receiver* (see details in [3]), factor 1/2 appears because $E\{\bar{\mathbf{b}}\bar{\mathbf{b}}^H\} = \tfrac{1}{2}\mathbf{I}$ and $\mathbf{C}_{\bar{\mathbf{s}}} \in \mathbb{R}^{2N_r\times 2N_r}$ accounts for the covariance matrix of $\bar{\mathbf{s}}$, which can be written in terms of $\mathbf{C}_\mathbf{s}$ and $\tilde{\mathbf{C}}_\mathbf{s}$ as follows

$$\mathbf{C}_{\bar{\mathbf{s}}} = E\{\bar{\mathbf{s}}\bar{\mathbf{s}}^H\} = \mathbf{T}\begin{bmatrix}\mathbf{C}_\mathbf{s} & \tilde{\mathbf{C}}_\mathbf{s}\\ \tilde{\mathbf{C}}_\mathbf{s}^* & \mathbf{C}_\mathbf{s}^*\end{bmatrix}\mathbf{T}^H, \quad \mathbf{T} = \tfrac{1}{2}\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ -j\mathbf{I} & j\mathbf{I}\end{bmatrix} \tag{8}$$

Let us recall that $\mathrm{tr}(\mathbf{E}) = \mathrm{tr}(\bar{\mathbf{E}})$, being tr(.) the trace operator. The relation between the MSE-matrix of the transmitted symbols and the achievable rate can be extracted from the double-sized decomposition. The result is given in the following Lemma 1.

*Lemma 1* (see proof in Appendix 8.1): The achievable rate ($R$) is related to the MSE-matrix in the double-sized form $\bar{\mathbf{E}}$ in (7) and to the MSE-matrix of the transmitted symbols $\mathbf{E}$ through

$$R = -\tfrac{1}{2}\log_2|2\bar{\mathbf{E}}| = -\tfrac{1}{2}\log_2|\mathbf{E}| - \tfrac{1}{2}\log_2|\mathbf{E}^* - \mathbf{Y}^*\mathbf{E}^{-1}\mathbf{Y}| \tag{9}$$

where $\mathbf{Y} = E\{(\mathbf{b}-\hat{\mathbf{b}})(\mathbf{b}-\hat{\mathbf{b}})^T\}$ and |.| is the determinant operator. Note that in case of proper Gaussian signaling (i.e. $\mathbf{Y}=\mathbf{0}$), the conventional relation between the MSE-matrix and the achievable rate is obtained: $R = -\log_2|\mathbf{E}|$ [11]. Furthermore, using the Schur complement of $|\bar{\mathbf{E}}|$ and applying the improved Fischer determinant inequality [12], the achievable rate in (9) is lower bounded by

$$R \underset{(a)}{\geq} -\log_2|\mathbf{E}| + \tfrac{1}{2}\log_2\left(1 + \tfrac{1}{2}|\mathbf{Y}||\mathbf{Y}^*||\bar{\mathbf{E}}|^{-1}\right) \underset{(b)}{\geq} -\log_2|\mathbf{E}| \tag{10}$$

where ($a$) is satisfied with equality if $\mathbf{E}$ is a scalar (i.e. for SISO, MISO or SIMO systems) and ($b$) is satisfied with equality for proper Gaussian signaling case. The lower bound indicates that, for a given MSE-matrix $\mathbf{E}$, the rate achieved with improper signaling is equal or larger than the rate obtained with proper signaling. Moreover, it also shows that there is not enough with $\mathbf{E}$ in order to determine the modulation and coding scheme to be adopted, but also $\mathbf{Y}$ is required when using improper Gaussian signaling.

## 3. Optimal MSE-Based Transmission Scheme

For a MIMO-P2P system, the optimal transmission scheme with improper Gaussian signaling for the MSE-based defined designs [2] is obtained as solution of the following optimization problem

$$(\mathrm{P}_0):\ \underset{\bar{\mathbf{W}}}{\text{minimize}}\ \ f(\bar{\mathbf{E}}) \quad \text{subject to}\ \ \tfrac{1}{2}\mathrm{tr}(\bar{\mathbf{W}}\bar{\mathbf{W}}^H) - \mathrm{P}^{\max} = 0 \tag{11}$$

where $f$(.) is any Schur-concave function of the double-sized MSE-matrix in (7) and $\mathrm{P}^{\max}$ is the maximum available power at the transmitter. For example, $f(\bar{\mathbf{E}}) = \mathrm{tr}(\bar{\mathbf{E}})$ defines the minimum MSE problem, while $f(\bar{\mathbf{E}}) = \tfrac{1}{2}\log_2|2\bar{\mathbf{E}}| = -R$ defines the maximum achievable rate problem. The optimal transmit precoder $\bar{\mathbf{W}}^{opt}$ (from majorization theory [2]) presents the following structure

$$\bar{\mathbf{W}}^{opt} = \bar{\mathbf{V}}\bar{\mathbf{P}}^{\frac{1}{2}} \tag{12}$$

where $\bar{\mathbf{V}}$ is a unitary matrix that diagonalizes $\bar{\mathbf{E}}$ and it is obtained from the following singular value decomposition (SVD)

$$\tfrac{1}{2}\bar{\mathbf{H}}^H\mathbf{C}_{\bar{\mathbf{s}}}^{-1}\bar{\mathbf{H}} = \bar{\mathbf{V}}\bar{\mathbf{\Lambda}}\bar{\mathbf{V}}^H \tag{13}$$

Finally, diagonal matrix $\bar{\mathbf{P}}$ in (12) describes the power allocation per mode and depends on the optimization criterion (see details in [13]). For uniform power allocation: $\bar{\mathbf{P}} = \bar{P}\mathbf{I}$, where $\bar{P} = \mathrm{P}^{\max}/(2m)$. The resulting diagonal MSE-matrix in (7) and the rate in (9) are

$$\bar{\mathbf{E}}^{opt} = \tfrac{1}{2}\left(\mathbf{I} + \bar{P}\bar{\mathbf{\Lambda}}\right)^{-1}, \qquad R^{opt} = \tfrac{1}{2}\sum_{i=1}^{2m}\log_2\left(1 + \bar{P}\bar{\lambda}_i\right) \tag{14}$$

being $\bar{\lambda}_i$ the $i$th diagonal value of $\bar{\mathbf{\Lambda}}$ in (13). Denoting by $[\bar{\mathbf{W}}^{opt}]_{kl}$ the $N_t\times m$ ($k,l$)-block matrix of $\bar{\mathbf{W}}^{opt}$ in (12) ($k,l\in\{1,2\}$), the equivalent optimal transmit precoders in (2) are

$$\begin{aligned}\mathbf{W}_1^{opt} &= \tfrac{1}{2}\left([\bar{\mathbf{W}}^{opt}]_{11} + [\bar{\mathbf{W}}^{opt}]_{22}\right) + j\tfrac{1}{2}\left([\bar{\mathbf{W}}^{opt}]_{21} - [\bar{\mathbf{W}}^{opt}]_{12}\right)\\ \mathbf{W}_2^{opt} &= \tfrac{1}{2}\left([\bar{\mathbf{W}}^{opt}]_{11} - [\bar{\mathbf{W}}^{opt}]_{22}\right) + j\tfrac{1}{2}\left([\bar{\mathbf{W}}^{opt}]_{21} + [\bar{\mathbf{W}}^{opt}]_{12}\right)\end{aligned} \tag{15}$$

The key aspect of the optimum solution in (12) is the SVD in (13) for which two cases are differentiated: proper noise-plus-interference RV $\mathbf{s}$ or improper noise-plus-interference RV $\mathbf{s}$.

In the former case (i.e. *proper* interference, $\tilde{\mathbf{C}}_\mathbf{s} = \mathbf{0}$), the solution for $\bar{\mathbf{V}}$ in (12) can be obtained without using the equivalent double-sized real-valued decomposition. By developing $\tfrac{1}{2}\bar{\mathbf{H}}^H\mathbf{C}_{\bar{\mathbf{s}}}^{-1}\bar{\mathbf{H}}$ with the structure of $\bar{\mathbf{H}}$ in (4) we get

$$\tfrac{1}{2}\bar{\mathbf{H}}^H\mathbf{C}_{\bar{\mathbf{s}}}^{-1}\bar{\mathbf{H}} = \begin{bmatrix}\Re\{\mathbf{H}^H\mathbf{C}_\mathbf{s}^{-1}\mathbf{H}\} & -\Im\{\mathbf{H}^H\mathbf{C}_\mathbf{s}^{-1}\mathbf{H}\}\\ \Im\{\mathbf{H}^H\mathbf{C}_\mathbf{s}^{-1}\mathbf{H}\} & \Re\{\mathbf{H}^H\mathbf{C}_\mathbf{s}^{-1}\mathbf{H}\}\end{bmatrix} \tag{16}$$

Consequently, the solution for $\bar{\mathbf{V}}$ in (13) is obtained from the singular-vectors of $\mathbf{H}^H\mathbf{C}_\mathbf{s}^{-1}\mathbf{H} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ stacked as follows

$$\bar{\mathbf{V}} = \begin{bmatrix}\Re\{\mathbf{V}\} & -\Im\{\mathbf{V}\}\\ \Im\{\mathbf{V}\} & \Re\{\mathbf{V}\}\end{bmatrix} \tag{17}$$

By comparing (17) and (5), the following transmit precoders in (15) are obtained: $\mathbf{W}_1^{opt} = \mathbf{V}\mathbf{P}^{\frac{1}{2}}$, $\mathbf{W}_2^{opt} = \mathbf{0}$, and hence $\tilde{\mathbf{C}}_\mathbf{x} = \mathbf{0}$ in (3) so that a proper Gaussian signaling scheme is optimum.

When received interference is *improper* (i.e. $\tilde{\mathbf{C}}_\mathbf{s} \neq \mathbf{0}$), relation in (16) is not satisfied and the optimal solution has to be obtained from the SVD in (13). As a consequence, the optimum transmit strategy leads to a precoder $\bar{\mathbf{W}}^{opt}$ in (12) with non-equal block diagonal matrices for $\mathbf{C}_{\bar{\mathbf{x}}}$ in (6) and hence $\tilde{\mathbf{C}}_{\mathbf{x}} \neq \mathbf{0}$. Therefore, the optimal transmit strategy is given by improper Gaussian signaling as it is the only one able to diagonalize the MSE-matrix $\bar{\mathbf{E}}$ in (7).

## 4. Superiority of Optimal Improper Signaling

Previous optimum schemes can be compared for a fixed noise-plus-interference covariance matrix $\mathbf{C}_{\mathbf{s}}$ when interference is proper or improper. The comparison reduces to relating the singular values of the two following SVD (obtained from (13)):

$$\text{Proper interference:} \quad \tfrac{1}{2}\bar{\mathbf{H}}^{H}\mathbf{T}^{-H}\begin{bmatrix}\mathbf{C}_{\mathbf{s}} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{\mathbf{s}}^{*}\end{bmatrix}^{-1}\mathbf{T}^{-1}\bar{\mathbf{H}} = \bar{\mathbf{V}}^{P}\bar{\mathbf{\Lambda}}^{P}\bar{\mathbf{V}}^{PH}$$
$$\text{Improper interference:} \quad \tfrac{1}{2}\bar{\mathbf{H}}^{H}\mathbf{T}^{-H}\begin{bmatrix}\mathbf{C}_{\mathbf{s}} & \tilde{\mathbf{C}}_{\mathbf{s}}\\ \tilde{\mathbf{C}}_{\mathbf{s}}^{*} & \mathbf{C}_{\mathbf{s}}^{*}\end{bmatrix}^{-1}\mathbf{T}^{-1}\bar{\mathbf{H}} = \bar{\mathbf{V}}^{I}\bar{\mathbf{\Lambda}}^{I}\bar{\mathbf{V}}^{IH} \qquad (18)$$

*Lemma 2* (see proof in Appendix 8.2): Singular values in (18) are related by the following weakly log-majorization inequality

$$\bar{\boldsymbol{\lambda}}^{I} \succ_{w\log} \bar{\boldsymbol{\lambda}}^{P} \qquad (19)$$

where $\bar{\boldsymbol{\lambda}}^{I}$ and $\bar{\boldsymbol{\lambda}}^{P}$ are the vectors containing the diagonal values of $\bar{\mathbf{\Lambda}}^{I}$ and $\bar{\mathbf{\Lambda}}^{P}$ in (18), respectively, and weakly log-majorization in (19) implies [14]

$$\prod_{i=1}^{n}\bar{\lambda}_i^{I} \geq \prod_{i=1}^{n}\bar{\lambda}_i^{P} \quad \text{and} \quad \sum_{i=1}^{n}\bar{\lambda}_i^{I} \geq \sum_{i=1}^{n}\bar{\lambda}_i^{P} \quad n=1,\ldots,2m \qquad (20)$$

The proof is valid for SISO, MISO and SIMO systems. In the general MIMO case, equation (19) is observed to be always satisfied through simulations. Therefore, thanks to the reception of improper interference and the use of improper Gaussian signaling, the singular values of the equivalent channel are more spread out having a large sum and a large product. Lemma 3 allows us determining how this fact impacts on the achievable rate of the user.

*Lemma 3* ([15]): If $\mathbf{a} \succ_{w\log} \mathbf{b}$ then $\sum_{i=1}^{n}\log(1+a_i) \geq \sum_{i=1}^{n}\log(1+b_i)$, as $\log(1+x)$ is an increasing function on $x \in [0, \infty)$ and $\log(1+e^{x})$ is a convex function.

From (19), we have: $\bar{P}\bar{\boldsymbol{\lambda}}^{I} \succ_{w\log} \bar{P}\bar{\boldsymbol{\lambda}}^{P}$, as $\bar{P}$ is a positive scalar. Hence, according to Lemma 3, the achievable rate $R^{opt}$ in (14) is outperformed when received interference is improper (i.e. $\tilde{\mathbf{C}}_{\mathbf{s}} \neq \mathbf{0}$):

$$R^{opt}(\bar{\boldsymbol{\lambda}}^{I}) \geq R^{opt}(\bar{\boldsymbol{\lambda}}^{P}) \qquad (21)$$

*Theorem 1* (proof through Lemma 1, 2 and 3): Assume a MIMO-P2P channel receiving noise-plus-interference with a given covariance matrix $\mathbf{C}_{\mathbf{s}}$. When applying the optimal transmission scheme, the achievable rate is improved in the improper interference scenario as compared to the proper interference case.

## 5. Application to the Z-Interference Channel

By focusing in the MIMO Z-channel displayed in Fig. 1, RX1 could benefit from the reception of improper interference from TX2, as it is shown in Section 4. However, the fact that TX2 transmits improper signals implies a degradation of the rate performance of RX2 because of the sub-optimality of the signaling scheme [1]. Assuming that we can tolerate a certain rate loss in RX2, performance gains at RX1 would be guaranteed with improper Gaussian signaling provided that TX2 uses the same transmit covariance matrix $\mathbf{C}_{\mathbf{x}}$ as in the optimum proper scheme (so that the received interference-plus-noise covariance matrix $\mathbf{C}_{\mathbf{s}}$ at RX1 is the same and, as $\tilde{\mathbf{C}}_{\mathbf{s}} \neq \mathbf{0}$, the rate of RX1 would be increased).

Improper Gaussian signaling schemes at TX2 can be derived from the optimal proper scheme by right-multiplying the optimal transmit precoder $\mathbf{W}_1^{opt}$ obtained with $\bar{\mathbf{V}}$ in (17) by two scaling factors and two unitary matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as follows

$$\mathbf{W}_1^{imp} = \sqrt{1-\alpha}\,\mathbf{W}_1^{opt}\mathbf{Z}_1, \quad \mathbf{W}_2^{imp} = \sqrt{\alpha}\,\mathbf{W}_1^{opt}\mathbf{Z}_2 \qquad (22)$$

where $\alpha \in [0,\ 0.5]$. This improper-based transmission scheme at TX2 has the same transmit covariance matrix than the optimum proper scheme provided that $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are unitary matrices: $\mathbf{C}_{\mathbf{x}} = \mathbf{W}_1^{opt}\mathbf{W}_1^{opt\,H}$, but the transmit pseudo-covariance matrix does not vanish to $\mathbf{0}$ for $\alpha>0$: $\tilde{\mathbf{C}}_{\mathbf{x}} = \sqrt{\alpha(1-\alpha)}\,\mathbf{W}_1^{opt}(\mathbf{Z}_1\mathbf{Z}_2^{T} + \mathbf{Z}_2\mathbf{Z}_1^{T})\mathbf{W}_1^{opt\,T}$.

The suitable selection of parameter $\alpha$ allows controlling the level of *improperness*: if $\alpha=0$ the generated signal at TX2 is proper, while if $\alpha=0.5$ the signal is improper and the maximum level of improper interference is generated towards RX1 in Fig. 1. Matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are included in (22) so as to get different spatial structures of $\{\mathbf{W}_1^{imp}, \mathbf{W}_2^{imp}\}$, however $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{I}$ is also an option. The proposed scheme is valid for any $m=\min(N_r,N_t)$, but in case that $m=2$ the following unitary matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are proposed

$$\mathbf{Z}_1 = \sqrt{\tfrac{1}{2}}\begin{bmatrix} j & -j\\ -j & -j\end{bmatrix},\ \mathbf{Z}_2 = \sqrt{\tfrac{1}{2}}\begin{bmatrix} j & j\\ -j & j\end{bmatrix},\ \mathbf{Z}_1\mathbf{Z}_2^{T} = \mathbf{Z}_2\mathbf{Z}_1^{T} = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix} \qquad (23)$$

## 6. Evaluation and Results

In this section we evaluate the performance of the optimum transmission schemes for each user in the Z-interference channel scenario displayed in Fig. 1 through Montecarlo simulations. It is assumed that both transmitters have the same available power $P^{max}$. Signal-to-noise ratio is defined as $SNR=P^{max}/\sigma_n^2$ and signal-to-interference ratio by $SIR=1/\eta_I^2$, where factor $0 \leq \eta_I \leq 1$ denotes the average ratio between interfering and direct channel attenuations. Channels are modelled through a Rayleigh distribution, such that $\mathbf{H}_{11}, \mathbf{H}_{22} \sim CN(\mathbf{0}, \mathbf{I})$ and $\mathbf{H}_{12} \sim CN(\mathbf{0}, \eta_I\mathbf{I})$. The antenna configuration is $N_t=N_r=2$ (so that $m=2$).

Fig. 2 displays the achievable rate of users in Fig. 1 vs. SIR for SNR=10dB. 1000 random channel realizations are evaluated. The rate of each user is shown when: 1) transmitters use proper Gaussian signaling ('proper' in legend) and 2) transmitters use improper Gaussian signaling ('improper' in legend), whereby TX2 uses the transmission scheme proposed in Section 5 with different values of $\alpha$ in (22) so that the generated interference towards RX1 is improper. The rate of RX1 is reduced with the level of received interference from the TX2. It is verified that the rate of RX1 is always increased when received interference is improper (according to Theorem 1) and the relative gain is larger at low SIR. However, the use of an improper scheme imposes consistent losses over the rate of RX2 and hence it is beneficial when interference is non-negligible, as it allows trading in transmission fairness through a single parameter $\alpha$ without adjusting the transmitted power.

Another way to achieve a similar behavior is by reducing the power used by TX2 (denoted by $P_2$). In such a case, the improper Gaussian signaling scheme proposed in Section 5 would have two variables to be adjusted: $\alpha$ and $P_2$. In this regard, the achievable rate region for the Z-interference channel in Fig. 1 is plotted in Fig. 3 for SNR=10dB and 20dB when $\eta_I=1$. The random MIMO channel matrices are: $\mathbf{H}_{11}=[1.01e^{-j174.9}\ 0.74e^{j152.8}\ ;\ 0.86e^{-j55.5}\ 0.82e^{j166.7}]$, $\mathbf{H}_{22}=[0.49e^{j162.3}\ 1.30e^{-j101.0}\ ;\ 0.70e^{-j43.8}\ 0.46e^{j9.8}]$ and $\mathbf{H}_{21}=[1.16e^{-j132.6}\ 0.90e^{-j88.5}\ ;\ 0.93e^{-j141.5}\ 0.70e^{j67.1}]$. The achievable rate region when using a proper scheme varying $P_2$ is compared to the achievable rate region when using the proposed improper scheme varying $\alpha$ in (22) and/or $P_2$. It is shown that the achievable rate region is significantly enlarged when using improper Gaussian signaling. Moreover, both sum-rate (i.e. $R_1+R_2$) and system fairness (i.e. $\min(R_1,R_2)$) are improved. Even with time-sharing, the improper scheme outperforms the proper scheme, as it is shown with dashed lines in Fig. 3.

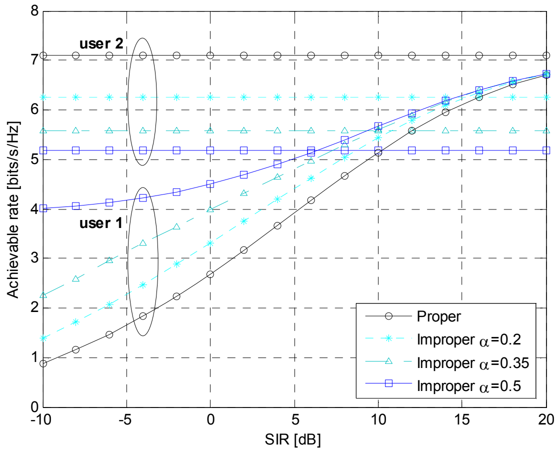


**Fig. 2**. Achievable rate of each user in Fig. 1 (in bits/s/Hz) vs. SIR (in dB) when using proper Gaussian signaling or improper Gaussian signaling varying $\alpha$, with $N_t$=2, $N_r$=2, and SNR=10dB.

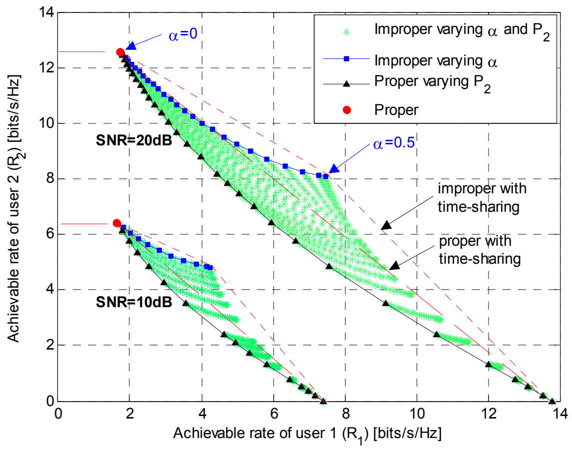


**Fig. 3**. Achievable rate region for the MIMO Z-interference channel scenario in Fig. 1 with channel realization $\mathbf{H}_{11}$, $\mathbf{H}_{22}$, $\mathbf{H}_{21}$, $N_t$=2, $N_r$=2, for SNR=10dB and SNR=20dB.

## 7. Conclusions

In this article the use of improper Gaussian signaling at transmission is analyzed for the simplest MIMO-P2P scenario, showing that improper signaling is beneficial in terms of achievable rate performance if received interference is improper. Such property is exploited in the Z-interference channel scenario so as to improve the achievable rate of the most impaired user. Simulation results show that a single parameter allows improving the system performance and control the system fairness in terms of bitrate.

## 8. Appendix

### 8.1. Proof of Lemma 1

The achievable rate $R$ with improper Gaussian signaling can be obtained as [4]

$$R = I(\mathbf{x};\mathbf{y}) = I(\bar{\mathbf{x}};\bar{\mathbf{y}}) = h(\bar{\mathbf{y}}) - h(\bar{\mathbf{y}}/\bar{\mathbf{x}}) = h(\bar{\mathbf{y}}) - h(\bar{\mathbf{s}}) = \\ = \tfrac{1}{2}\log_2 |\mathbf{I} + \tfrac{1}{2}\bar{\mathbf{H}}\bar{\mathbf{W}}\bar{\mathbf{W}}^H\bar{\mathbf{H}}^H\mathbf{C}_{\bar{s}}^{-1}| \qquad (24)$$

where $I(\bar{\mathbf{x}};\bar{\mathbf{y}})$ denotes the mutual information between RV $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, $h(\bar{\mathbf{y}})$ refers to the entropy of the RV $\bar{\mathbf{y}}$ [4] and the $\frac{1}{2}$ factor comes from $E\{\bar{\mathbf{b}}\bar{\mathbf{b}}^H\} = \frac{1}{2}\mathbf{I}$. Therefore, the expression in (9) that relates $R$ in (24) and the MSE-matrix $\bar{\mathbf{E}}$ in (7) can be obtained. Further, the MSE-matrix in the double-sized decomposition $\bar{\mathbf{E}}$ is related to the MSE-matrix of the transmitted symbols (i.e. $\mathbf{E}$) as

$$\bar{\mathbf{E}} = \mathbf{T}\tilde{\mathbf{E}}\mathbf{T}^H, \quad \tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{E} & \mathbf{Y} \\ \mathbf{Y}^* & \mathbf{E}^* \end{bmatrix} \qquad (25)$$

where $\mathbf{Y} = E\{(\mathbf{b}-\hat{\mathbf{b}})(\mathbf{b}-\hat{\mathbf{b}})^T\}$ and $\mathbf{T}$ is defined in (8). So, by plugging (25) into (9) and applying the properties of the determinant and $\mathbf{T}^{-1}\mathbf{T}^{-H} = 2\mathbf{I}$, we get the second equality of equation (9).

Finally, the achievable rate can be lower bounded by using the Fischer improved determinant inequality [12] $|\tilde{\mathbf{E}}| \leq |\mathbf{E}|^2 - |\mathbf{Y}||\mathbf{Y}^*|$:

$$-\tfrac{1}{2}\log_2|\tilde{\mathbf{E}}| + \tfrac{1}{2}\log_2|\mathbf{E}|^2 \geq \tfrac{1}{2}\log_2(1 + |\mathbf{Y}||\mathbf{Y}^*||\tilde{\mathbf{E}}|^{-1}) \qquad (26)$$

and then, using $|\tilde{\mathbf{E}}| = |2\bar{\mathbf{E}}|$, inequality in (10) is derived from (26).

### 8.2. Proof of Lemma 2

By using the pinching inequality [16], which states that the "stronger" the off-diagonal blocks are the more spread out the singular values become, we have [3]

$$\mathbf{sv}(\mathbf{A}) \succ \mathbf{sv}(\mathbf{B}), \quad \mathbf{A} = \begin{bmatrix} \mathbf{C}_s & \tilde{\mathbf{C}}_s \\ \tilde{\mathbf{C}}_s^* & \mathbf{C}_s^* \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{C}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_s^* \end{bmatrix} \qquad (27)$$

where $\mathbf{A}$ and $\mathbf{B}$ in (27) are Hermitian and positive semidefinite, $\mathbf{sv}(\mathbf{A})$ and $\mathbf{sv}(\mathbf{B})$ denotes the vector of singular values of $\mathbf{A}$ and $\mathbf{B}$ in decreasing order, respectively. As the inverse function is a convex function, the sum of inverses is a Schur-convex function and hence majorization in (27) applies for the inverse component-wise of vectors of singular values in (27) [14] but with weak majorization because the sum of inverses is not equal to the inverse of the sum: $\mathbf{sv}(\mathbf{A}^{-1}) \succ_w \mathbf{sv}(\mathbf{B}^{-1})$. In addition, due to the "block Hadamard inequality" ($|\mathbf{A}^{-1}| \geq |\mathbf{B}^{-1}|$), we get weakly log-majorization [17]

$$\mathbf{sv}(\mathbf{A}^{-1}) \succ_{w\log} \mathbf{sv}(\mathbf{B}^{-1}) \qquad (28)$$

Let us consider the SVD of the double-sized real-valued MIMO channel matrix $\bar{\mathbf{H}} = \bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^H$ in (4), where $\bar{\mathbf{S}} = diag(\bar{s}_1, \bar{s}_1, ..., \bar{s}_m, \bar{s}_m)$ is a diagonal matrix containing real values repeated twice. As for square positive semidefinite matrices $\mathbf{X}$ and $\mathbf{Y}$: $\mathbf{sv}(\mathbf{XY})$=$\mathbf{sv}(\mathbf{YX})$ [16], multiplying $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ by a unitary matrix does not affect the majorization rule

$$\mathbf{sv}(\bar{\mathbf{A}}) \succ_{w\log} \mathbf{sv}(\bar{\mathbf{B}}) \qquad (29)$$

where $\bar{\mathbf{A}} = \frac{1}{2}\bar{\mathbf{U}}^H\mathbf{T}^{-H}\mathbf{A}^{-1}\mathbf{T}^{-1}\bar{\mathbf{U}}$ and $\bar{\mathbf{B}} = \frac{1}{2}\bar{\mathbf{U}}^H\mathbf{T}^{-H}\mathbf{B}^{-1}\mathbf{T}^{-1}\bar{\mathbf{U}}$ are both real, symmetric and positive semidefinite matrices.

In case that $m$=1 (i.e. SISO, MISO or SIMO systems), as $\bar{\mathbf{S}} = \bar{s}_1\mathbf{I}$, equation (29) is equivalent to $\mathbf{sv}(\bar{\mathbf{S}}^H\bar{\mathbf{A}}\bar{\mathbf{S}}) \succ_{w\log} \mathbf{sv}(\bar{\mathbf{S}}^H\bar{\mathbf{B}}\bar{\mathbf{S}})$ due to the fact that for any square matrix $\mathbf{X}$ and scalar $\alpha$: $\mathbf{sv}(\alpha\mathbf{X})$=$|\alpha|\mathbf{sv}(\mathbf{X})$. Then, as unitary transformations do not affect the majorization rule [14], equation (19) is derived.

In case that $m$>1 (i.e. the rest of MIMO cases), it is important to realize that both $\bar{\mathbf{S}}\bar{\mathbf{S}}^H$ and $\bar{\mathbf{B}}$ have real singular values with multiplicity 2. For that reason, the product $\bar{\mathbf{S}}\bar{\mathbf{S}}^H\bar{\mathbf{B}}$ would also have real singular values with multiplicity 2, which can be proved by using the Kronecker product properties [16]. Moreover, due to $|\mathbf{A}^{-1}| \geq |\mathbf{B}^{-1}|$, we have $|\bar{\mathbf{S}}^H\bar{\mathbf{A}}\bar{\mathbf{S}}| \geq |\bar{\mathbf{S}}^H\bar{\mathbf{B}}\bar{\mathbf{S}}|$.

*Conjecture 1*: Given $2m \times 2m$ real positive semidefinite matrices $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{Y}$ with all singular values of multiplicity 2, such that $\mathbf{sv}(\mathbf{X}) \succ_{w\log} \mathbf{sv}(\mathbf{Y})$. For any $2m \times 2m$ real diagonal and positive semidefinite matrix $\mathbf{D}$ with all singular values of multiplicity 2, the following majorization inequality holds: $\mathbf{sv}(\mathbf{DX}) \succ_{w\log} \mathbf{sv}(\mathbf{DY})$.

As consequence, from (29) and by using Conjecture 1 for $m$>1

$$\mathbf{sv}(\bar{\mathbf{S}}^H\bar{\mathbf{A}}\bar{\mathbf{S}}) \succ_{w\log} \mathbf{sv}(\bar{\mathbf{S}}^H\bar{\mathbf{B}}\bar{\mathbf{S}}) \qquad (30)$$

Finally, as unitary transformations do not affect the weakly log-majorization inequality [14], equation (19) is obtained

$$\mathbf{sv}(\bar{\mathbf{V}}\bar{\mathbf{S}}^H\bar{\mathbf{A}}\bar{\mathbf{S}}\bar{\mathbf{V}}^H) \succ_{w\log} \mathbf{sv}(\bar{\mathbf{V}}\bar{\mathbf{S}}^H\bar{\mathbf{B}}\bar{\mathbf{S}}\bar{\mathbf{V}}^H) \qquad (31)$$

## 9. References


[1] E. Telatar, “Capacity of multi-antenna Gaussian channels”, *European Trans. on Telecommunications*, vol. 10, no. 6, pp. 585–595, November-December 1999.

[2] D. P. Palomar, “A Unified Framework for Communications through MIMO channels”, Ph.D. dissertation, Technical University of Catalonia (UPC), Barcelona, Spain, 2003.

[3] P. J. Schreier, L. L. Scharf, *Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals*, Cambridge (UK), Cambridge University Press, 2010.

[4] G. Taubock, “Complex-valued random vectors and channels: entropy, divergence, and capacity”, *IEEE Trans. on Information Theory*, vol. 58, no. 5, pp. 2729–2744, May 2012.

[5] M. Costa, “On the Gaussian interference channel”, *IEEE Trans. on Information Theory*, vol. 31, no. 5, pp. 607–615, September 1985.

[6] S. A. Jafar, *Interference Alignment: A New Look at Signal Dimensions in a Communication Network*, Foundations and Trends in Communications and Information Theory, vol. 7, no. 1, 2010.

[7] V. R. Cadambe, S. A. Jafar, C. Wang, “Interference alignment with asymmetric complex signaling - settling the Host-Madsen-Nosratinia conjecture”, *IEEE Trans. on Information Theory*, vol. 56, no. 9, pp. 4552–4565, September 2010.

[8] Z. K. M. Ho, E. Jorswieck, “Improper Gaussian signaling on the two-user SISO interference channel”, in *Proc. IEEE International Symposium on Wireless Commun. Systems*, pp. 824–828, November 2011.

[9] Y. Zeng, C. M. Yetis, E. Gunawan, Y. L. Guan, R. Zhang, “Transmit Optimization with Improper Gaussian Signaling for Interference Channels”, *IEEE Trans. on Signal Processing*, vol. 61, no. 11, pp. 2899–2913, June 2013.

[10] Y. Zeng, R. Zhang, E. Gunawan, and Y. L. Guan, “Optimized transmission with improper Gaussian signaling in the K-user MISO interference channel”, *IEEE Trans. on Wireless Communications*, vol. 12, no. 12, pp. 6303-6313, December 2013.

[11] D. Guo, S. Shamai, S. Verdú, “Mutual Information and Minimum Mean-Square Error in Gaussian Channels”, *IEEE Trans. on Information Theory*, vol. 52, no. 4, pp. 1261–1282, April 2005.

[12] R. Merris, “An Improvement of the Fischer Inequality”, *Journal of Research of the Notional Bureau of Standards - B. Mathematical Sciences*, vol. 75B, nos. 1 and 2, January-June 1971.

[13] S. Barbarossa, *Multiantenna wireless communication systems*, Mobile Communications series, 2005.

[14] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer Series in Statistics, 2009.

[15] F. Hiai, “Log-majorizations and norm inequalities for exponential operators”, *Banach Center Publications*, vol. 38, no.1, pp.119–181, 1997.

[16] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[17] F. Zhang, “Matrix inequalities by means of block matrices”, *Mathematical Inequalities and Applications*, vol. 4, no. 4, pp. 481-490, 2001.