A DOPPLER FOCUSING APPROACH FOR SUB-NYQUIST RADAR

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ABSTRACT

We investigate the problem of a monostatic radar transceiver trying to detect a sparse target scene. Several past works employ compressed sensing (CS) algorithms to this type of problem, but either do not address sample rate reduction, impose constraints on the radar transmitter, or propose CS recovery methods with prohibitive dictionary size. Here, using the Xampling framework, we describe a sub-Nyquist sampling approach which overcomes the shortcomings of previous methods. Xampling allows reducing the number of samples needed to accurately represent the signal, directly in the analog-to-digital conversion process. After sampling, the entire digital recovery process is performed on the low rate samples without having to return to the Nyquist rate. With our recovery method we are able to obtain good detection performance even at SNRs as low as -25dB.

Index Terms— compressed sensing, radar, sparse recovery, sub-Nyquist sampling, delay-Doppler estimation.

1. INTRODUCTION

We consider target detection and feature extraction in a single transceiver, monostatic, narrow-band pulse-train radar system, using sub-Nyquist sampling rates. Targets are non-fluctuating point targets, sparsely populated in the radar's unambiguous time-frequency region: delays up to the Pulse Repetition Interval (PRI) and Doppler frequencies up to its reciprocal the Pulse Repetition Frequency (PRF). We propose a recovery method which can detect and estimate targets' delay and Doppler, using a linear, non-adaptive sampling technique at a rate significantly lower than the radar signal's Nyquist frequency, assuming the number of targets L is small.

Current state-of-the-art radar systems sample at the signal's Nyquist rate, which can be hundreds of MHz and higher. The goal of this work is to break the link between radar signal bandwidth and sampling rate. The sub-Nyquist Xampling ("compressed sampling") [1] method we use is an analog-to-digital conversion (ADC) which performs analog prefiltering of the signal before taking pointwise samples. These compressed samples ("Xamples") contain the information needed to recover the desired signal parameters. This work expands [2], adding Doppler to the target model and proposing a new digital recovery method to estimate it.

Past works employ compressed sensing (CS) algorithms to this type of problem, but do not address sample rate reduction and continue sampling at the Nyquist rate [3,4]. Other works combine radar and CS in order to reduce the receiver's sampling rate, but in doing so impose constraints on the radar transmitter and do not treat noise [5], or require an infinite number of samples [6]. Another line of work proposes single stage CS recovery methods with dictionary size proportional to the product of delay and Doppler grid sizes, making them infeasible for many realistic scenarios [4,7].

Our approach is based on the observation that the received signal can be modeled with $3L$ degrees of freedom (DOF). Signals which can be described with a fixed number of DOF per unit of time are known as Finite Rate of Innovation [8] signals. Our proposed method recovers these DOF from low rate samples.

At the crux of our approach is a coherent superposition of time shifted and modulated pulses, the Doppler focusing function $\Phi(t;\nu)$. For any Doppler frequency ν, this function combines the received signals from different pulses so targets with appropriate Doppler frequencies come together in phase. For each ν, $\Phi(t;\nu)$ is processed as a simple one-dimensional CS problem and the appropriate delays are recovered. The gain from this method is both in terms of signal-to-noise ratio (SNR) and Doppler resolution. For P pulses adding coherently, we obtain a factor P SNR improvement over white noise (which adds incoherently, i.e. in power). In addition, denoting the PRI as τ, the width of the Doppler focus for each $\Phi(t;\nu)$ is $2\pi/P\tau$, meaning that delays of targets separated in Doppler by more than $2\pi/P\tau$ will not interfere with each other.

Simulations provided in Section 6 show that when sampling at one tenth the Nyquist rate, our Doppler focusing recovery method outperforms both two-stage CS recovery and classic radar processing [9]. When the SNR reaches -25dB, our approach achieves the performance of classic processing operating at the full Nyquist rate.

The main merits of our proposed method are as follows:

1. **Low rate ADC and DSP** – we acquire the sub-Nyquist samples containing information needed for target recovery, and then digitally recover the unknown target parameters using low rate processing, without returning to the higher Nyquist rate.
2. **Scaling with problem size** – many CS delay-Doppler estimation methods construct a dictionary with a column for each delay-Doppler hypothesis, creating prohibitive memory requirements. Our method separates the Doppler from delay recovery, making each CS delay recovery indifferent to the underlying Doppler and requiring for less memory.
3. **Transmitter compatibility** – our recovery method does not impose any restrictions on the transmitted signal, provided it meets the assumptions stated in Section 2.

2. RADAR MODEL

We consider a radar transceiver that transmits a pulse train

$$x_T(t) = \sum_{p=0}^{P-1} h(t - p\tau), \quad 0 \leq t \leq P\tau \tag{1}$$

consisting of P equally spaced pulses $h(t)$. The pulse-to-pulse delay τ is referred to as the PRI, and its reciprocal $1/\tau$ is the PRF. The pulse $h(t)$ is a known time-limited baseband function with continuous-time Fourier transform (CTFT) $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$. We assume that $H(\omega)$ has negligible energy at frequencies beyond $B_h/2$ and we refer to B_h as the bandwidth of $h(t)$. The target scene is composed of L non-fluctuating point targets (Swerling 0 model, see [10]), where we assume that L is known, although this assumption can easily be relaxed. The pulses reflect off the L targets and propagate back to the transceiver. Each target l is defined by three parameters: a delay τ_l, a Doppler frequency ν_l and a complex amplitude α_l, proportional to the target's radar cross section (RCS) and all propagation factors.

Throughout, we make the following assumptions on the targets' location and motion, which allow us to obtain a simplified expression for the received signal.

A1. "Far targets" - target-radar distance is large compared to the distance change during observation interval which allows for constant α_l.

A2. "Slow targets" - small target velocity allows for constant τ_l during observation interval and constant Doppler phase during pulse time T_p.

A3. "Small acceleration" - target velocity remains approximately constant during observation interval allowing for constant ν_l.

These assumptions all rely on slow "enough" relative motion between the radar and its targets. Radar systems tracking people, ground vehicles and sea vessels usually comply quite easily. As for airborne targets, care must be taken to ensure compliance.

Using these assumptions, we can write the received signal as

$$x(t) = \sum_{p=0}^{P-1} \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l - p\tau) e^{-j\nu_l p\tau}. \tag{2}$$

It will be convenient to express the signal as a sum of single frames

$$x(t) = \sum_{p=0}^{P-1} x_p(t) \tag{3}$$

where

$$x_p(t) = \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l - p\tau) e^{-j\nu_l p\tau}. \tag{4}$$

In reality $x(t)$ will be contaminated by additive noise. We will take this into account in our simulations.

Our goal in this work is to accurately detect the L targets, i.e. to estimate the $3L$ DOF $\{\alpha_l, \tau_l, \nu_l\}_{l=0}^{L-1}$ in (2), using the least possible number of digital samples.

3. DOPPLER FOCUSING

We now introduce and explain the main idea in this paper, called Doppler Focusing. This processing technique uses target echoes from different pulses to create a single superimposed pulse, improving SNR for robustness against noise and implicitly estimating targets' Doppler in the process. Using (4), we define the following time shift and modulation operation on the received signal:

$$\begin{aligned}\Phi(t;\nu) &= \sum_{p=0}^{P-1} x_p(t + p\tau) e^{j\nu p\tau} \\ &= \sum_{p=0}^{P-1} \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l) e^{j(\nu - \nu_l)p\tau} \\ &= \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l) \sum_{p=0}^{P-1} e^{j(\nu - \nu_l)p\tau}. \end{aligned} \tag{5}$$

We now analyze the sum of exponents in (5). For any given ν, targets with Doppler frequency ν_l in a band of width $2\pi/P\tau$ around ν, i.e. in $\Phi(t;\nu)'s$ "focus zone", will achieve coherent integration and an SNR boost of approximately

$$g(\nu|\nu_l) = \sum_{p=0}^{P-1} e^{j(\nu - \nu_l)p\tau} \overset{|\nu - \nu_l| < 2\pi/P\tau}{\cong} P \tag{6}$$

compared with a single pulse. On the other hand, since the sum of P equally spaced points covering the unit circle is generally close to zero, targets with ν_l not "in focus" will approximately cancel out. Thus $g(\nu|\nu_l) \cong 0$ for $|\nu - \nu_l| > 2\pi/P\tau$. See Fig. 1 for an example of $g(\nu|\nu_l)$. Hence we can approximate (5) by

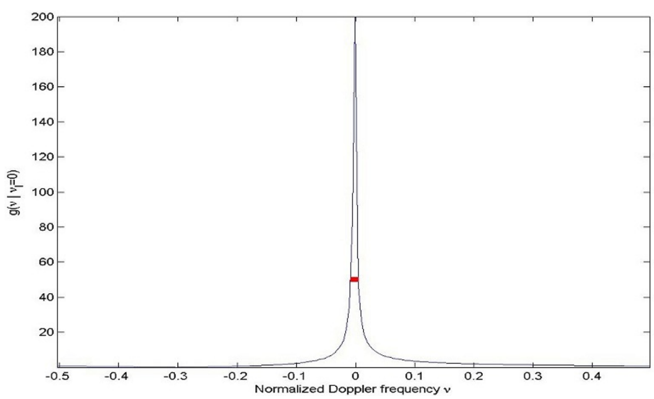

Fig. 1: Example of $g(\nu|\nu_l)$ for $P = 200$ pulses and $\nu_l = 0$. Red line marks "focus zone", i.e. $|\nu| < 2\pi/P\tau$. Frequencies outside focus zone are severely attenuated.

$$\Phi(t;\nu) \cong P \sum_{l:|\nu - \nu_l| < 2\pi/P\tau} \alpha_l h(t - \tau_l). \tag{7}$$

Instead of trying to estimate delay and Doppler together, we have reduced our problem to delay only estimation for a small range of Doppler frequencies, with increased amplitude for improved performance against noise.

We now show how Doppler focusing can also be performed in the frequency domain, paving the way towards sub-Nyquist Doppler focusing. Using (4), and denoting $X_p(\omega)$ as the CTFT of $x_p(t+p\tau)$:

$$X_p(\omega) = H(\omega) \sum_{l=0}^{L-1} \alpha_l e^{-j\omega\tau_l} e^{-j\nu_l p\tau}. \tag{8}$$

Taking the CTFT of $\Phi(t;\nu)$ as a function of t:

$$\begin{aligned}\Psi(\omega;\nu) = \text{CTFT}(\Phi(t;\nu)) &= \sum_{p=0}^{P-1} X_p(\omega) e^{j\nu p\tau} \\ &= H(\omega) \sum_{l=0}^{L-1} \alpha_l e^{-j\omega\tau_l} \sum_{p=0}^{P-1} e^{j(\nu - \nu_l)p\tau}. \end{aligned} \tag{9}$$

All $3L$ of the problem's parameters appear in (9). Its structure is that of a delay estimation problem as we will see in Section 4, combined with the familiar sum of exponents term from (5).

We have seen that Doppler focusing reduces a delay-Doppler estimation problem to a delay-only problem for a specific Doppler frequency. In the next section we describe delay recovery from sub-Nyquist sampling rates using Xampling.

4. SUB-NYQUIST DELAY RECOVERY

The problem of recovering the $2L$ amplitudes and delays in

$$\phi(t) = \sum_{l=0}^{L-1} \alpha_l h(t - \tau_l), \quad 0 \le t < \tau \tag{10}$$

from sub-Nyquist samples has been previously studied in [2, 8, 11, 12]. Since Doppler focusing yields such a problem, we now review how to solve (10) at a sub-Nyquist sampling rate.

4.1. Xampling

Xampling [1, 13, 14] can be interpreted as "compressed sampling", in the sense that we are performing data compression inherently in the sampling stage. To do this, we do not simply reduce sampling rate, since this is bound to cause loss of information. Instead, we perform an analog prefiltering operation on our signal and only then sample it, in order to extract the required information for recovery. We now show how the signal's Fourier series coefficients are related to the problem's unknown parameters [6, 8, 11, 12], and how to acquire these Fourier coefficients via Xampling.

Since $\phi(t)$ is confined to the interval $t \in [0, \tau]$, it can be expressed by its Fourier series

$$\phi(t) = \sum_{k \in \mathbb{Z}} c[k] e^{j2\pi kt/\tau}, \quad t \in [0, \tau], \tag{11}$$

where

$$\begin{aligned} c[k] &= \frac{1}{\tau} \int_0^\tau \phi(t) e^{-j2\pi kt/\tau} dt \\ &= \frac{1}{\tau} \sum_{l=0}^{L-1} \alpha_l \int_0^\tau h(t - \tau_l) e^{-j2\pi kt/\tau} dt \\ &= \frac{1}{\tau} H(2\pi k/\tau) \sum_{l=0}^{L-1} \alpha_l e^{-j2\pi k\tau_l/\tau}. \end{aligned} \tag{12}$$

From (12) we see that the unknown parameters $\{\alpha_l, \tau_l\}_{l=0}^{L-1}$ are embodied in the Fourier coefficients $c[k]$ in the form of a complex sinusoid problem. For these problems, if there is no noise, $2L$ samples are enough to recover the unknown α's and τ's [8], using spectral analysis methods such as the annihilating filter [15] or matrix pencil [16]. The lower bound can be achieved only when the noise is negligible. When there is substantial noise in the problem, having more than $2L$ coefficients will allow the recovery to be more robust.

Our signals exist in the time domain, and therefore we do not have direct access to $c[k]$. We can use the Direct Multichannel Sampling scheme described in [11] in order to obtain the Fourier series coefficients.

4.2. Compressed Sensing Recovery

We now describe a CS-based recovery method, operating on the Xamples $c[k]$, which is more robust to noise.

Assume the delays are aligned to a grid $\tau_l = q_l \Delta_\tau$ where $0 \le q_l < N_\tau$ and we choose Δ_τ so that $N_\tau = \tau/\Delta_\tau$ is an integer. Choose a set of indices $\kappa = \{k_0, ..., k_{|\kappa|-1}\}$, and define the corresponding vector of Fourier coefficients

$$\mathbf{c} = [c[k_0] \; ... \; c[k_{|\kappa|-1}]]^T \in \mathbb{C}^{|\kappa|}. \tag{13}$$

We can then write (12) in vector form as $\mathbf{c} = \frac{1}{\tau}\mathbf{HVx}$ where $\mathbf{H}$ is a $|\kappa| \times |\kappa|$ diagonal matrix with diagonal elements $H(2\pi k_i/\tau)$ and $\mathbf{V}$ is a $|\kappa| \times N_\tau$ matrix with elements $\mathbf{V}_{mq} = e^{-j2\pi k_m q/N_\tau}$, i.e. it is composed of $|\kappa|$ rows of the $N_\tau \times N_\tau$ DFT matrix. The vector $\mathbf{x} \in \mathbb{C}^{N_\tau}$ is L-sparse, where each index q contains the amplitude of a target with delay $q\Delta_\tau$ if it exists, or zero otherwise. Defining the CS dictionary $\mathbf{A} = \frac{1}{\tau}\mathbf{HV} \in \mathbb{C}^{|\kappa| \times N_\tau}$ we obtain the CS equation

$$\mathbf{c} = \mathbf{Ax}. \tag{14}$$

Estimating delays can be carried out by solving (14) and finding $\mathbf{x}$'s support - any nonzero index q denotes a target with delay $q\Delta_\tau$.

For any set of sampled Fourier coefficients, a variety of CS techniques can be employed for recovery, see [17–19] and references within. Also, choosing the coefficients at random produces favorable conditions for CS, aiding recovery in the presence of noise. If the indices in κ are selected uniformly at random, it can be shown that if $|\kappa| \ge CL(\log N_\tau)^4$, for some positive constant C, then we are able to recover $\mathbf{x}$, using a CS recovery algorithm with high probability.

5. DELAY-DOPPLER RECOVERY USING DOPPLER FOCUSING

We now show how Xampling can be performed on the multi-pulse signal (2). We then describe the Doppler focusing method, and compare it to previous CS approaches.

5.1. Applying Doppler Focusing and CS Recovery

Similarly to the Xampling technique of Section 4 which obtained $c[k]$, we can extend this technique to each of the pulses $x_p(t)$ of the multi-pulse signal (2) to obtain $c_p[k]$. Since $x_p(t)$ is confined to the interval $t \in [p\tau, (p+1)\tau]$, we can replace $t \to t + p\tau$ and $\alpha_l \to \alpha_l e^{-j\nu_l p\tau}$ in (12) to obtain

$$c_p[k] = \frac{1}{\tau} H(2\pi k/\tau) \sum_{l=0}^{L-1} \alpha_l e^{-j\nu_l p\tau} e^{-j2\pi k\tau_l/\tau}, \tag{15}$$

where we used the fact that since both $k, p \in \mathbb{Z}$ we have $e^{-j2\pi kp} \equiv 1$. From (15) we see that all $3L$ unknown parameters $\{\alpha_l, \tau_l, \nu_l\}_{l=0}^{L-1}$ are embodied in the Fourier coefficients $c_p[k]$ in the form of a complex sinusoid problem.

Having acquired $c_p[k]$ using Xampling, we now perform the Doppler focusing operation for a specific frequency ν

$$\begin{aligned} \Psi_\nu[k] &= \sum_{p=0}^{P-1} c_p[k] e^{j\nu p\tau} \\ &= \frac{1}{\tau} H(2\pi k/\tau) \sum_{l=0}^{L-1} \alpha_l e^{-j2\pi k\tau_l/\tau} \sum_{p=0}^{P-1} e^{j(\nu-\nu_l)p\tau}. \end{aligned} \tag{16}$$

From (9) we see that $\Psi_\nu[k] = \Psi(\omega; \nu)|_{\omega=2\pi k/\tau}$.

Following the same arguments as in (6), for any target l satisfying $|\nu - \nu_l| < 2\pi/P\tau$ we have

$$\sum_{p=0}^{P-1} e^{j(\nu-\nu_l)p\tau} \cong P. \tag{17}$$

Therefore, Doppler focusing can be performed on the low rate sub-Nyquist samples:

$$\Psi_\nu[k] \cong \frac{P}{\tau} H(2\pi k/\tau) \sum_{l: |\nu-\nu_l| < 2\pi/P\tau} \alpha_l e^{-j2\pi k\tau_l/\tau}. \tag{18}$$

Equation (18) is identical in form to (12) except it is scaled by P, increasing SNR for improved performance with noise. Furthermore, we reduced the number of active delays. For each ν we now have a delay estimation problem, which can be written in vector form using the same notations of Section 4 as

$$\mathbf{\Psi}_\nu = \frac{P}{\tau}\mathbf{H}\mathbf{V}\mathbf{x}_\nu \tag{19}$$

where

$$\mathbf{\Psi}_\nu = [\Psi_\nu[k_0] \; ... \; \Psi_\nu[k_{|\kappa|-1}]]^T \in \mathbb{C}^{|\kappa|}. \tag{20}$$

This is exactly the CS problem we have already shown how to solve, with one important difference. In the delay-only problem of Section 4, the model order L was known. Here, since there are L targets but we have no prior knowledge of their distribution in the delay-Doppler grid, for each ν we must either estimate the model order $0 \leq L_\nu \leq L$, or take a worst case approach and assume $L_\nu = L$. The problem of estimating the number of sinusoids in a noisy sequence has been studied extensively [20–22]. Solving (19) with an accurate model order can decrease computation time (although estimating model order is also time consuming) and possibly reduce detection of spurious targets. In our simulations, to avoid model order errors which influence recovery performance, we employ the worst case approach.

The Doppler focusing technique is a continuous operation on ν, and can be performed for any Doppler frequency. Since the focus zone for each ν is of width $2\pi/P\tau$, we can find various finite sets of ν's spanning $[0, 2\pi/\tau]$. For any such set, define its size as N_ν. For each ν in the set, we solve (19) assuming $\mathbf{x}_\nu$'s support is of size L. This problem can be solved using an abundance of CS algorithms as described in Section 4. After solving N_ν separate CS problems with dictionary of size $|\kappa| \times N_\tau$, we hold at most LN_ν estimated amplitudes. Since the absolute value of amplitudes recovered in the support is indicative of true target existence as opposed to noise, we take the L strongest ones as true target locations.

5.2. Previous CS Approaches

A possible approach to the problem at hand could be a two-stage CS recovery technique, first estimating delays and afterwards for each delay estimating Doppler (or vice versa). These two-stage methods tend to be suboptimal since the problem is not separable, and a mistake in the first estimation stage propagates to the second stage where it cannot be undone. We compare our method to this type of recovery in Section 6.

To overcome this inefficiency, several works [4, 7] employ a single stage CS recovery technique. Instead of estimating delays and Doppler frequencies separately and sequentially, they estimate the most likely (τ_l, ν_l) pairs. The drawback of this technique is that it requires using a dictionary with dimensions proportional to $N_\tau N_\nu F_s$, where F_s is the problem's sample rate. Since grid sizes can easily reach 10^3, and sample rates are on the order of MHz, the dictionary grows rapidly rendering these methods infeasible for even moderate problem size.

6. SIMULATION RESULTS

We now present some numerical experiments illustrating recovery performance. We corrupt the received signal $x(t)$ with additive white Gaussian noise $n(t)$ with power spectral density $S_n(f) = N_0/2$, bandlimited to $x(t)$'s bandwidth B_h. We define the signal to noise power ratio for target l as $SNR_l = \int_0^{T_p} |\alpha_l h(t)|^2 dt / T_p N_0 B_h$ where T_p is the pulse time. The scenario parameters used were number of targets L=5, number of pulses P=100, PRI τ=10μsec, and B_h=200MHz. The classic time and frequency resolutions ("Nyquist bins"), defined as $1/B_h$ and $1/P\tau$, are 5$n$sec and 1KHz accordingly. In order to demonstrate a 1:10 sampling rate reduction, our sub-Nyquist Xampling scheme generated 200 Fourier coefficients per pulse, as opposed to the 2000 Nyquist rate samples. We tested Doppler focusing with two types of Fourier coefficient sets κ, a consecutive set and a random set. We compared Doppler focusing recovery with classic processing and a two-stage recovery method as described in [6] (where we use a CS algorithm instead of ESPRIT) using a **Hit-Rate** criterion: we define a "hit" as a delay-Doppler estimate which is circumscribed by an ellipse around the true target position in the time-frequency plane. We used an ellipse with axes equivalent to ± 3 times the Nyquist bins.

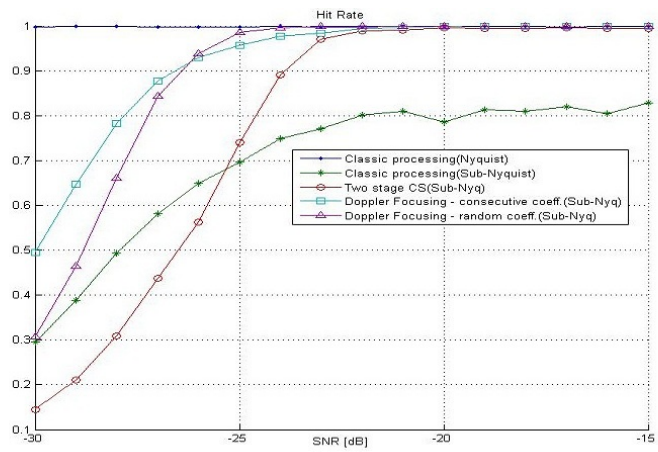

Fig. 2: Hit Rate for classic processing, two-stage CS recovery and Doppler focusing as function of SNR. Sub-Nyquist sampling rate was one tenth the Nyquist rate.

Fig. 2 demonstrates the hit-rate performance of the different recovery methods for various SNR values. It is evident that Doppler focusing is superior to the other sub-Nyquist recovery techniques. Between the two Doppler focusing approaches, consecutive coefficients are better suited for lower SNR, while choosing coefficients randomly improves performance as SNR increases. Also, random coefficients, when producing a hit, have very small delay errors even compared with Nyquist rate classic processing.

7. CONCLUSION

We demonstrated a radar sampling and recovery method called Doppler focusing, which employs the techniques of Xampling and CS, and is independent of the radar signal's bandwidth. Doppler focusing allows for low rate sampling and digital processing. It also leads to CS recovery with dictionary size scaling with delay grid size only, and imposes no constraints on transmitted signal. We compared our method to other sub-Nyquist recovery techniques and have seen its clear advantage in simulations. When sampling at one tenth the Nyquist rate, and for SNR above -25dB, Doppler focusing achieves results almost identical to classic recovery working at the Nyquist rate.

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