FUSION OF QUANTIZED DATA FOR BAYESIAN ESTIMATION AIDED BY CONTROLLED NOISE

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ABSTRACT

In this paper, we consider a Bayesian estimation problem in a sensor network where the local sensor observations are quantized before their transmission to the fusion center (FC). Inspired by Widrow's statistical theory on quantization, at the FC, instead of fusing the quantized data directly, we propose to fuse the post-processed data obtained by adding independent controlled noise to the received quantized data. The injected noise acts like a low-pass filter in the characteristic function (CF) domain such that the output is an approximation of the original raw observation. The optimal minimum mean squared error (MMSE) estimator and the posterior Cramér-Rao lower bound for this estimation problem are derived. Based on the Fisher information, the optimal controlled Gaussian noise and the optimal bit allocation are obtained. In addition, a near-optimal linear MMSE estimator is derived to reduce the computational complexity significantly.

***Index Terms*—** Bayesian estimation, data fusion, quantization, bit allocation, Fisher information, sensor networks

1. INTRODUCTION

For a sensor network (SN) with limited resources (bandwidth and/or energy), it is important to limit the communication within the network. Therefore, transmission of binary or multi-bit quantized data is a desirable solution. For a centralized sensor network architecture with quantized data, each sensor node sends its quantized data to a FC, where all the quantized sensor data are fused to perform parameter estimation (e.g. target localization). Our previous work has focused on target localization with quantized sensor data using static quantizers when no prior on the target's location is available. These include [1–3]. In [1, 2], target localization methods based on quantized sensor data have been developed assuming perfect communication channels between the sensors and the FC, while in [3], wireless channel statistics are taken into consideration. In this paper, instead of estimation of deterministic target location, we are interested in estimating a random variable (RV) with known prior based on quantized data collected at the FC, keeping the assumption of perfect communication channels.

The novelty of this paper is that the quantized data are not fused directly by the FC for parameter estimation, but preprocessed by injecting independent controlled noise. The basic idea was inspired by Widrow's statistical theory of quantization [4]. The addition of noise after quantization is equivalent to low pass filtering in the CF domain, such that the original analog observation can be recovered. Therefore, the whole process of quantizing and injecting controlled noise can be theoretically modeled as an additive disturbance, whose distribution is analytically derived in this paper. This theoretical model facilitates the derivation of the optimal minimum mean squared error (MMSE) estimator, the posterior Cramér-Rao lower bound (PCRLB), the near-optimal linear MMSE (LMMSE) estimator, and its corresponding mean squared error (MSE), all of which are in *exact* forms. Furthermore, the numerical results in this paper show that the LMMSE estimator can provide comparable performance to that of the optimal MMSE estimator while saves a lot of computation effort. A similar idea (injecting controlled noise to quantized data) has been applied to solve a distributed detection problem in our previous work [5], and promising preliminary results were obtained therein. A related but different work is documented in [6] where the problem of estimating a deterministic parameter in noise using quantized observations has been discussed. However, in [6], the authors proposed to add the dither noise before quantization at local sensors which amounts to anti-alias filtering [6], while we propose to add it post quantization, which is performed by the FC.

Since quantizers are involved in the problem, the issue of bit allocation naturally arises, which has been formulated as an optimization problem in several publications [7, 8]. In this paper, we will also address the bit allocation problem. There are two major differences between our work and that in [8]: 1) In this paper, the probability density function (PDF) of the equivalent additive disturbance, which models the whole quantization and noise injection process, is derived. Based on this, as discussed earlier, *exact* solutions for estimators and their performance measures are derived. In [8], a quasi-MMSE estimator that fuses quantized data directly was proposed in an ad-hoc manner, by simply replacing the analog data with the quantized ones in the MMSE formula designed for unquantized analog data. This may incur large estimation error and severe suboptimality in many cases, as clearly shown in the numerical examples in [8]. 2) In [8], the bit allocation problem was solved by minimizing either an upper bound on the MSE of the quasi-MMSE, or an approximated difference between the Fisher information of the analog data and that of the quantized data, the latter of which is based on the strong assumption that the quantization interval is approaching zero. In contrast, in this paper, the bit allocation problem is solved based on the *exact* Fisher information.

2. A REVIEW OF WIDROW'S STATISTICAL THEORY OF QUANTIZATION

In [4], the uniform quantization of a RV is interpreted as sampling of its PDF, and it was shown that the PDF of the quantized RV is the

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convolution of the input RV's PDF with the PDF of a uniform distribution followed by conventional sampling. Thus, at the ith sensor, the PDF of the quantizer output, u_i, is

$$p_{U_i}(u) = [p_{W_i}(u) * p_{Z_i}(u)] \cdot \sum_{k \in \mathcal{Z}} q_i \delta \left(u - kq_i - \frac{q_i}{2} \right), \quad (1)$$

where $p_{Z_i}(z)$ is the PDF of the input RV z_i, $p_{W_i}(w)$ denotes the PDF of a uniform distribution over $[-q_i/2,\ q_i/2]$, and q_i is the quantization step-size of the uniform quantizer. Thus, uniform quantization introduces two types of distortions or errors: (a) the additive noise w_i, and (b) the aliasing error due to sampling. However, if the input PDF is bandlimited so that its CF $\phi_{Z_i}(v) = 0$ for $|v| > \frac{\pi}{q_i}$, the aliasing error can be avoided and, in principle, the original PDF can be reconstructed from the knowledge of p_{U_i}. This is Widrow's first quantization theorem:

Theorem 1. *If the CF of the input variable Z_i is bandlimited, i.e.,*

$$\phi_{Z_i}(v) = 0,\ |v| > \frac{\pi}{q_i} \quad (2)$$

then the replicas of $\phi_{U_i}(v)$ do not overlap, and in principle, the original PDF p_{Z_i} can be recovered from p_{U_i}.

3. PROBLEM FORMULATION

Let us consider the estimation of a RV θ in a wireless sensor network, where $\theta \sim \mathcal{N}(0, \sigma_\theta^2)$. It is also assumed that N sensors are observing the parameter θ, and each local sensor's observation is corrupted by independent additive Gaussian noise, *i.e.*, observation model for sensor i is

$$z_i = \theta + n_i,\ i = 1, 2, \cdots, N \quad (3)$$

where $n_i \sim \mathcal{N}(0, \sigma_{n_i}^2)$.

Each sensor performs uniform quantization before transmission and the step-size of quantizer i is set as q_i. Denote the quantized data as u_i. In [5], for hypothesis testing problems, the fusion process is simplified by adding *controlled noise* to the observations received at the FC. For the Bayesian estimation problem considered in this paper, we propose a similar fusion system as shown in Fig. 1. An externally generated noise (d_i) with a band-limited CF, is added to the quantized observations from the ith sensor to filter out the repeated and phase-shifted CF side lobes in the CF of u_i. This is analogous to low pass filtering in signal processing. We, therefore, call the noise d_i, the LPF-noise.

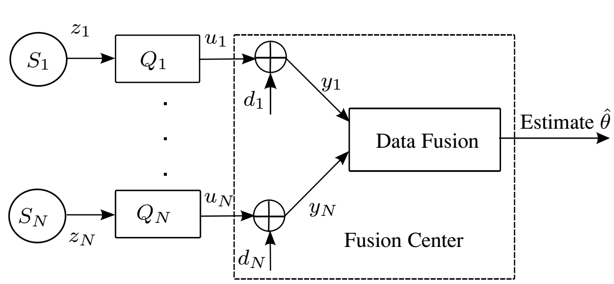

Fig. 1. Bayesian estimation aided by controlled noise. S: sensor; Q: quantizer; z: sensor data; u: quantized data; d: controlled noise; y: data received at fusion center.

Then, the received data at the FC is given as

$$y_i = u_i + d_i \quad (4)$$

Note that an ideal noise source would be one with a rectangular CF in the pass-band, $-\frac{\pi}{q_i} \leq v \leq \frac{\pi}{q_i}$. However, a rectangular function in the CF domain corresponds to a PDF whose shape corresponds to a sinc function, which is obviously an invalid PDF. Therefore, we limit our consideration to only Gaussian noise in this paper. That is, $d_i \sim \mathcal{N}(0, \sigma_{d_i}^2)$, and the variance $\sigma_{d_i}^2$ controls the bandwidth of the filter.

Note that once (2) is satisfied, we have

$$y_i = u_i + d_i = z_i + w_i + d_i \quad (5)$$

where $w_i \sim U(-\frac{q_i}{2}, \frac{q_i}{2})$. One needs to carefully design the PDF of d_i so that it causes minimal distortion while transforming the discrete-valued RV, u_i, into a continuous variable, y_i.

4. CONTROLLED NOISE AIDED MMSE ESTIMATION

In this section, the design of the controlled noise and allocation of bits across the network will be solved jointly, such that the estimation performance of the system is optimized. Since the PCRLB is the lower bound on the MSE, it is used as the metric in the paper for optimization.

4.1. Bayesian Estimators and Fisher Information

Let us denote the received data vector at the FC as $\mathbf{y} = [y_1, \cdots, y_N]^T$. Since z_i and d_i are Gaussian RV respectively, we have $p_{(z_i+d_i)|\theta} = \mathcal{N}(\theta, \sigma_{n_i}^2 + \sigma_{d_i}^2)$. Then, using (5), we can express the likelihood as

$$\begin{aligned} p(y_i|\theta) &= p_{(z_i+d_i)|\theta} * p_{w_i} \quad (6) \\ &= \frac{1}{q_i} \left[\Phi \left(\frac{y_i - \theta + q_i/2}{\sqrt{\sigma_{n_i}^2 + \sigma_{d_i}^2}} \right) - \Phi \left(\frac{y_i - \theta - q_i/2}{\sqrt{\sigma_{n_i}^2 + \sigma_{d_i}^2}} \right) \right] \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a Gaussian RV with zero mean and unit variance. Since sensors' observations are conditionally independent, we have

$$p(\mathbf{y}|\theta) = \prod_{i=1}^{N} p(y_i|\theta) \quad (7)$$

4.1.1. Optimal MMSE Estimator

The optimal MMSE estimator, *i.e.*, the posterior conditional mean is given as follows

$$\hat{\theta}_{\text{MMSE}} = \int \theta \frac{p(\mathbf{y}|\theta)p(\theta)}{\int p(\mathbf{y}|\theta)p(\theta)d\theta} d\theta \quad (8)$$

For any Bayesian estimator, its MSE is bounded below by the PCRLB, which is the inverse of the Bayesian Fisher information. The Bayesian Fisher information is derived and provided in the following theorem.

Theorem 2. *For a sensor network with N sensors, and the observation model given by (3), if z_i and q_i, for $i = 1, 2, \cdots, N$, satisfy the condition specified in (2), and the controlled noise d_i is Gaussian, then the Fisher information is given as*

$$\mathcal{J} = \sum_{i=1}^{N} \tilde{J}_i + \sigma_\theta^{-2} \quad (9)$$

where

$$\tilde{J}_i = \int_{\theta} p(\theta) \int_{y_i} \frac{1}{q_i 2\pi(\sigma_{n_i}^2 + \sigma_{d_i}^2)} \cdot \frac{\left[e^{-\frac{1}{2}\xi_{i,1}^2} - e^{-\frac{1}{2}\xi_{i,2}^2}\right]^2}{[\Phi(\xi_{i,1}) - \Phi(\xi_{i,2})]} dy_i d\theta \tag{10}$$

and $\xi_{i,1} = \frac{y_i - \theta + q_i/2}{\sqrt{(\sigma_{n_i}^2 + \sigma_{d_i}^2)}}$, $\xi_{i,2} = \frac{y_i - \theta - q_i/2}{\sqrt{(\sigma_{n_i}^2 + \sigma_{d_i}^2)}}$.

Proof. Once we obtain the likelihood (6) and the prior $p(\theta)$, the Fisher information can be derived using standard procedures. The detailed proof is omitted for brevity. ■

It is clear that both the implementation of the MMSE estimator and the evaluation of Fisher information involve integrals.

4.1.2. Sub-Optimal LMMSE Estimator

Though (8) is optimal, it requires the evaluation of two integrals. We would like to seek a computationally more efficient estimator. Combining (3) and (5), we have

$$y_i = \theta + g_i \tag{11}$$

where $g_i \triangleq n_i + w_i + d_i$. It is easy to show that

$$E\{g_i\} = E\{n_i + w_i + d_i\} = 0 \tag{12}$$

and the variance is given as

$$\sigma_{g_i}^2 = \sigma_{n_i}^2 + \sigma_{d_i}^2 + q_i^2/12 \tag{13}$$

since noise n_i, w_i and q_i are independent of each other. g_i does not follow a Gaussian distribution. However, due to the linear relationship between y_i and θ as in (11), it is natural to use the LMMSE estimator for θ [9], which is derived and provided as follows.

$$\hat{\theta}_{\text{LMMSE}} = \mu_\theta + \left(\frac{1}{\sigma_\theta^2} + \sum_{i=1}^{N} \frac{1}{\sigma_{g_i}^2}\right)^{-1} \sum_{i=1}^{N} \frac{y_i - \mu_\theta}{\sigma_{g_i}^2} \tag{14}$$

where $\mu_\theta = 0$ is the mean of the prior PDF of θ. The corresponding estimation MSE is

$$E\{(\theta - \hat{\theta}_{\text{LMMSE}})^2\} = \left(\frac{1}{\sigma_\theta^2} + \sum_{i=1}^{N} \frac{1}{\sigma_{g_i}^2}\right)^{-1} \tag{15}$$

As can be seen, the LMMSE and its MSE have closed-form solutions and are computationally efficient.

Proposition 1. *Given q_i such that (2) holds, the MSE of the LMMSE estimator is a monotonic increasing function of σ_{d_i}.*

Remark 1: Proposition 1 can be easily proved using (15), since a larger σ_{d_i} means a smaller $1/\sigma_{g_i}$, and hence a larger MSE. Due to Proposition 1, we conjecture that the Fisher information J_i should be a monotonic decreasing function of σ_{d_i}, which will be shown numerically in Section 5. This is intuitively true, because smaller σ_{d_i} means larger signal-to-noise ratio (SNR). Thus, once q_i is fixed, σ_{d_i} can be determined, $i.e.$, the smallest acceptable one. One should note that σ_{d_i} cannot be infinitely small. This can be interpreted from the CF domain of u_i. Since the bandwidth of d_i in CF domain is inversely proportional to σ_{d_i}, the largest bandwidth acceptable is the one that completely covers the central lobe while does not cover the second side lobe to make sure that no aliasing error is introduced.

4.2. LPF-noise design and bit allocation

We are interested in simultaneously designing the local quantizer parameter q_i and controlled noise d_i such that the performance is optimized. In fact, quantizer design is equivalent to the bit allocation problem, since a uniform quantizer is used. Note that sensors considered in this problem are not identical, in the sense that the variances of the observation noises are different from each other, $i.e.$, $\sigma_{n_i} \neq \sigma_{n_j}$, for $i \neq j$. If $\mathcal{J}$ is used as the optimization metric, then the design problem can be formulated as

Optimization Formulation:

$$\max_{\vec{B}, \vec{\sigma}_d} \mathcal{J}(\vec{B}, \vec{\sigma}_d) \tag{16}$$

$$s.t. \sum_{i=1}^{N} b_i = R, \quad \text{and}$$
$$\phi_{Z_i}(v) = 0, \ |v| > \frac{\pi}{q_i}, \ \text{for } i = 1, 2, \cdots, N \tag{17}$$

where $\vec{B} = (b_1, b_2, \cdots, b_N)$, $\vec{\sigma}_d = (\sigma_{d_1}, \sigma_{d_2}, \cdots, \sigma_{d_N})$, R is the total number of bits, and Z_i is the input variable of the quantizer.

Remark 2: A) the problem can be solved without exhaustive search over the entire space of $\{\vec{B}, \vec{\sigma}_d\}$, due to Remark 1. The optimal solution for this problem can be obtained as follows: 1) for any possible $\vec{B}$, determine $\vec{\sigma}_d$ first, and then compute its corresponding Fisher information; 2) the optimal solution is the combination which provides the maximum Fisher information in step 1). B) When $N = 2$, there are only $R+1$ different bit allocation solutions, and one can find the optimal solution by brute force. However, when N is large, the brute force method is not practical, and suboptimal solutions are more desirable. Algorithms such as the GBFOS algorithm [10], the convex relaxation [11], and the approximate dynamic programming method [12] can be used for this purpose.

5. NUMERICAL RESULTS

In this section, we first numerically show that J_i is a monotonic decreasing function of σ_{d_i} (Remark 1). Then, when $N = 2$, the optimal bit allocation scheme in the sense of maximizing the Fisher information by brute force is obtained. Besides, numerical results show that the optimal solution obtained by the proposed mechanism indeed yields the minimum MSE, compared to the method of equally distributing the bits and the method of allocating all the bits to the better sensor. We also show that the proposed sub-optimal LMMSE estimator can achieve comparable performance to the optimal one while alleviating the computational complexity.

5.1. Experiment 1

In this experiment, only 1 sensor is considered. We set $\sigma_n = 1$, and $q = 0.3$, such that (2) holds. The RV θ is Gaussian distributed with zero mean and variance $\sigma_\theta^2 = 4$. We can observe that, from Fig. 2, the Fisher information is a monotonic decreasing function of σ_d.

5.2. Experiment 2

There are a total of $N = 2$ sensors in the network, and R bits are available to be allocated between the two sensors. The two sensors are different from each other, with $\sigma_{n1} = 0.6$, and $\sigma_{n2} = 3$. Since Gaussian noise is considered in this paper, and theoretically

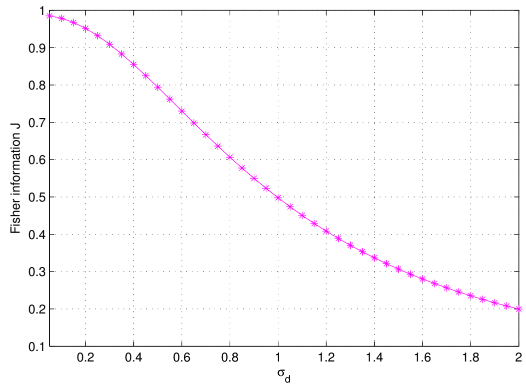

Fig. 2. Fisher information as a function of σ_d

its bandwidth in the CF domain is not limited, we will truncate the bandwidth in the experiments in this section. That is, $\phi_x(v) \approx 0$, if $|v| \geq 4\sigma_{x_v}$, where $x \sim \mathcal{N}(0, \sigma_x^2)$ and $\sigma_{x_v} = 1/\sigma_x$ (which is because the CF of a Gaussian RV is still in the form of Gaussian, and the variance of the former is the inverse of that of the latter). To ensure that (2) holds, we have $\frac{\pi}{q_i} \geq 4(\frac{1}{\sigma_{n_i}})$, *i.e.*, $q_i \leq \frac{\pi\sigma_{n_i}}{4}$. According to Remark 1, the smallest standard derivation σ_{d_i} should satisfy the condition $4(\frac{1}{\sigma_{d_i}}) = \frac{2\pi}{q_i} - 4(\frac{1}{\sigma_{n_i}})$, *i.e.*, $\sigma_{d_i} = \frac{1}{\frac{2\pi}{4q_i} - \frac{1}{\sigma_{n_i}}}$. Since the uniform quantizer is used at each local sensor, the relationship between the quantizer resolution q_i and the number of bits b_i for sensor i is given as $q_i = \frac{L_i}{2^{b_i}}$, where L_i is the observation data z_i's range for sensor i, and $L_i = 8\sqrt{\sigma_\theta^2 + \sigma_{n_i}^2}$.

Table 1. $R = 20$, Fisher Information Comparison

Bit alloc.	$\mathcal{J}$	Bit alloc.	$\mathcal{J}$
(20, 0)	3.0231	(10, 10)	3.1376
(16, 4)	3.1062	(9, 11)	3.1172
(15, 5)	3.1318	(8, 12)	3.1204
(14, 6)	3.1331	(7, 13)	3.0492
(13, 7)	3.1366	(6, 14)	2.7506
(12, 8)	3.1385	(0, 20)	0.3610
(11, 9)	3.1313	–	–

Table 1 shows the Fisher information comparison, when $R = 20$, for all the feasible bit allocation solutions. In the table, the combination (a, b) has the following meaning: a is the number of bits allocated to the first sensor while b is that allocated to the second one, and $a + b = R$. Note that non-feasible solutions (solutions violating (2)) are not listed in the table. Note also that, $(20, 0)$ and $(0, 20)$ are considered as feasible solutions, since 0 means one sensor is not active. We can observe that, the equal allocation $(10, 10)$ (which is usually used in SNs) does not yield the maximum Fisher information. Another interesting observation from Table 1 is that allocating all bits to the better sensor, *i.e.*, $(20, 0)$ is not the optimal solution, while the optimal one is $(12, 8)$, which implies that even if sensor 1 is better than sensor 2 in the sense of higher SNR, it is better to assign a few bits to sensor 2 to achieve the diversity gain. However, the Fisher information begins to decrease if more bits are assigned to sensor 2. Nevertheless, the difference between the Fisher information yielded by the optimal bit allocation solution $(12, 8)$ and that yielded by the equal allocation $(10, 10)$ is very small. This is because 10 is a large number of bits, which implies very high resolution of the quantizer.

In Table 2, R is reduced to 12. As in Table 1, the solution $(12, 0)$ does not yield the optimal performance, neither does the solution $(6, 6)$. And the optimal solution in the sense of maximizing the Fisher information is $(8, 4)$. Obviously, when the total number of bits is decreased to a smaller number, the difference between the Fisher information yielded by the optimal bit allocation solution $(8, 4)$ and that by the equal allocation $(6, 6)$ is much larger than that in Table 1. We would like to justify the optimality of the proposed bit allocation scheme. Also, we would like to show that the performance of the sub-optimal LMMSE estimator is comparable to the optimal MMSE estimator. Therefore, in Table 2, we also provide the MSE comparison between different bit allocation solutions and between different estimators as well as the corresponding PCRLBs. Note that 1000 Monte Carlo runs are performed to compute the MSEs. Note also that MSE1 is the MSE of the optimal MMSE estimator, MSE2 is that of the sub-optimal LMMSE estimator and MSE3 is computed according to (15). Obviously, the optimal solution $(8, 4)$ yields the minimum MSE, which justifies the proposed bit allocation scheme. Another observation is that the MSE2 is comparable to MSE1, which means that the proposed LMMSE estimator can provide performance that is very close to the optimal estimator while saving a lot of computation efforts. By comparing MSE2 to MSE3, we can observe that the experimental MSEs are close to the analytic ones. Note that MSE3 is very close to the PCRLBs obtained by inverting (9), which further justifies that the LMMSE estimator is a very good alternative to the optimal MMSE estimator.

Table 2. $R = 12$, Fisher information comparison

Bit alloc.	$\mathcal{J}$	PCRLB	MSE1	MSE2	MSE3
(12, 0)	3.0276	0.3303	0.3351	0.3353	0.3303
(8, 4)	3.0883	0.3238	0.3312	0.3310	0.3238
(7, 5)	3.0519	0.3277	0.3374	0.3375	0.3277
(6, 6)	2.7492	0.3637	0.3686	0.3685	0.3637
(0, 12)	0.3610	2.7698	2.7780	2.7756	2.7692

6. CONCLUSION

In this paper, we have proposed a controlled noise aided Bayesian estimation scheme. The controlled noise acts like a low-pass filter in the domain of CF. Assuming that the controlled noise is Gaussian, the problems of the optimal controlled noise design and bit allocation, in the sense of maximizing the Fisher information at the FC, were solved jointly. A near-optimal linear MMSE estimator was also proposed in this paper, which is computationally efficient. Numerical results justify our theoretical derivation. One interesting future work is to relax the Gaussian assumption on the controlled noise while designing the optimal realizable low-pass filter, so that the performance can be further improved.

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