ONLINE EM ESTIMATION OF THE DIRICHLET PROCESS MIXTURES SCALE PARAMETER TO MODEL THE GPS MULTIPATH ERROR

Vincent Pereira[⋆] *Audrey Giremus*[⋆] *Asma Rabaoui*[†] *Eric Grivel*[⋆]

[⋆]Université Bordeaux 1 - IPB - UMR CNRS 5218 IMS
351 cours de la Libération 33405, TALENCE Cedex, France
[†]Institut Fresnel - Phyti Team - Domaine universitaire de Saint-Jérôme
Avenue Escadrille Normandie-Nimen 13397, MARSEILLE Cedex 20, France

ABSTRACT

The performance of GPS is strongly degraded in a multipath environment. The multipath impact the distribution of the additive noise corrupting the distance measurements between the satellites and the GPS receiver. In this paper, this distribution is assumed unknown and modeled in a flexible way by using the Bayesian non parametric framework and more precisely the Dirichlet process mixtures. Nevertheless, these latter depend on the so-called scale parameter which can be difficult to tune *a priori*. The originality of our approach consists in adapting a recent version of the online EM algorithm, developed by Cappé for hidden Markov models, to compute a maximum *a posteriori* estimate of the scale parameter. Then, as the proposed model is non linear and non Gaussian, the EM-based scale parameter estimation is coupled with a Rao-Blackwellized particle filter for the joint estimation of the mobile location and the distance measurement noise distribution.

***Index Terms*—** GPS navigation, multipath, Dirichlet process mixtures, online EM, Rao-Blackwellized particle filter.

1. INTRODUCTION

Thanks to the Global Positioning System (GPS), every user can obtain his position anywhere on earth. For that purpose, the GPS receiver estimates the propagation delays of signals transmitted by a constellation of satellites of known locations. Then, distance measurements are computed through multiplication by the velocity of light in vacuum. In the sequel, these latter will be referred to as pseudo-ranges.

Today, an accuracy of about 10 m can be nominally achieved. However, the GPS performance can strongly deteriorate in urban environments due to the multipath phenomenon. It happens when different replicas of the satellite signal, incoming from reflections on nearby obstacles, reach the receiver. Chapter 7 in [1] details several methods that have been designed to mitigate multipath effects. A class of approaches deals with multipath effects directly at the level of the navigation algorithm which estimates the position from the pseudo-ranges. They have the advantage of avoiding any modification of the receiver architecture.

In [2], a Bernoulli-Gaussian distribution is considered to model multipath-induced errors that contribute to the additive noise term within the pseudo-ranges. Therefore, the multipath errors can be directly detected and compensated within a particle filter (PF) that estimates the mobile dynamics. The estimation of the multipath model parameters is based on the correlation in time of the multipath. However, such an assumption is no longer realistic in an environment with a high density of obstacles and fast moving vehicles.

As an alternative, in [3], Rabaoui *et al.* suggest modeling the unknown distribution of the noises disturbing the pseudo-ranges by using non parametric approaches based on Dirichlet Process Mixtures (DPM). This makes it possible to capture any multimodal distribution without having to set *a priori* the number of modes. Then, the unknown distribution is directly estimated by a PF from the gathered pseudo-ranges along with the position of the GPS receiver. Nevertheless, the performance of the estimation strongly depends on the so-called scale parameter of the DPM. Tuning this parameter can be a difficult task for the practitioner. A solution is to estimate it jointly with the other unknown variables.

However, online estimation of static parameters cannot be performed by applying a standard PF. Indeed, this problem leads to convergence issues since the parameter space is only explored at the initialization of the algorithm [4]. A first solution would consist in introducing artificial dynamics for the unknown parameters at the expense of the estimation accuracy [5]. As an alternative, the seminal work of Gilks [6] consists in adding Markov Chain Monte Carlo steps within the PF to reintroduce diversity. This approach was used in [3] to jointly estimate the parameters of the DPM and the unknown pseudo-range noise distribution. However, this solution results in particle impoverishment [4].

In this paper, we propose to adjust the scale parameter by means of a recursive online expectation-maximisation (EM) procedure. Compared to the classical EM, the principle is to update the parameter estimate whenever a new measurement becomes available. Different formulations have been recently developed for finite state hidden Markov models (HMM) [7] and continuous state HMM [8] when the complete-data likelihood belongs to the exponential family of distributions (EFD) [9]. In this paper, we take advantage of the latter to perform a maximum *a posteriori* (MAP) estimation in place of a maximum likelihood (ML) estimation. The introduction of a prior distribution regularizes the estimation of the scale parameter by preventing degeneracies. Then, as the proposed model is weakly non linear and conditionally Gaussian given some latent variables characterizing the unknown noise distribution, a Rao-Blackwellized PF (RBPF) is used to perform the estimation.

The paper is organized as follows: in section 2, DPM are presented. The so-called Polya urn and the scale parameters are also introduced. Section 3 details the Bayesian hierarchical model of the GPS navigation problem in the presence of multipath. Section 4 describes the proposed RBPF algorithm. Section 5 deals with the online EM algorithm and its application to the DPM scale parameter estimation. Finally, the results obtained on simulated GPS data are discussed in section 6.

2. DIRICHLET PROCESS MIXTURES

DPM are Bayesian non parametric models that make it possible to fit all kinds of probability distributions.
Let us consider a random variable v_t distributed according to an unknown distribution F. In a non parametric framework, F can be expressed by the following integral expression:

$$F(v_t) = \int_{\Theta} f(v_t|\boldsymbol{\theta}_t) G(\boldsymbol{\theta}_t) d\boldsymbol{\theta}_t, \tag{1}$$

where $\boldsymbol{\theta}_t \in \Theta$ is the so-called latent variable. $\boldsymbol{\theta}_t$ contains the parameters of f, the user-chosen mixed probability density function (PDF). Usually, a Gaussian PDF is chosen for f with its mean μ_t and variance ϕ_t stored in $\boldsymbol{\theta}_t = [\mu_t, \phi_t]^T$ and $f(v_t|\boldsymbol{\theta}_t) = \mathcal{N}(v_t; \mu_t, \phi_t)$. Finally, G is the unknown mixing distribution. It is assumed to be a random distribution. In a Bayesian framework, its prior distribution must be chosen by the practitioner. A classical choice for the prior of G is the Dirichlet process (DP).

The DP can be seen as a probability distribution over the space of probability distributions. G is distributed according to a DP of base distribution G_0 and positive scale parameter α, denoted as $G \sim \mathcal{DP}(G_0, \alpha)$.
Furthermore, the realizations G of a DP are discrete but infinite distributions. According to the stick-breaking representation, they can be expressed as infinite mixtures of Dirac measures as follows:

$$G(\boldsymbol{\theta}_t) = \sum_{j=1}^{+\infty} \pi_j \delta_{\mathbf{U}_j}(\boldsymbol{\theta}_t), \;\; \pi_j = \beta_j \prod_{l=1}^{j-1}(1-\beta_l), \tag{2}$$

with the clusters $\mathbf{U}_j \overset{\text{iid}}{\sim} G_0$ and where iid stands for "independent and identically distributed". The weights π_j in (2) are defined sequentially with $\beta_j \overset{\text{iid}}{\sim} \mathcal{B}(1, \alpha)$ where $\mathcal{B}$ denotes the Beta law.
Combining (1) and (2), we obtain the following alternative expression for the unknown distribution F:

$$F(v_t) = \sum_{j=1}^{+\infty} \pi_j f(v_t|\mathbf{U}_j). \tag{3}$$

Thus, F consists of an infinite mixture of the PDF f with mixture weights π_j and parameters contained in $\mathbf{U}_j$.
An advantage of the DP is that the computations involved in the Bayesian estimation procedure can be considerably simplified. Indeed, it has been shown in [10] that the latent variables $\boldsymbol{\theta}_t$ can be directly sampled sequentially without explicitly involving G. Marginalizing G leads to the Polya urn representation as follows:

$$p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{1:t-1};\alpha) = \frac{1}{\alpha+t-1}\sum_{j=1}^{t-1}\delta_{\boldsymbol{\theta}_j}(\boldsymbol{\theta}_t) + \frac{\alpha}{\alpha+t-1}G_0(\boldsymbol{\theta}_t). \tag{4}$$

An equivalent formulation, which is used in the sequel, consists in introducing an auxiliary variable $c_t \in \{0, 1\}$. The value $c_t = 1$ indicates that the sample $\boldsymbol{\theta}_t$ is identical to a previous one with probability $\Pr[c_t = 1; \alpha, t] = (t-1)/(\alpha+t-1)$. Conversely, the value $c_t = 0$ indicates that $\boldsymbol{\theta}_t$ takes a new value distributed according to G_0 with probability $\Pr[c_t = 0; \alpha, t] = \alpha/(\alpha+t-1)$. Thus, the joint distribution of $\boldsymbol{\theta}_t$ and c_t given $\boldsymbol{\theta}_{1:t-1} = \{\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_{t-1}\}$ can be written as follows:

$$p(\boldsymbol{\theta}_t, c_t|\boldsymbol{\theta}_{1:t-1};\alpha) = p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{1:t-1}, c_t)\Pr[c_t;\alpha,t], \tag{5}$$

where:

$$p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{1:t-1}, c_t) = G_0(\boldsymbol{\theta}_t)\delta_0(c_t) + \frac{1}{t-1}\sum_{j=1}^{t-1}\delta_{\boldsymbol{\theta}_j}(\boldsymbol{\theta}_t)\delta_1(c_t).$$

Note that a high value of the scale parameter α introduced in (4) favors the occurrence of a high number of different modes.

3. BAYESIAN MODELING OF THE PROBLEM

The problem is to jointly estimate the dynamics of the mobile equipped by the GPS receiver and the pseudo-range noise distribution. In the following, the dynamics of the mobile is described by a set of variables contained in the state vector $\mathbf{x}_t$ of size n_x. Three of its components include the position coordinates of the mobile $\mathbf{p}_t = [x_t, y_t, z_t]^T$ in the ECEF coordinate system, but the others depend on the considered motion model.

3.1. GPS measurement equation

The estimation of the state vector $\mathbf{x}_t$ is based on the measurement vector $\mathbf{Z}_t = [Z_t^1, \cdots, Z_t^{n_t}]^T$ storing the pseudo-ranges from n_t satellites at time instant t. In the following, the upper-script k denotes the kth satellite. The kth component of the column vector $\mathbf{Z}_t$ can be expressed as follows:

$$Z_t^k = h_t^k(\mathbf{x}_t) + v_t^k, \;\; h_t^k(\mathbf{x}_t) = \|\mathbf{p}_t - \mathbf{p}_t^k\| + b_t, \tag{6}$$

where $\mathbf{p}_t^k = [x_t^k, y_t^k, z_t^k]^T$ contains the 3 position coordinates of the satellite k and $\|.\|$ is the Euclidian norm. Furthermore, b_t denotes the GPS receiver clock offset with respect to the GPS reference time.
Finally, v_t^k is the additive noise affecting the kth pseudo-range whose PDF is influenced by the multipath and hence assumed unknown. According to (1), we model this PDF denoted F^k as a DPM of mixing distribution $G^k \sim \mathcal{DP}(G_0^k, \alpha^k)$, where G_0^k and α^k are, respectively, the base distribution and the scale parameter. Note that unlike Rabaoui's work, a different DPM is used for each pseudo-range noise. In the following we use the compact notations $\underline{\boldsymbol{\theta}}_t = \left[(\boldsymbol{\theta}_t^1)^T, \cdots, (\boldsymbol{\theta}_t^{n_t})^T\right]$, $\boldsymbol{\alpha} = \left[\alpha^1, \cdots, \alpha^{n_t}\right]^T$ and $\mathbf{c}_t = [c_t^1, \cdots, c_t^{n_t}]^T$.
By taking advantage of the Polya urn representation in (5), the density estimation problem reduces to computing the joint posterior distribution of the latent and auxiliary variables $p(\underline{\boldsymbol{\theta}}_{1:t}, \mathbf{c}_{1:t}|\mathbf{Z}_{1:t})$, where $\mathbf{Z}_{1:t} = \{\mathbf{Z}_1, \cdots, \mathbf{Z}_t\}$ and $\mathbf{c}_{1:t} = \{c_1, \cdots, c_t\}$.
Therefore, our purpose in the following is to recursively estimate from the sets of measurements $\mathbf{Z}_{1:t}$ the extended state vector $\mathbf{X}_t = [\mathbf{x}_t^T, \underline{\boldsymbol{\theta}}_t^T, \mathbf{c}_t^T]^T$. Before expressing the transition distribution of $\mathbf{X}_t$, note that it can be factorized by using Bayes' rule and by taking into account the independence between the random variables as follows:

$$p(\mathbf{X}_t|\mathbf{X}_{1:t-1};\boldsymbol{\alpha}) = p(\mathbf{x}_t|\mathbf{x}_{t-1})p(\underline{\boldsymbol{\theta}}_t|\underline{\boldsymbol{\theta}}_{1:t-1}, \mathbf{c}_t)\Pr[\mathbf{c}_t;\boldsymbol{\alpha},t]. \tag{7}$$

3.2. Prior distributions of the state vector components

A) The state vector $\mathbf{x}_t = \left[\mathbf{p}_t^T, \dot{\mathbf{p}}_t^T, b_t, d_t\right]^T$ is assumed to satisfy a second-order model, with $\dot{\mathbf{p}}_t = [\dot{x}_t, \dot{y}_t, \dot{z}_t]^T$ a vector containing the 3 velocity coordinates of the mobile and d_t the receiver clock drift with respect to the GPS reference time. The evolution of $\mathbf{x}_t$ is described by the transition law:

$$p(\mathbf{x}_t|\mathbf{x}_{t-1}) = \mathcal{N}(\mathbf{x}_t; \mathbf{F}\mathbf{x}_{t-1}, \mathbf{Q}), \tag{8}$$

where $\mathbf{F}$ and $\mathbf{Q}$ are block diagonal matrices. These latter are for instance detailed in [11].

B) The transition PDF for the kth latent variable $\boldsymbol{\theta}_t^k$ is given by (4). A classical choice for the base distribution $G_0^k(\boldsymbol{\theta}_t^k)$ is a Normal Inverse Gamma ($\mathcal{NIG}$) distribution that makes it possible to jointly define a prior model for a mean and a variance as follows:

$$G_0^k(\boldsymbol{\theta}_t^k) = \mathcal{NIG}(\boldsymbol{\theta}_t^k; \mu_0, \nu_0, \alpha_0, \beta_0), \tag{9}$$

with $\mathcal{NIG}(\boldsymbol{\theta}_t^k;\mu_0,\nu_0,\alpha_0,\beta_0) = \mathcal{N}(\mu_t^k;\mu_0,\frac{\phi_t^k}{\nu_0})\mathcal{IG}(\phi_t^k;\alpha_0,\beta_0)$ and $\mathcal{IG}$ the Inverse-Gamma PDF. Taking advantage of the independence between the components of $\underline{\boldsymbol{\theta}}_t$ given $\underline{\boldsymbol{\theta}}_{1:t-1}$ and $\mathbf{c}_t$, one can write the joint transition PDF of $\underline{\boldsymbol{\theta}}_t$ as follows:

$$p(\underline{\boldsymbol{\theta}}_t|\underline{\boldsymbol{\theta}}_{1:t-1},\mathbf{c}_t) = \prod_{k=1}^{n_t} p(\boldsymbol{\theta}_t^k|\boldsymbol{\theta}_{1:t-1}^k, c_t^k). \tag{10}$$

4. RAO-BLACKWELLIZED PARTICLE FILTER

The above-described hierarchical model allows us to apply an efficient estimation algorithm known as RBPF. It is based on the following factorization of the posterior PDF:

$$p(\mathbf{X}_{1:t}|\mathbf{Z}_{1:t}) = p(\mathbf{x}_{1:t}|\underline{\boldsymbol{\theta}}_{1:t},\mathbf{Z}_{1:t})p(\underline{\boldsymbol{\theta}}_{1:t},\mathbf{c}_{1:t}|\mathbf{Z}_{1:t}). \tag{11}$$

In our case, the transition model in (8) is linear/Gaussian and conditionally on $\underline{\boldsymbol{\theta}}_{1:t}$, the measurement model in (6) is weakly non linear/Gaussian. Then, conditionally on $\underline{\boldsymbol{\theta}}_{1:t}$, the EKF can be used as a nearly optimal estimator of the sequence of states $\mathbf{x}_{1:t}$. As for the latent variables $\underline{\boldsymbol{\theta}}_{1:t}$ and the indicator variables $\mathbf{c}_{1:t}$, they can be estimated using a PF as follows:

$$\hat{P}_N(\underline{\boldsymbol{\theta}}_{1:t},\mathbf{c}_{1:t}|\mathbf{Z}_{1:t}) = \sum_{i=1}^{N} w_t^{(i)}\delta_{\underline{\boldsymbol{\theta}}_{1:t}^{(i)},\mathbf{c}_{1:t}^{(i)}}(\underline{\boldsymbol{\theta}}_{1:t},\mathbf{c}_{1:t}), \tag{12}$$

where $\underline{\boldsymbol{\theta}}_{1:t}^{(i)}$ and $\mathbf{c}_{1:t}^{(i)}$ are the so-called particles and $w_t^{(i)}$ are the positive weights the sum of which is equal to 1. Each particle is associated with an EKF that recursively computes the conditional posterior mean estimate of $\mathbf{x}_t$, denoted $\hat{\mathbf{x}}_{t|t}(\underline{\boldsymbol{\theta}}_{1:t}^{(i)})$, along with the estimation error covariance matrix $\mathbf{P}_{t|t}(\underline{\boldsymbol{\theta}}_{1:t}^{(i)})$.
Finally, the estimated marginal PDF of $\mathbf{x}_t$ is computed as a mixture of Gaussian distributions:

$$\hat{P}_N(\mathbf{x}_t|\mathbf{Z}_{1:t}) = \sum_{i=1}^{N} w_t^{(i)}\mathcal{N}\left(\mathbf{x}_t;\hat{\mathbf{x}}_{t|t}(\underline{\boldsymbol{\theta}}_{1:t}^{(i)}),\mathbf{P}_{t|t}(\underline{\boldsymbol{\theta}}_{1:t}^{(i)})\right). \tag{13}$$

This algorithm requires the knowledge of the scale vector $\boldsymbol{\alpha}$. The proposed approach for its estimation is detailed in the next section.

5. ONLINE EM ALGORITHM

5.1. Block EM and exponential family of distributions

The EM algorithm is an iterative optimization procedure that makes it possible to solve ML and MAP estimation problems when the likelihood $p(\mathbf{Z}_{1:T};\boldsymbol{\alpha})$ is not available in a closed form. The EM algorithm involves the complete-data likelihood $p(\mathbf{X}_{1:T},\mathbf{Z}_{1:T};\boldsymbol{\alpha})$ that can be factorized as follows:

$$p(\mathbf{X}_{1:T},\mathbf{Z}_{1:T};\boldsymbol{\alpha}) = \prod_{n=1}^{T}\underbrace{p(\mathbf{Z}_n|\mathbf{X}_n)p(\mathbf{X}_n|\mathbf{X}_{1:n-1};\boldsymbol{\alpha})}_{p(\mathbf{X}_n,\mathbf{Z}_n|\mathbf{X}_{1:n-1};\boldsymbol{\alpha})}. \tag{14}$$

The first step, known as E-step, consists in computing the Q-function as follows:

$$\begin{aligned} &Q(\boldsymbol{\alpha};\boldsymbol{\alpha}_{i-1}) = \\ &\quad E_{p(\mathbf{X}_{1:T}|\mathbf{Z}_{1:T};\boldsymbol{\alpha}_{i-1})}[\log p(\mathbf{X}_{1:T},\mathbf{Z}_{1:T};\boldsymbol{\alpha})] + \log p(\boldsymbol{\alpha}), \end{aligned} \tag{15}$$

with $\boldsymbol{\alpha}_{i-1}$ the estimate of $\boldsymbol{\alpha}$ from the previous iteration and $p(\boldsymbol{\alpha})$ an user-chosen prior distribution on $\boldsymbol{\alpha}$ necessary when a MAP estimate is desired. Note that computing the expectation involved in (15) requires to solve an optimal smoothing problem.
Then, the M-step consists in maximizing (15) over $\boldsymbol{\alpha}$ to yield the parameter estimate $\boldsymbol{\alpha}_i$ at iteration i. In practice, the M-step cannot always be performed explicitly by a simple function evaluation. Usually, this problem is alleviated when the complete-data likelihood belongs to the EFD as it is often the case in practice. In this case, we can write:

$$\begin{aligned} p(\mathbf{X}_T,\mathbf{Z}_T|\mathbf{X}_{1:T-1};\boldsymbol{\alpha}) &= h(\mathbf{X}_{1:T},\mathbf{Z}_T) \\ &\times \exp\left(\langle\mathbf{s}(\mathbf{X}_{1:T},\mathbf{Z}_T),\boldsymbol{\Psi}(\boldsymbol{\alpha})\rangle - A(\boldsymbol{\alpha})\right), \end{aligned} \tag{16}$$

where $\mathbf{s}(\mathbf{X}_{1:T},\mathbf{Z}_T)$ is the vector of sufficient statistics, $\boldsymbol{\Psi}(\boldsymbol{\alpha})$ and $A(\boldsymbol{\alpha})$ are two functions of $\boldsymbol{\alpha}$ and $\langle .,.\rangle$ denotes the inner product operator.
Combining (14), (15) and (16), one can write a normalized version of the E-step as follows:

$$\frac{Q(\boldsymbol{\alpha};\boldsymbol{\alpha}_{i-1})}{T} = \Big\langle\mathbf{S}(\boldsymbol{\alpha}_{i-1}),\boldsymbol{\Psi}(\boldsymbol{\alpha})\Big\rangle - A(\boldsymbol{\alpha}) + \frac{\log p(\boldsymbol{\alpha})}{T}, \tag{17}$$

where we omitted the first term that does not depend on $\boldsymbol{\alpha}$ and with:

$$\mathbf{S}(\boldsymbol{\alpha}_{i-1}) = \frac{1}{T}E_{p(\mathbf{X}_{1:T}|\mathbf{Z}_{1:T};\boldsymbol{\alpha}_{i-1})}\left[\sum_{n=1}^{T}\mathbf{s}(\mathbf{X}_{1:n},\mathbf{Z}_n)\right], \tag{18}$$

the so-called smoothed additive functional.
Often, the M-step can be performed explicitly by evaluating a function Λ as follows:

$$\boldsymbol{\alpha}_i = \Lambda(\mathbf{S}(\boldsymbol{\alpha}_{i-1})). \tag{19}$$

It should be noted that this EFD formulation is a key ingredient in order to build an online version of the EM algorithm.

5.2. Online EM formulation

The online EM consists in processing sequentially the measurements. In [7] the author introduces an auxiliary function that makes it possible to develop a recursive resolution of the smoothing problem involved in (18). At time instant t, it takes the following form:

$$\boldsymbol{\rho}(\mathbf{X}_t;\boldsymbol{\alpha}_{t-1}) = \frac{1}{t}E_{p(\mathbf{X}_{1:t-1}|\mathbf{Z}_{1:t},\mathbf{X}_t;\boldsymbol{\alpha}_{t-1})}\left[\sum_{n=1}^{t}\mathbf{s}(\mathbf{X}_{1:n},\mathbf{Z}_n)\right]. \tag{20}$$

It should be noted that contrary to [7] we have to keep the whole history $\mathbf{X}_{1:n}$ in the sufficient statistics vector due to the DPM Polya urn representation in (5). Equation (20) can be combined with the filtering density at time t denoted $p(\mathbf{X}_t|\mathbf{Z}_{1:t};\boldsymbol{\alpha}_{t-1})$ as follows to compute the functional in (18):

$$\mathbf{S}(\boldsymbol{\alpha}_{t-1}) = \int\boldsymbol{\rho}(\mathbf{X}_t;\boldsymbol{\alpha}_{t-1})p(\mathbf{X}_t|\mathbf{Z}_{1:t};\boldsymbol{\alpha}_{t-1})d\mathbf{X}_t. \tag{21}$$

After some simplifications based on Bayes' rule and defining $K_t = 1/t$ one obtains from (20) the following recursion:

$$\begin{aligned} \boldsymbol{\rho}(\mathbf{X}_t;\boldsymbol{\alpha}_{t-1}) = \int &\left[K_t\mathbf{s}(\mathbf{X}_{1:t},\mathbf{Z}_t) + \left(1-K_t\right)\boldsymbol{\rho}(\mathbf{X}_{t-1};\boldsymbol{\alpha}_{t-1})\right] \\ &\times p(\mathbf{X}_{t-1}|\mathbf{Z}_{1:t-1},\mathbf{X}_t;\boldsymbol{\alpha}_{t-1})d\mathbf{X}_{t-1}. \end{aligned} \tag{22}$$

However, (22) cannot be computed directly because it depends on $\rho(\mathbf{X}_{t-1};\boldsymbol{\alpha}_{t-1})$ whereas we only have access to $\rho(\mathbf{X}_{t-1};\boldsymbol{\alpha}_{t-2})$ computed at the previous iteration. This problem can be alleviated by using a stochastic approximation (SA) as in [7] and [8]. Thus, (22) is replaced by the following recursion:

$$\boldsymbol{\rho}(\mathbf{X}_t;\boldsymbol{\alpha}_{t-1}) = \int \Big[\gamma_t \mathbf{s}(\mathbf{X}_{1:t},\mathbf{Z}_t) + \big(1-\gamma_t\big)\boldsymbol{\rho}(\mathbf{X}_{t-1};\boldsymbol{\alpha}_{t-2})\Big] \times p(\mathbf{X}_{t-1}|\mathbf{Z}_{1:t-1},\mathbf{X}_t;\boldsymbol{\alpha}_{t-1})d\mathbf{X}_{t-1}, \tag{23}$$

where $\gamma_t = t^{-\beta}$ denotes the SA step-size to be chosen by the practitioner, with $\beta \in]0.5,1]$. Furthermore, the integral in (21) is intractable. It can be computed by using the weighted samples of the PF that estimates the posterior distribution of the state vector $\mathbf{X}_t$ in (12):

$$\hat{\mathbf{S}}(\boldsymbol{\alpha}_{t-1}) = \sum_{i=1}^{N} w_t^{(i)} \boldsymbol{\rho}_t^{(i)}, \tag{24}$$

with $\boldsymbol{\rho}_t^{(i)} = \gamma_t \mathbf{s}(\mathbf{X}_{1:t}^{(i)},\mathbf{Z}_t) + (1-\gamma_t)\,\boldsymbol{\rho}_{t-1}^{(i)}$. Finally, the estimate of $\boldsymbol{\alpha}$ at time instant t is obtained as $\boldsymbol{\alpha}_t = \Lambda(\hat{\mathbf{S}}(\boldsymbol{\alpha}_{t-1}))$.

5.3. Estimation of the scale parameter

In our case, the conditional complete-data likelihood can be written as follows:

$$p(\mathbf{X}_t,\mathbf{Z}_t|\mathbf{X}_{1:t-1};\boldsymbol{\alpha}) = \underbrace{p(\mathbf{Z}_t|\mathbf{X}_t)p(\mathbf{x}_t|\mathbf{x}_{t-1})p(\underline{\boldsymbol{\theta}}_t|\underline{\boldsymbol{\theta}}_{1:t-1},\mathbf{c}_t)}_{h(\mathbf{X}_{1:t},\mathbf{Z}_t)}\Pr[\mathbf{c}_t;\boldsymbol{\alpha},t]. \tag{25}$$

Thus, by noting that:

$$\log\Pr[\mathbf{c}_t;\boldsymbol{\alpha},t] = \sum_{k=1}^{n_t}\left[c_t^k \log\left(\frac{t-1}{\alpha^k+t-1}\right) + (1-c_t^k)\log\left(\frac{\alpha^k}{\alpha^k+t-1}\right)\right], \tag{26}$$

the equation (26) can be rewritten in the EFD form:

$$\log\Pr[\mathbf{c}_t;\boldsymbol{\alpha},t] = \Big\langle \mathbf{s}(\mathbf{X}_{1:t},\mathbf{Z}_t), \boldsymbol{\Psi}(\boldsymbol{\alpha})\Big\rangle - A(\boldsymbol{\alpha}), \tag{27}$$

with

$$\begin{aligned}\mathbf{s}(\mathbf{X}_{1:t},\mathbf{Z}_t) &= \left[c_t^1,\cdots,c_t^{n_t},1-c_t^1,\cdots,1-c_t^{n_t}\right]^T,\\ \boldsymbol{\Psi}(\boldsymbol{\alpha}) &= \Big[\log\big(\frac{t-1}{\alpha^1+t-1}\big),\cdots,\log\big(\frac{t-1}{\alpha^{n_t}+t-1}\big),\\ &\log\big(\frac{\alpha^1}{\alpha^1+t-1}\big),\cdots,\log\big(\frac{\alpha^{n_t}}{\alpha^{n_t}+t-1}\big)\Big]^T.\end{aligned} \tag{28}$$

Finally, by inserting (28) in (24), we can compute the smoothed additive functional $\hat{\mathbf{S}}(\boldsymbol{\alpha}_{t-1})$. The latter is used to compute the Q-function to be maximized in (17). To prevent the scale parameter estimates from taking negative values, we select a Gamma prior for $\boldsymbol{\alpha}$ such as in [12]. In this case, it can be shown that these estimates are obtained as the only positive root of a second order polynomial.

6. SIMULATION RESULTS

Our algorithm is tested on simulated GPS data corresponding to a nearly constant velocity trajectory of 100 s in an urban environment. The receiver has 6 satellites in view during the whole trajectory. The pseudo-ranges are generated by a routine of our own where GPS almanac data are used to compute the satellite positions. To simulate the multipath, a white noise is added on 3 of the satellite pseudo-ranges. This noise is distributed according to a 3-component Gaussian mixture:

$$0.4\mathcal{N}(0,10^2+40^2) + 0.3\mathcal{N}(0,10^2+80^2) + 0.3\mathcal{N}(50,10^2).$$

The remainder of the pseudo-ranges are assumed to be in their nominal state where the satellite pseudo-ranges are affected by an additive white Gaussian noise of standard deviation equal to 10 m.

Firstly, to show the relevance of estimating the PDF of each pseudo-range noise we compare our algorithm with an EKF that does not consider the influence of multipath. Secondly, to illustrate the interest of estimating the scale parameter of the DPM associated with each pseudo-range noise, we also run the RBPF described in section 4 with a fixed value of α^k set equal to 10, for each DPM. For our algorithm as for the standard RBPF, 5000 particles were used. Fig. 1 shows the estimated PDF along with the real PDF for one of the pseudo-ranges degraded by multipath. With initial values of α^k equal to 10, the final values estimated by the EM algorithm are $\alpha^1 = 1.04$, $\alpha^2 = 1.01$ and $\alpha^3 = 1.03$. Finally, Fig. 2 shows the evolution of the square root of the horizontal estimation mean-square error (RMSE) associated with each tested algorithms. It was computed from 50 runs corresponding to different realizations of the pseudo-range noises. It should be noted that the proposed approach outperforms the other algorithms. Furthermore, it guarantees good performance even in our simulation protocol that considers 3 out of 6 pseudo-ranges permanently affected by multipath.

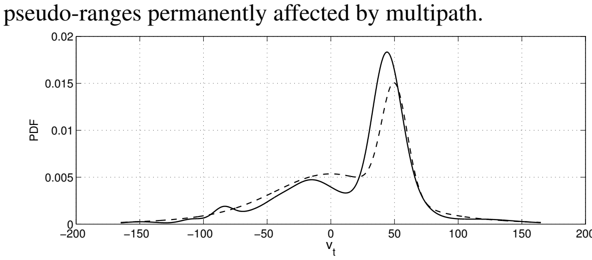

Fig. 1. Estimated PDF (plain curve) and true PDF (dashed curve) of a pseudo-range noise affected by multipath.

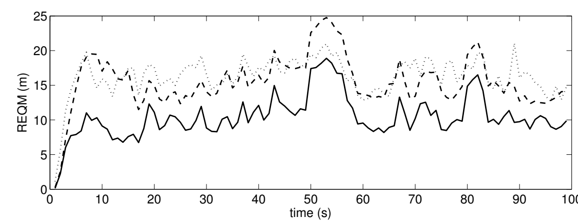

Fig. 2. Evolution of the RMSE for the algorithm with the estimation of $\boldsymbol{\alpha}$ (plain curve), with a fixed value of $\boldsymbol{\alpha} = 10$ (dashed curve) and for the EKF-based algorithm (dotted curve).

7. CONCLUSIONS AND PERSPECTIVE

This paper addresses the problem of multipath in GPS navigation. In the continuity of recent works, we propose to model the GPS pseudo-range noises as DPM. The latter are well-suited to capture the multiple modes of their distributions in a dense urban area. Our contribution is to estimate the DPM scale parameters by means of a sequential online Monte Carlo EM algorithm. To ensure the positivity of the estimates, we regularize the ML estimates by introducing a Gamma prior. Simulation results show the efficiency of our algorithm in a difficult multipath scenario. As a perspective, it would be of interest to consider time-varying DPM due to the variability of the multipath environment. Furthermore, an approximation of the optimal simulation law for the latent variables could be designed to reduce the number of particles needed in our algorithm.

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