ACHIEVING THE DEGREES OF FREEDOM OF $2 \times 2 \times 2$ INTERFERENCE NETWORK WITH ARBITRARY ANTENNA CONFIGURATIONS

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ABSTRACT

This paper studies the degrees of freedom (DoF) region of the $2 \times 2 \times 2$ interference network, which is comprised of two sources, two relays and two destinations, each with arbitrary number of antennas. We prove that with linear transceivers, the cut-set outer bound can be achieved without any symbol extensions, except for one specific system setup, which has one-DoF gap to the cut-set bound. We show that to achieve the outer-bound, the transceivers include interference avoidance, cancelation, neutralization and alignment, depending on the antenna configuration.

1. INTRODUCTION

To characterize and explore the potential of interference networks, the degrees of freedom (DoF) has been studied extensively, and many novel interference management mechanisms are proposed, such as interference alignment (IA) [1], interference neutralization (IN) [2] and interference shaping [3].

In spite of rapid progress on the DoF analysis for single-hop interference networks, until now little is known for the maximal DoF and the transceivers achieving the DoF of multi-hop interference networks, especially when each node is equipped with multiple antennas [4]-[10]. For multi-antenna two-hop interference network with multiple sources and destination each with arbitrary number of antennas, a DoF upper bound was presented in [11], but the achievability of the bound was not shown. In [12], a strategy of aligned IN including IA and IN was proposed for single antenna $2\times2\times2$ interference network, which achieves the cut-set outer bound of the system almost surely in the asymptotic sense (i.e., with infinite symbol extensions). In [13], the DoF region of $2 \times 2 \times 2$ interference network with arbitrary number of antennas at each node was proposed, which coincides with the cut-set outer-bound and can be achieved by beamforming and the aligned IN. As a result, the outer-bound is achieved in the asymptotic sense.

In this paper, we study the same $2 \times 2 \times 2$ interference network with arbitrary antenna configuration as in [13]. We prove that the cut-set outer bound of the network can be achieved without symbol extension except for one specific setup, for which there is a DoF gap of one to the outer bound.

Notations: For matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, we denote $\boldsymbol{A}^+$ and $\boldsymbol{A}^\perp$ as the pseudo inverse and the basis of the null space of $\boldsymbol{A}$, respectively. When $m \geq n$, we have $\boldsymbol{A}^+ \in \mathbb{C}^{n\times m}$, $\boldsymbol{A}^\perp \in \mathbb{C}^{(m-n)\times m}$ and $\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{I}_n$, $\boldsymbol{A}^\perp\boldsymbol{A} = \boldsymbol{0}$; otherwise, we have $\boldsymbol{A}^+ \in \mathbb{C}^{n\times m}$, $\boldsymbol{A}^\perp \in \mathbb{C}^{n\times(n-m)}$ and $\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{I}_m$, $\boldsymbol{A}\boldsymbol{A}^\perp = \boldsymbol{0}$.

2. SYSTEM MODEL AND MAIN RESULT

The $2\times2\times2$ interference network with arbitrary antenna configuration is denoted as $((d_1, M_1, N_1), (d_2, M_2, N_2,), R_1, R_2)$, where M_i, N_i and R_i are the numbers of antennas at each source, destination and relay, and d_i is the number of data streams transmitted from source i to destination i, $i = 1, 2$.

Denote $\boldsymbol{V}_i \in \mathbb{C}^{M_i\times d_i}$ as the transmit matrix at source i, $\boldsymbol{U}_i^H \in \mathbb{C}^{d_i\times N_i}$ as the receive matrix at destination i, $\boldsymbol{\Gamma}_i \in \mathbb{C}^{R_i\times R_i}$ as the processing matrix at relay i, $\boldsymbol{F}_{ij} \in \mathbb{C}^{R_i\times M_j}$ and $\boldsymbol{G}_{ij} \in \mathbb{C}^{N_i\times R_j}$ as the channel matrices from source j to relay i and from relay j to destination i, respectively. All elements in the channel matrices are independent and identically distributed ($i.i.d.$) random variables.

The DoF tuple (d_1, d_2) is achievable by linear transceivers without symbol extension if the following interference-free transmission constraints can be satisfied,

$$\boldsymbol{E}_{ij} \triangleq \boldsymbol{U}_i^H (\boldsymbol{G}_{i1}\boldsymbol{\Gamma}_1\boldsymbol{F}_{1j} + \boldsymbol{G}_{i2}\boldsymbol{\Gamma}_2\boldsymbol{F}_{2j})\boldsymbol{V}_j = \boldsymbol{0}, i \neq j, \tag{1a}$$

$$rank\left\{\boldsymbol{U}_i^H (\boldsymbol{G}_{i1}\boldsymbol{\Gamma}_1\boldsymbol{F}_{1i} + \boldsymbol{G}_{i2}\boldsymbol{\Gamma}_2\boldsymbol{F}_{2i})\boldsymbol{V}_i\right\} = d_i, \tag{1b}$$

where $\boldsymbol{E}_{ij}$ is the interference generated from source j to destination i, (1a) ensures interference-free and (1b) ensures reliable data transmission.

Our main result is shown in the following theorem.

Theorem 1: For the $((d_1, M_1, N_1), (d_2, M_2, N_2,), R_1, R_2)$ network, the following DoF region is achievable by linear transceivers without symbol extension,

$$\begin{aligned}\mathfrak{D} = \{(d_1, d_2)| & 1 \leq d_i \leq min\{M_i, N_i\}, i = 1, 2; \\ & d_1 + d_2 \leq R_1 + R_2 - 1, \text{ when } M_i = N_i \text{ and } R_1 + R_2 = M_1 + M_2; \\ & d_1 + d_2 \leq R_1 + R_2, \quad \text{otherwise}\}.\end{aligned} \tag{2}$$

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Compared with the cut-set outer bound, which is [13],

$$\mathfrak{C} = \{(d_1, d_2)| 1 \leq d_i \leq min\{M_i, N_i\}, i = 1, 2; \\ d_1 + d_2 \leq R_1 + R_2\}. \tag{3}$$

we can see that there is a DoF gap of one between our result and the outer bound only in a setup when $M_i = N_i$ and $R_1 + R_2 = M_1 + M_2$. In all other setups, our result coincides with the cut-set outer bound.

3. PROOF OF THE MAIN RESULT

In this section, we prove and explain the main result. We refer the system with setup of $M_i = N_i = d_i$ and $R_1 + R_2 = M_1 + M_2$ as the *standard* system. We first prove that for the *standard* system, linear transceivers can not ensure interference-free transmission, which results in the DoF loss from the cut-set outer bound. Then we prove that when one additional antenna is added to any one of the nodes in the *standard* system, interference-free transmission can be ensured, from which we obtain the DoF region for the considered network with arbitrary antenna configuration. Without loss of generality, we assume that $d_1 \leq d_2$ and $R_1 \leq R_2$ in the rest of the paper.

3.1. Transceivers for The Standard System

In the sequel, we show that the solution of interference-free equation (1a) for the *standard* system is $\mathbf{\Gamma}_1 = \mathbf{0}$ and $\mathbf{\Gamma}_2 = \mathbf{0}$, hence data transmission constraint (1b) can not be ensured.

For notational simplicity, we consider symmetric *standard* systems $((d, d, d), (d, d, d), d, d)$. For asymmetric *standard* systems, after some trivial transformations to the interference-free equation the same result can be obtained, which are omitted due to the lack of space.

In the symmetric standard system, the transmit and receive matrices and all the channel matrices are full rank square matrices of size $d \times d$. Therefore, (1a) reduces to

$$\mathbf{\Gamma}_1 = \underbrace{-\boldsymbol{G}_{11}^{-1}\boldsymbol{G}_{12}}_{\boldsymbol{A}_1}\mathbf{\Gamma}_2\underbrace{\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{-1}}_{\boldsymbol{B}_1} = \underbrace{-\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}}_{\boldsymbol{A}_2}\mathbf{\Gamma}_2\underbrace{\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{-1}}_{\boldsymbol{B}_2}, \tag{4}$$

where $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are all full rank square matrices.

Denote $\boldsymbol{S}$ as the right eigenvectors of matrix $\boldsymbol{A}_1^{-1}\boldsymbol{A}_2$ and $\mathbf{\Sigma}_1$ as a diagonal matrix composed of the eigenvalues of the matrix such that $\boldsymbol{A}_2\boldsymbol{S} = \boldsymbol{A}_1\boldsymbol{S}\mathbf{\Sigma}_2$. Denote $\boldsymbol{W}$ as the left eigenvectors of matrix $\boldsymbol{B}_2\boldsymbol{B}_1^{-1}$ and $\mathbf{\Sigma}_2$ as a diagonal matrix composed of the eigenvalues of the matrix such that $\boldsymbol{W}\boldsymbol{B}_2 = \mathbf{\Sigma}_1\boldsymbol{W}\boldsymbol{B}_1$. Note that $\boldsymbol{S}$ and $\boldsymbol{W}$ are full rank square matrices. After substituting $\mathbf{\Gamma}_2 = \boldsymbol{S}\hat{\mathbf{\Gamma}}_2\boldsymbol{W}$ into (4), we obtain an equation $\mathbf{\Sigma}_1\hat{\mathbf{\Gamma}}_2\mathbf{\Sigma}_2 = \hat{\mathbf{\Gamma}}_2$, which can be rewritten as a set of element-wise equations: $\sigma_{1,i}\sigma_{2,j}\gamma_{ij} = \gamma_{ij}$, where $\sigma_{1,i}$ is the ith diagonal element of $\mathbf{\Sigma}_1$, $\sigma_{2,j}$ is the jth diagonal element of $\mathbf{\Sigma}_2$, and γ_{ij} is the element in the ith row and jth column of $\hat{\mathbf{\Gamma}}_2$. From the equations we have $\gamma_{ij} = 0$, and hence $\mathbf{\Gamma}_2 = \mathbf{0}$. Further considering (4) we have $\mathbf{\Gamma}_1 = \mathbf{0}$. This implies that for the *standard* system the cut-set bound can not be achieved by linear transceivers and a DoF gap from it exists.

3.2. Transceivers With Additional Antenna at Relay

When one extra antenna is added to one of the relays of the *standard* system, the network becomes $((d_1, d_1, d_1), (d_2, d_2, d_2), R_1, R_2)$ with $R_1 + R_2 = d_1 + d_2 + 1$. We prove that for such a network the interference-free transmission constraints (1a) and (1b) can be satisfied, by finding the transceivers at each source, destination and relay. We first find the transceivers for two special systems to ensure interference-free transmission, where the interference are removed by interference neutralization. Then by introducing interference avoidance at sources and interference cancelation at destinations, we show that a general system can transmit without interference by using the result for the two special systems.

A. Special system 1: $((d, d, d), (d, d, d), d, d + 1)$

In this case, $\boldsymbol{U}_i$, $\boldsymbol{V}_i$ and $\boldsymbol{G}_{i1}$ and $\boldsymbol{F}_{1j}$ are all full rank square matrices of size $d \times d$, while $\boldsymbol{G}_{i2}$ is of size $d \times (d + 1)$ and $\boldsymbol{F}_{2j}$ is of size $(d + 1) \times d$, $i, j = 1, 2$.

To ensure data transmission constraint (1b), we set the transmit and receive matrices at sources and destinations as

$$\boldsymbol{V}_1 = \boldsymbol{F}_{11}^{-1}, \boldsymbol{V}_2 = \boldsymbol{F}_{12}^{-1}, \boldsymbol{U}_1^H = \boldsymbol{G}_{11}^{-1}, \boldsymbol{U}_2^H = \boldsymbol{G}_{21}^{-1}.$$

To ensure interference-free constraint (1a), the processing matrices at the relays are with the same form as in (4), i.e.,

$$\mathbf{\Gamma}_1 = \boldsymbol{A}_1\mathbf{\Gamma}_2\boldsymbol{B}_1 = \boldsymbol{A}_2\mathbf{\Gamma}_2\boldsymbol{B}_2, \tag{5}$$

where $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are of size $d \times (d + 1)$, and $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are of size $(d + 1) \times d$, all defined in (4). (5) ensures that the interference received at each destination are summed to be zero, i.e., removed by interference neutralization.

Denote $\mathbf{\Gamma}_2 = \boldsymbol{P}\boldsymbol{Q}$ where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are square matrices of dimension $d + 1$ and $\mathbf{\Sigma} = diag\{\sigma_1, \cdots, \sigma_{d+1}\}$ as a random diagonal matrix with $\sigma_r \neq \sigma_{r'}$, $r \neq r'$. We can find a matrix $\boldsymbol{Q} = \left[\boldsymbol{q}_1, \cdots, \boldsymbol{q}_{d+1}\right]^H$ to ensure $\mathbf{\Sigma}\boldsymbol{Q}\boldsymbol{B}_1 = \boldsymbol{Q}\boldsymbol{B}_2$ by setting the rth row of $\boldsymbol{Q}$ as

$$\boldsymbol{q}_r^H = (\sigma_r\boldsymbol{B}_1 - \boldsymbol{B}_2)^{\perp}, r = 1, \cdots, d + 1. \tag{6}$$

Considering the dimension of $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, the obtained $\boldsymbol{q}_r^H$ is non-zero and it is easy to show that $\boldsymbol{q}_r^H$ and $\boldsymbol{q}_{r'}^H$ are linearly independent when $\sigma_r \neq \sigma_{r'}$. Thus, the matrix $\boldsymbol{Q}$ is full rank. Then (5) becomes

$$(\boldsymbol{A}_1\boldsymbol{P} - \boldsymbol{A}_2\boldsymbol{P}\mathbf{\Sigma})\boldsymbol{Q}\boldsymbol{B}_1 = \mathbf{0}. \tag{7}$$

Because $\boldsymbol{Q}\boldsymbol{B}_1$ is of size $(d + 1) \times d$, there exists a row vector $\boldsymbol{b}^H = \left[b_1, \cdots, b_{d+1}\right] \in \mathbb{C}^{1 \times (d+1)}$ such that $\boldsymbol{b}^H\boldsymbol{Q}\boldsymbol{B}_1 = \mathbf{0}$. Therefore, (7) can be solved when $\boldsymbol{A}_1\boldsymbol{P} - \boldsymbol{A}_2\boldsymbol{P}\mathbf{\Sigma} = \boldsymbol{p}\boldsymbol{b}^H$, where $\boldsymbol{p} \in \mathbb{C}^{d \times 1}$ is an arbitrary random vector. This can be achieved by setting the rth column of $\boldsymbol{P}$ as

$$\boldsymbol{p}_r = b_r\left(\boldsymbol{A}_1 - \sigma_r\boldsymbol{A}_2\right)^{+}\boldsymbol{p}, r = 1, \cdots, d + 1. \tag{8}$$

Using (6) and (8) we can obtain $\boldsymbol{P}$ and $\boldsymbol{Q}$, then the relay processing matrix at the 2^{nd} relay $\mathbf{\Gamma}_2$ is obtained. Furthermore, $\mathbf{\Gamma}_1$ can be obtained from (5). By substituting the relay processing matrices into (1b), it is easy to verify that the data transmission constraint can be satisfied.

B. Special system 2: $((d-1,d-1,d-1),(d,d,d),d,d)$

To satisfy data transmission constraint (1b), we set the transmit matrices at both sources and the receive matrices at both destinations as identity matrices. To enure interference-free constraint (1a), we can show that the processing matrices at the two relays should satisfy the following relationship,

$$\begin{aligned}\boldsymbol{\Gamma}_1 = &\left[\boldsymbol{G}_{11}^{\perp}, \boldsymbol{G}_{11}^{+}\right] \begin{bmatrix} t & \\ & \boldsymbol{T}\end{bmatrix} \begin{bmatrix}\boldsymbol{F}_{11}^{\perp} \\ \boldsymbol{F}_{11}^{+}\end{bmatrix} \\ &- \boldsymbol{G}_{11}^{+}\boldsymbol{G}_{12}\boldsymbol{\Gamma}_2\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{-1} - \boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}\boldsymbol{\Gamma}_2\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{+},\end{aligned} \tag{9}$$

where t is a random variable and the square matrix satisfies

$$\boldsymbol{T} \overset{(a)}{=} \underbrace{\boldsymbol{G}_{11}\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}}_{(d-1)\times d}\boldsymbol{\Gamma}_2 \underbrace{\boldsymbol{F}_{21}}_{d\times(d-1)} \overset{(b)}{=} \underbrace{\boldsymbol{G}_{12}}_{(d-1)\times d} \boldsymbol{\Gamma}_2 \underbrace{\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{-1}\boldsymbol{F}_{11}}_{d\times(d-1)}. \tag{10}$$

When $\boldsymbol{T}$ satisfies equation (a) in (10), the corresponding relay processing matrices in (9) ensure $\boldsymbol{E}_{12} = \mathbf{0}$, which neutralizes the interference at 1^{st} destination. When $\boldsymbol{T}$ satisfies equation (b), $\boldsymbol{E}_{21} = \mathbf{0}$, neutralizing the interference at 2^{nd} destination. Using the same method as for the special system 1, we can find $\boldsymbol{\Gamma}_2$ by solving the equation in (10) and further obtain $\boldsymbol{\Gamma}_1$ from (9), which can be shown also satisfy constraint (1b).

In the following, we show that a general system can transmit without interference by using the result for special system 1 when $R_1 \le d_1$ and that for special system 2 when $R_1 > d_1$.

3.2.1. *General case 1:* $R_1 \le d_1$

Recalling the assumption that $d_1 \le d_2$ and $R_1 \le R_2$ and there is an additional relay at relay, which is $R_1 + R_2 = d_1 + d_2 + 1$, in this case, the relationship of the system parameters should satisfy $R_1 \le d_1 < d_2 + 1 \le R_2$. To ensure (1b), we set the transmit and receive matrices as

$$\boldsymbol{V}_1 = [\boldsymbol{F}_{11}^{+}, \boldsymbol{F}_{11}^{\perp}], \boldsymbol{V}_2 = [\boldsymbol{F}_{12}^{+}, \boldsymbol{F}_{12}^{\perp}], \boldsymbol{U}_1^H = \begin{bmatrix}\boldsymbol{G}_{11}^{+} \\ \boldsymbol{G}_{11}^{\perp}\end{bmatrix}, \boldsymbol{U}_2^H = \begin{bmatrix}\boldsymbol{G}_{21}^{+} \\ \boldsymbol{G}_{21}^{\perp}\end{bmatrix}. \tag{11}$$

To enure (1a), we set the processing matrix at the relays as

$$\begin{aligned}\boldsymbol{\Gamma}_1 &\overset{(a)}{=\!=} \underbrace{-\left(\boldsymbol{G}_{11}^{+}\boldsymbol{G}_{12}\right)\boldsymbol{M}}_{\boldsymbol{A}_1}\hat{\boldsymbol{\Gamma}}_2\underbrace{\boldsymbol{N}\left(\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{+}\right)}_{\boldsymbol{B}_1} \\ &\overset{(b)}{=\!=} \underbrace{-\left(\boldsymbol{G}_{21}^{+}\boldsymbol{G}_{22}\right)\boldsymbol{M}}_{\boldsymbol{A}_2}\hat{\boldsymbol{\Gamma}}_2\underbrace{\boldsymbol{N}\left(\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{+}\right)}_{\boldsymbol{B}_2}, \\ \boldsymbol{\Gamma}_2 &= \left[\boldsymbol{G}_{22}^{\perp}\boldsymbol{P}_1, \quad \boldsymbol{G}_{12}^{\perp}\boldsymbol{P}_2, \quad \boldsymbol{M}\hat{\boldsymbol{\Gamma}}_2\right]\begin{bmatrix}\boldsymbol{Q}_1\boldsymbol{F}_{22}^{\perp} \\ \boldsymbol{Q}_2\boldsymbol{F}_{21}^{\perp} \\ \boldsymbol{N}\end{bmatrix},\end{aligned} \tag{12}$$

where $\boldsymbol{P}_1 \in \mathbb{C}^{(R_2-d_2)\times(d_1-R_1)}$, $\boldsymbol{Q}_1 \in \mathbb{C}^{(d_1-R_1)\times(R_2-d_2)}$, $\boldsymbol{P}_2 \in \mathbb{C}^{(R_2-d_1)\times(d_2-R_1)}$, $\boldsymbol{Q}_2 \in \mathbb{C}^{(d_2-R_1)\times(R_2-d_1)}$ are arbitrary random matrices, and $\boldsymbol{M}, \boldsymbol{N}$ are defined as

$$\begin{aligned}\boldsymbol{M} &= \underbrace{\left(\boldsymbol{G}_{21}^{\perp}\boldsymbol{G}_{22}\right)^{\perp}\left(\left(\boldsymbol{G}_{11}^{\perp}\boldsymbol{G}_{12}\right)\left(\boldsymbol{G}_{21}^{\perp}\boldsymbol{G}_{22}\right)^{\perp}\right)^{\perp}}_{R_2\times(R_1+R_1+R_2-d_1-d_2)}, \\ \boldsymbol{N} &= \underbrace{\left(\left(\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{\perp}\right)^{\perp}\left(\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{\perp}\right)\right)^{\perp}\left(\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{\perp}\right)^{\perp}}_{(R_1+R_1+R_2-d_1-d_2)\times R_1}.\end{aligned} \tag{13}$$

$\boldsymbol{A}_1, \boldsymbol{A}_2$ are of size $R_1 \times (R_1+1)$ and $\boldsymbol{B}_1, \boldsymbol{B}_2$ are of size $(R_1+1)\times R_1$, same as those in (5). Hence, $\hat{\boldsymbol{\Gamma}}_2$ can be designed using the same way as from (5). Then we can obtain $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ from (12) ensuring data constraint as in special system 1.

From $\boldsymbol{V}_i$ in (11) we observe that the transmit antennas at source i can be divided into two parts. From the first R_1 antennas the signals are transmitted with the sub-precoder $\boldsymbol{F}_{1i}^{+} \in \mathbb{C}^{d_i\times R_1}$ and can be received by 1^{st} relay. From the rest of $d_i - R_1$ antennas the signals are transmitted with the sub-precoder $\boldsymbol{F}_{1i}^{\perp} \in \mathbb{C}^{d_i\times(d_i-R_1)}$ and are avoiding to be received by 1^{st} relay because $\boldsymbol{F}_{1i}\boldsymbol{F}_{1i}^{\perp} = \mathbf{0}$. From the receive matrix $\boldsymbol{U}_i^H$, we can see that the first R_1 antennas with $\boldsymbol{G}_{i1}^{+} \in \mathbb{C}^{R_1\times d_i}$ receive signals from both relays and the rest of $d_i - R_1$ antennas with $\boldsymbol{G}_{i1}^{\perp} \in \mathbb{C}^{(d_i-R_1)\times d_i}$ can cancel the receive signals from 1^{st} relay because $\boldsymbol{G}_{i1}^{\perp}\boldsymbol{G}_{i1} = \mathbf{0}$.

From the number of rows of $\boldsymbol{Q}_1\boldsymbol{F}_{22}^{\perp}$, $\boldsymbol{Q}_2\boldsymbol{F}_{21}^{\perp}$ and $\boldsymbol{N}$ in $\boldsymbol{\Gamma}_2$, we can see that the antennas at relay 2^{nd} can be divided into three parts: (1) only the first $d_1 - R_1$ antennas receive signals from the $d_1 - R_1$ antennas at the 1^{st} source because $\boldsymbol{Q}_2\boldsymbol{F}_{21}^{\perp}\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{\perp} = \mathbf{0}$ and $\boldsymbol{N}\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{\perp} = \mathbf{0}$; then these signals are forwarded by $\boldsymbol{G}_{22}^{\perp}\boldsymbol{P}_1$ such that they do not interfere with the 2^{nd} destination, because $\boldsymbol{G}_{22}^{\perp}\boldsymbol{G}_{22} = \mathbf{0}$; (2) only the second $d_2 - R_1$ antennas receive signals from the $d_2 - R_1$ antennas at the 2^{nd} source and forward them such that they do not interfere with the 1^{st} destination; (3) the rest $R_1 + 1$ antennas receive signals with $\boldsymbol{N}$ from the R_1 antennas of the two sources and forward them with $\boldsymbol{M}\hat{\boldsymbol{\Gamma}}_2$ to the two destinations, these signals will be neutralized with their counterparts forwarded through the 1^{st} relay using the processing matrix $\boldsymbol{\Gamma}_1$.

In summary, by using interference avoidance at sources, interference cancelation at destinations, and interference neutralization through relays, the networks under this general case can be decoupled into three sub-systems: (1) $d_1 - R_1$ antennas at the 1^{st} source transmit $d_1 - R_1$ data streams through $d_1 - R_1$ antennas at the 2^{nd} relay to $d_1 - R_1$ antennas at the 1^{st} destination without interfering the 2^{nd} destination; (2) $d_2 - R_1$ antennas at the 2^{nd} source transmit $d_2 - R_1$ data streams through $d_2 - R_1$ antennas at the 2^{nd} relay to $d_2 - R_1$ antennas at the 2^{nd} destination without interfering the 1^{st} destination; (3) the rest R_1 antennas at both sources and destinations transmit R_1 data streams each through R_1 antennas at the 1^{st} relay and $R_1 + 1$ antennas at the 2^{nd} relay, which is the special system 1 $((R_1, R_1, R_1), (R_1, R_1, R_1), R_1, R_1 + 1)$.

3.2.2. *General case 2:* $d_1 < R_1$

Recalling that $d_1 \le d_2$, $R_1 \le R_2$ and $R_1 + R_2 = d_1 + d_2 + 1$, in this case, the relationship of the system parameters should be $d_1 < R_1 \le R_2 < d_2 + 1$. To ensure (1b), we set the transmit and receive matrices as

$$\begin{aligned}&\boldsymbol{V}_1 = \boldsymbol{I}_{d_1}, \boldsymbol{V}_2 = \left[\boldsymbol{F}_{12}^{\perp}, \boldsymbol{F}_{22}^{\perp}, \boldsymbol{F}_{12}^{+}\bar{\boldsymbol{V}}_2\right], \\ &\boldsymbol{U}_1^H = \boldsymbol{I}_{d_1}, \boldsymbol{U}_2^H = \left[(\boldsymbol{G}_{21}^{\perp})^H, (\boldsymbol{G}_{22}^{\perp})^H, (\bar{\boldsymbol{U}}_2^H\boldsymbol{G}_{21}^{+})^H\right]^H,\end{aligned} \tag{14}$$

where $\bar{\boldsymbol{V}}_2 \in \mathbb{C}^{R_1\times(d_1+1)}$ and $\bar{\boldsymbol{U}}_2^H \in \mathbb{C}^{(d_1+1)\times R_1}$ are arbitrary random matrices.

To enure (1a), we set the processing matrices at relays as

$$\begin{aligned}\boldsymbol{\Gamma}_1 &= \left[\boldsymbol{G}_{11}^{\perp}\boldsymbol{P}_1, (\boldsymbol{G}_{22}^{\perp}\boldsymbol{G}_{21})^{\perp}\hat{\boldsymbol{\Gamma}}_1\right]\begin{bmatrix}\boldsymbol{Q}_1\boldsymbol{F}_{11}^{\perp}\\ (\boldsymbol{F}_{12}\boldsymbol{F}_{22}^{\perp})^{\perp}\end{bmatrix},\\ \boldsymbol{\Gamma}_2 &= \left[\boldsymbol{G}_{12}^{\perp}\boldsymbol{P}_2, (\boldsymbol{G}_{21}^{\perp}\boldsymbol{G}_{22})^{\perp}\hat{\boldsymbol{\Gamma}}_2\right]\begin{bmatrix}\boldsymbol{Q}_2\boldsymbol{F}_{21}^{\perp}\\ (\boldsymbol{F}_{22}\boldsymbol{F}_{12}^{\perp})^{\perp}\end{bmatrix},\end{aligned} \tag{15}$$

where $\boldsymbol{P}_1 \in \mathbb{C}^{(R_1-d_1)\times(d_2-R_2)}$, $\boldsymbol{Q}_1 \in \mathbb{C}^{(d_2-R_2)\times(R_1-d_1)}$, $\boldsymbol{P}_2 \in \mathbb{C}^{(R_2-d_1)\times(d_2-R_1)}$, $\boldsymbol{Q}_1 \in \mathbb{C}^{(d_2-R_1)\times(R_2-d_1)}$ are arbitrary random matrices, and both $\hat{\boldsymbol{\Gamma}}_1$ and $\hat{\boldsymbol{\Gamma}}_2$ are of size $(R_1 + R_2 - d_2) \times (R_1 + R_2 - d_2)$. Substituting (14) and (15) into (1a), the interference-free constraints reduce to

$$\begin{aligned}&\underbrace{\hat{\boldsymbol{G}}_{11}}_{(d-1)\times d}\hat{\boldsymbol{\Gamma}}_1\underbrace{\hat{\boldsymbol{F}}_{12}}_{d\times d} + \underbrace{\hat{\boldsymbol{G}}_{12}}_{(d-1)\times d}\hat{\boldsymbol{\Gamma}}_2\underbrace{\hat{\boldsymbol{F}}_{22}}_{d\times d} = \boldsymbol{0},\\ &\underbrace{\hat{\boldsymbol{G}}_{21}}_{d\times d}\hat{\boldsymbol{\Gamma}}_1\underbrace{\hat{\boldsymbol{F}}_{11}}_{d\times(d-1)} + \underbrace{\hat{\boldsymbol{G}}_{22}}_{d\times d}\hat{\boldsymbol{\Gamma}}_2\underbrace{\hat{\boldsymbol{F}}_{21}}_{d\times(d-1)} = \boldsymbol{0},\end{aligned} \tag{16}$$

where the expressions of $\hat{\boldsymbol{G}}_{ij}$ and $\hat{\boldsymbol{F}}_{ij}$ are omitted due to the lack of space, from whose dimension we see that (16) can be viewed as the interference-free equation of a special system 2 with $d = d_1 + 1$. Hence, $\hat{\boldsymbol{\Gamma}}_1$ and $\hat{\boldsymbol{\Gamma}}_2$ can be obtained using the same way as in the special system 2.

Again, we can observe from (15) and (16) that the networks under this general case can be decoupled into three sub-systems, where the third sub-system is the special system 2, and the interference is avoided at sources, canceled at destinations, and neutralized through relays.

3.3. Transceivers With Additional Antenna at Destination or Source

Now we show that when one extra antenna is added to one of the sources or destinations of the *standard* system, interference-free transmission can be satisfied. We only present the transceivers for an example case of adding an antenna to the 1^{st} destination of the symmetric *standard* system. The same idea can be employed to find the transceivers to achieve the DoF region for other systems.

In the example system $((d, d, d+1), (d, d, d), d, d)$, the receive matrix $\boldsymbol{U}_1^H \in \mathbb{C}^{d\times(d+1)}$. Since the desired number of data streams at the 1^{st} destination is only d, the interference generated from the 2^{nd} source can be aligned to the one-dimension null space of $\boldsymbol{U}_1^H$. To see this, we denote $\boldsymbol{U}_1^H = (\boldsymbol{U}_1^{IC})^H(\boldsymbol{U}_1^{IA})^H$. $(\boldsymbol{U}_1^{IA})^H \in \mathbb{C}^{(d+1)\times(d+1)}$, together with the relay processing matrices and transmit matrix at the 2^{nd} source, ensures that the interference are aligned, and $(\boldsymbol{U}_1^{IC})^H \in \mathbb{C}^{d\times(d+1)}$ is the receive matrix that cancels the aligned interference. Denote $\boldsymbol{l} \in \mathbb{C}^{(d+1)\times 1}$ as the null space of $(\boldsymbol{U}_1^{IC})^H$, i.e., $(\boldsymbol{U}_1^{IC})^H\boldsymbol{l} = \boldsymbol{0}$. When all interference from the 2^{nd} source are aligned in the space spanned by $\boldsymbol{l}$, the 1^{st} destination can cancel all the aligned interference. In this case, the interference-free constraints (1a) are equivalent to

$$(\boldsymbol{U}_1^{IA})^H\left(\boldsymbol{G}_{11}\boldsymbol{\Gamma}_1\boldsymbol{F}_{12} + \boldsymbol{G}_{12}\boldsymbol{\Gamma}_2\boldsymbol{F}_{22}\right)\boldsymbol{V}_2 = \boldsymbol{l}\boldsymbol{r}^H, \tag{17a}$$

$$\boldsymbol{U}_2^H\left(\boldsymbol{G}_{21}\boldsymbol{\Gamma}_1\boldsymbol{F}_{11} + \boldsymbol{G}_{22}\boldsymbol{\Gamma}_2\boldsymbol{F}_{21}\right)\boldsymbol{V}_1 = \boldsymbol{0}. \tag{17b}$$

To enure (1b), we set the transmit and receive matrices as

$$(\boldsymbol{U}_1^{IA})^H = \begin{bmatrix}1 & \\ & \bar{\boldsymbol{G}}_{11}^{-1}\end{bmatrix}, \boldsymbol{U}_2^H = \boldsymbol{G}_{21}^{-1}, \boldsymbol{V}_1 = \boldsymbol{F}_{11}^{-1}, \boldsymbol{V}_2 = \boldsymbol{F}_{12}^{-1},$$

where $\bar{\boldsymbol{G}}_{11}^{-1} \in \mathbb{C}^{d\times d}$ is a sub-matrix in $\boldsymbol{G}_{1i} = \begin{bmatrix}\boldsymbol{g}_{1i}^H\\ \bar{\boldsymbol{G}}_{1i}\end{bmatrix}$.

To ensure (17b), the processing matrix at the 2^{nd} relay can be obtained as $\boldsymbol{\Gamma}_1 = -\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}\boldsymbol{\Gamma}_2\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{-1}$, which has the same form as in equation (b) in (5). By substituting these transceivers to (17a), we know that $\boldsymbol{\Gamma}_2$ should be designed to satisfy the following equations,

$$\boldsymbol{g}_{11}^H\boldsymbol{A}_1\boldsymbol{\Gamma}_2\boldsymbol{B}_1 + \boldsymbol{g}_{12}^H\boldsymbol{\Gamma}_2\boldsymbol{B}_2 = l_1\boldsymbol{r}^H, \tag{18a}$$

$$\boldsymbol{A}_1\boldsymbol{\Gamma}_2\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{\Gamma}_2\boldsymbol{B}_2 = \boldsymbol{l}_2\boldsymbol{r}^H, \tag{18b}$$

where $\boldsymbol{A}_1 = -\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}$, $\boldsymbol{A}_2 = -\bar{\boldsymbol{G}}_{11}^{-1}\bar{\boldsymbol{G}}_{12}$, $\boldsymbol{B}_1 = \boldsymbol{F}_{21}\boldsymbol{F}_{11}^{-1}$, $\boldsymbol{B}_2 = \boldsymbol{F}_{22}\boldsymbol{F}_{12}^{-1}$ are of size $d \times d$, and vector $\boldsymbol{l} = \left[l_1^H, \quad \boldsymbol{l}_2^H\right]^H$.

Let $\boldsymbol{W}$ be the left eigenvectors of matrix $\boldsymbol{B}_2\boldsymbol{B}_1^{-1}$ and $\boldsymbol{\Sigma}$ be the diagonal matrix composed of its eigenvalues, then substitute $\boldsymbol{\Gamma}_2 = \hat{\boldsymbol{\Gamma}}_2\boldsymbol{W}$ into (18a) and (18b). By solving (18b), we can obtain an expression of each column in $\hat{\boldsymbol{\Gamma}}_2$ as a function of $\boldsymbol{l}_2$; then by substituting the expression into (18a), we can obtain the value of $\boldsymbol{l}_2$ and therefore $\hat{\boldsymbol{\Gamma}}_2$ and $\boldsymbol{\Gamma}_1$.

From the designed transceivers we can see that now IA is necessary except for interference avoidance, interference cancelation and IN.

3.4. DoF Region of the Network

In the following, we show the DoF region for the $2 \times 2 \times 2$ interference network with arbitrary antenna configurations.

When $M_i = N_i$ and $R_1 + R_2 = M_1 + M_2$, the cut-set bound implies that the system can transmit maximum $d_1 + d_2 = R_1 + R_2$ data streams and $d_i = M_i$. From our analysis in section 3.1 we know that this DoF region can not be achieved by linear transceivers because the interference-free transmission can not be ensured. With one DoF loss from the cut-set bound, totally $R_1 + R_2 - 1$ data streams can be transmitted according to section 3.2, because one additional antenna is now available at the relays.

For all the other systems, there will be at least one more antenna than the total data streams at either a source, a destination or a relay. According to section 3.2 and 3.3, $d_1 + d_2 \leq R_1 + R_2$ data streams can be transmit without interference.

This completes the proof of Theorem 1.

4. CONCLUSIONS

In this paper, we proved that the achievable DoF region of the $2 \times 2 \times 2$ interference network of arbitrary antenna configurations coincides with the cut-set outer bound even without symbol extension, except for one-DoF gap in one special configuration. We showed that the out-bound can be achieved by linear transceivers, where the interference is removed by avoidance, neutralization and cancelation or further by alignment, depending on the antenna configurations.

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