

POUTINE: A CORRELATION ESTIMATOR FOR ERGODIC STATIONARY SIGNALS

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ABSTRACT

In this work, we present POUTINE, a novel estimator of the auto-correlation function (or more generally, the cross-correlation function) of ergodic stationary signals, an important task in a variety of applications. This estimator sparsely and non-adaptively samples the process via Bernoulli selection, generalizing the classical estimator in a natural way, and offering significant sampling reductions while sacrificing a modest degree of accuracy. Both the mean and variance of our estimator are explicitly analyzed, and in particular, we show that POUTINE gives an unbiased estimate of the classical estimator, which in turn gives an unbiased estimate of the underlying second-order statistics of interest. Furthermore, we show that POUTINE is a consistent estimator with variance approaching zero asymptotically. We demonstrate favorable performance of this approach for a simple stochastic process.

Index Terms— POUTINE, sparsity, ergodicity, non-adaptive measurements, cross-correlation

1. INTRODUCTION

Correlation-based methods [1] are effective techniques for characterizing the statistical properties of signals and systems. In many applications, one is interested in evaluating the power spectrum of a random signal, which is the Fourier transform of its auto-correlation. The power spectrum provides useful information about the expected power of a signal at each frequency in the spectrum. Spectral estimation can be used for speech analysis, pattern recognition, seismology, communications, radar, and sonar [2].

While auto-correlation is essential for spectral estimation, cross-correlation has recently generated much interest as a powerful tool for system characterization. For example, the cross-correlation of ambient noise signals in a multi-channel observation system can be used to estimate the Green's function of the wave equation in an inhomogeneous medium. This idea can be used for travel-time estimation, passive sensor imaging of reflectors, and structural health monitoring [3, 4].

In general, the cross-correlation between two random, uniformly-sampled signals $x^{(1)}, x^{(2)}$ is defined as

$$R_{x^{(1)}x^{(2)}}(n_1, n_2) = \mathbb{E} \left\{ x^{(1)}[n_1]x^{(2)}[n_2] \right\}.$$

For stationary ergodic signals, the cross-correlation only depends on the time difference $\tau = n_2 - n_1$.

Earlier theoretical results [5] have shown that random sampling strategies can accurately estimate the cross-correlation function regardless of the temporal spacing between the samples. In practice, small data sets can be processed in batch mode. For larger datasets, or in streaming data applications, partial correlation estimates can be generated by processing windows of the input data. For two ergodic input signals $x^{(i)}$, $i = 1, 2$, we first split each signal into W length- N observation windows, i.e., $x^{(i)} = [x_1^{(i)}, x_2^{(i)}, \dots, x_W^{(i)}]$, where $x_w^{(i)}$ is the w -th segment of $x^{(i)}$. The empirical cross-correlation between the signals in window w is defined as

$$\hat{R}_{x_w^{(1)}x_w^{(2)}}(\tau) = \frac{1}{N - |\tau|} \sum_{n=1}^N x_w^{(1)}[n]x_w^{(2)}[n + \tau]$$

where we use the convention that $x_w^{(i)}[k] = 0$ for indices k outside the window. This can be written in matrix form as

$$\hat{R}_{x_w^{(1)}x_w^{(2)}}(\tau) = \frac{1}{N - |\tau|} x_w^{(1)T} S^\tau x_w^{(2)},$$

where S^τ is the time-shift operator.¹ The classical cross-correlation estimate can then be formed as an average over the correlations within each window

$$\hat{R}_{x^{(1)}x^{(2)}}(\tau) = \frac{1}{W} \sum_{w=1}^W \hat{R}_{x_w^{(1)}x_w^{(2)}}(\tau). \quad (1)$$

In practical applications, architectural or temporal constraints might necessitate the compressive measurements of the signals of interest. The recent field of compressive sensing [6] studies how signals can be robustly acquired using such dimensionality-reducing operators Φ . Theoretical results in the field have shown that such a process could retain the relevant information of the original signal, even when Φ is highly under-determined. For instance, when the signal is sparse, it can be recovered exactly from its compressed measurements for a wide class of measurement operator Φ .

In this work, we present POUTINE: a Partially-Observing Unbiased Time-Integrating Non-Adaptive Estimator, which

¹Note that S^τ is linear shift instead of a circular shift.

estimates the cross-correlation of ergodic signals that have been observed by a random, sparse subsampling operator Φ_w within every window w . The measurements of the windowed segments are denoted $y_w^{(i)} = \Phi_w x_w^{(i)}$. Contrary to classical compressive sensing, we do not assume that the signals themselves are sparse. We show that the POUTINE estimator (defined carefully in Section 2) provides an unbiased and consistent estimation of the classical cross-correlation. This implies that even from highly under-sampled observations of the signals, we can obtain an estimator that performs in average as well as the classical estimator that uses all the samples. Although the random operator Φ_w is different for each window observation, our technique is non-adaptive and does not rely on any signal property other than its ergodicity.

2. APPROACH

In this section, we describe the POUTINE estimator that estimates the cross-correlation of two ergodic stationary signals $x^{(1)}, x^{(2)}$ from their undersampled measurements in each window. Specifically, we non-adaptively observe each sample of $x_w^{(i)}$ for $i = 1, 2$ with probability ρ . The observations can be compactly expressed $y_w^{(i)} = \Phi_w x_w^{(i)}$ for $i = 1, 2$.

The estimator proceeds as follows. First, we pad the unknown entries of the signals in each window with zeros. Thus, we use $\tilde{x}_w^{(i)} := \Phi_w^* y_w^{(i)}$. Then, the POUTINE estimator of $R_{x^{(1)}x^{(2)}}(\tau)$ is defined as

$$\tilde{R}_{x^{(1)}x^{(2)}}(\tau) = \frac{1}{W} \sum_{w=1}^W \tilde{R}_{x_w^{(1)}x_w^{(2)}}(\tau),$$

over the range $|\tau| \leq N - 1$ where

$$\tilde{R}_{x_w^{(1)}x_w^{(2)}}(\tau) = \alpha(\tau) \sum_{n=1}^N \tilde{x}_w^{(1)}[n] \tilde{x}_w^{(2)}[n + \tau]$$

and

$$\alpha(\tau) = \frac{1}{N - |\tau|} \begin{cases} \rho^{-1} & \text{when } \tau = 0 \\ \rho^{-2} & \text{when } \tau \neq 0. \end{cases}$$

Note that when the sampling matrix Φ_w is the identity (i.e., $\rho = 1$), the POUTINE estimator reduces to the classical estimator (1). The following theorem shows that the POUTINE estimator is unbiased and consistent with respect to the classical cross-correlation estimator.

Theorem 1. *The POUTINE estimator of the cross-correlation of two ergodic signals $x^{(1)}, x^{(2)}$ satisfies*

$$\mathbb{E} \left\{ \tilde{R}_{x^{(1)}x^{(2)}}(\tau) \right\} = \hat{R}_{x^{(1)}x^{(2)}}(\tau), \quad (2)$$

and

$$\text{Var} \left(\tilde{R}_{x^{(1)}x^{(2)}}(\tau) \right) = \frac{1}{W} \left(\rho^{-\beta(\tau)} - 1 \right) \times \sum_{n=1}^N \left(x_w^{(1)}[n] \right)^2 \left(x_w^{(2)}[n + \tau] \right)^2, \quad (3)$$

where $\beta(0) = 1$ and $\beta(\tau) = 2$ for $\tau \neq 0$.

From this theorem, we can make two observations about the POUTINE estimator. First, the expected value of the estimator is equal to the classical estimator of the cross-correlation. In particular, the POUTINE estimator gives an unbiased estimate of the classical estimator (with respect to randomness in Φ), and the classical estimator gives an unbiased estimate of the actual cross-correlation function (with respect to randomness in the process itself). By the transitivity of iterated expectations, the POUTINE estimator is itself an unbiased estimate of the actual underlying cross-correlation function. Also, through ergodicity and in the limit of large W , the classical estimator approaches the true cross-correlation function asymptotically as $W \rightarrow \infty$.

Second, (3) shows that the estimator is consistent, meaning that the variance of our estimator goes to zero as the number of drawn samples increases. This occurs when the number of windows W increases or when the compression ratio ρ goes to one.

3. RESULTS

In this section we compare the POUTINE estimate with the classical estimate generated from the complete set of samples. For this purpose we synthesize an ergodic wide-sense stationary (WSS) signal $x^{(1)}$ and its time-delayed counterpart $x^{(2)}$ using a first order auto-regressive model as:

$$x^{(1)}[n] = Cx^{(1)}[n - 1] + Dg[n] \quad (4)$$

$$x^{(2)}[n] = x^{(1)}[n - 20] \quad (5)$$

where $g[n] \sim \mathcal{N}(0, 1)$, $C = e^{-3/200}$, and $D = \sqrt{1 - C^2}$, so that $x^{(i)}[n] \sim \mathcal{N}(0, 1)$. Here we use $W = 300$ windows each of length $N = 200$ to compute the cross-correlation.

For the POUTINE estimate, we use a sparse non-adaptive selection of the samples. Specifically, in every window, a particular sample is drawn with probability ρ , which leads to an average sample count of ρN per window.

Figure 1 shows the result of estimating this cross-correlation classically (i.e., with $\rho = 1$) and compressively using POUTINE. We respectively denote them as $\hat{R}_{x^{(1)}x^{(2)}}(\tau)$ and $\tilde{R}_{x^{(1)}x^{(2)}}(\tau)$. The classical approach uses all samples within each window, and POUTINE uses approximately $\rho = 10\%$ of the available samples. The blue line in represents the classical estimate and the green dots represent the POUTINE estimate, which forms a good approximation to the classical estimate.

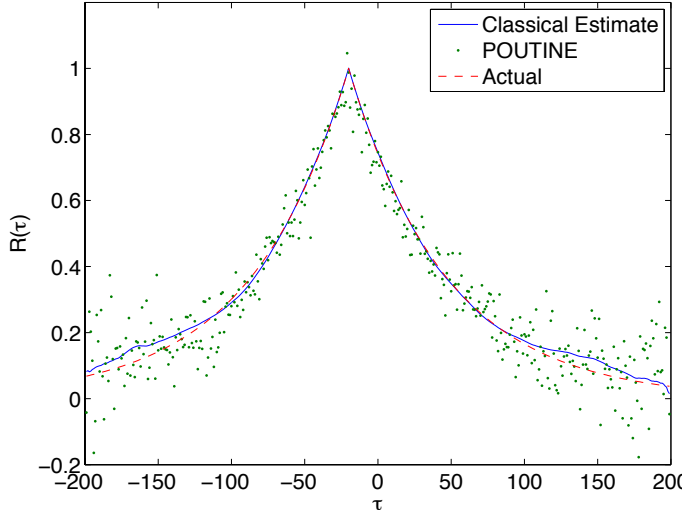


Fig. 1. Estimates for both classical $\hat{R}_{x^{(1)}x^{(2)}}(\tau)$ and POUTINE $\tilde{R}_{x^{(1)}x^{(2)}}(\tau)$ estimates of the true cross-correlation $R_{x^{(1)}x^{(2)}}(\tau)$

Figure 2 shows the relative root mean-squared errors (RMSE) of POUTINE ($E(\tau)$) for several values of ρ , calculated as:

$$E(\tau) = \frac{\sqrt{\mathbb{E} \left\{ |\tilde{R}_{x^{(1)}x^{(2)}}(\tau) - R_{x^{(1)}x^{(2)}}(\tau)|^2 \right\}}}{R_{x^{(1)}x^{(2)}}(\tau)}. \quad (6)$$

The curve for $\rho = 1$ is the classical estimator (i.e. when all samples are retained). It is apparent that as ρ increases, the RMSE of the POUTINE estimate approaches that of the classical estimate. For $\rho = 0.5$, which corresponds to roughly using half the samples in each window, the RMSE is reasonably close to the classic estimate.

4. DISCUSSION

In this paper, we presented the POUTINE estimator $\tilde{R}_{x^{(1)}x^{(2)}}(\tau)$ that accurately estimates the cross-correlation of two stationary and ergodic signals $x^{(1)}$, and $x^{(2)}$ for substantial subsampling ratios. This data reduction may facilitate the use of cross-correlation techniques in a wider variety of applications. We further showed that this estimate is unbiased and consistent with respect to the classical cross-correlation estimator, $\hat{R}_{x^{(1)}x^{(2)}}(\tau)$.

We foresee many extensions to our approach and results. First, we assumed a non-adaptive subsampling architecture for each time window of the signal. An adaptive measurement process that acquires future measurements based on past measurements may further reduce the compression ratio ρ without incurring any loss of the estimator unbiasedness or consistency. Moreover, it is possible that other types of subsampling

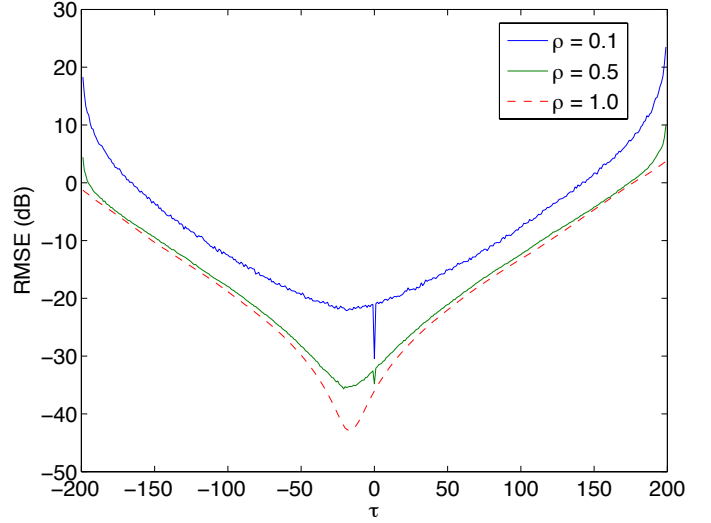


Fig. 2. Relative root mean-square error (RMSE) for both classical (dashed red) and POUTINE estimates for compression ratios of $\rho = 0.1$ and 0.5

architectures (e.g., random demodulator [7]) can be used in place of the sparse-sampling architecture proposed here.

5. PROOFS

5.1. Unbiased Estimator

First, we want to show that the POUTINE estimator is unbiased, i.e., $\mathbb{E} \left\{ \tilde{R}_{x_w^{(1)}x_w^{(2)}}(\tau) \right\} = R_{x_w^{(1)}x_w^{(2)}}(\tau)$ for every w . Observe that

$$\begin{aligned} \tilde{R}_{x_w^{(1)}x_w^{(2)}}(\tau) &= \alpha(\tau) \left\langle \Phi_w^* \Phi_w x_w^{(1)}, S^\tau \Phi_w^* \Phi_w x_w^{(2)} \right\rangle \\ &= \alpha(\tau) \left(x_w^{(1)} \right)^* \Phi_w^* \Phi_w S^\tau \Phi_w^* \Phi_w x_w^{(2)}, \end{aligned}$$

and

$$\Phi_w^* \Phi_w = \sum_{n=1}^N \epsilon_n E_n,$$

where E_n is a zero matrix with a '1' on the n -th diagonal element and

$$\epsilon_n = \begin{cases} 1 & \text{w.p. } \rho, \\ 0 & \text{w.p. } 1 - \rho. \end{cases}$$

With this, note that

$$\mathbb{E} \{ \epsilon_n \epsilon_{n'} \} = \begin{cases} \rho & \text{when } n = n', \\ \rho^2 & \text{when } n \neq n'. \end{cases}$$

Now, rewrite

$$\begin{aligned}\Phi_w^* \Phi_w S^\tau \Phi_w^* \Phi_w &= \left(\sum_{n=1}^N \epsilon_n E_n \right) S^\tau \left(\sum_{n'=1}^N \epsilon_{n'} E_{n'} \right) \\ &= \sum_{n,n'=1}^N \epsilon_n \epsilon_{n'} E_n S^\tau E_{n'}\end{aligned}$$

Taking expectation over the random vector ϵ , we get

$$\begin{aligned}\mathbb{E} \left\{ \tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau) \right\} &= \alpha(\tau) \left(x_w^{(1)} \right)^* \left(\sum_{n,n'=1}^N \mathbb{E} \{ \epsilon_n \epsilon_{n'} \} E_n S^\tau E_{n'} \right) x_w^{(2)} \\ &= \begin{cases} \alpha(\tau) \rho \left(x_w^{(1)} \right)^* S^\tau x_w^{(2)} & \text{when } \tau = 0 \\ \alpha(\tau) \rho^2 \left(x_w^{(1)} \right)^* S^\tau x_w^{(2)} & \text{when } \tau > 0 \end{cases} \\ &= \frac{1}{N - |\tau|} \left(x_w^{(1)} \right)^* S^\tau x_w^{(2)} = \hat{R}_{x_w^{(1)} x_w^{(2)}}(\tau).\end{aligned}$$

because when $\tau = 0$, all the non-zero entries of S^τ lie only on the diagonal while when $\tau \neq 0$, all the non-zero entries of S^τ will lie only on the off-diagonal. This shows that the POUTINE estimator is unbiased.

5.2. Consistent Estimator

We now show that the estimator is also consistent, meaning that the variance of the estimator goes to zero as the number of drawn samples increases (either with an increase in the number of windows W , or an increase in the compression ratio ρ). The variance of the POUTINE estimator can be written as

$$\begin{aligned}\text{Var} \left(\frac{1}{W} \sum_w \tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau) \right) &= \frac{1}{W^2} \sum_w \text{Var} \left(\tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau) \right) \\ &= \frac{1}{W} \text{Var} \left(\tilde{R}_{x_1^{(1)} x_1^{(2)}}(\tau) \right)\end{aligned}$$

where we used the fact that the measurement operators Φ_w are i.i.d.. The variance over one window can be expressed as

$$\begin{aligned}\text{Var} \left\{ \tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau) \right\} &= \mathbb{E} \left\{ \tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau)^2 \right\} \\ &\quad - \mathbb{E} \left\{ \tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau) \right\}^2.\end{aligned}$$

The second term was computed in Section 5.1. The first term can be computed as follows.

$$\begin{aligned}\mathbb{E} \left\{ \tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau)^2 \right\} &= \mathbb{E} \left\{ \left| \alpha(\tau) \left\langle \Phi_w^* \Phi_w x_w^{(1)}, S^\tau \Phi_w^* \Phi_w x_w^{(2)} \right\rangle \right|^2 \right\} \\ &= \alpha(\tau)^2 \mathbb{E} \left\{ \left| \sum_{n,n'} \epsilon_n \epsilon_{n'} \left(x_w^{(1)} \right)^T E_n S^\tau E_{n'} x_w^{(2)} \right|^2 \right\} \\ &= \alpha(\tau)^2 \mathbb{E} \left\{ \left| \sum_n \epsilon_n \epsilon_{n+\tau} x_w^{(1)}[n] x_w^{(2)}[n+\tau] \right|^2 \right\} \\ &= \alpha(\tau)^2 \mathbb{E} \left\{ \sum_{n,k} \epsilon_n \epsilon_{n+\tau} \epsilon_k \epsilon_{k+\tau} x_w^{(1)}[n] \right. \\ &\quad \times x_w^{(2)}[n+\tau] x_w^{(1)}[k] x_w^{(2)}[k+\tau] \left. \right\} \\ &= \alpha(\tau)^2 \sum_{n,k} \mathbb{E} \{ \epsilon_n \epsilon_{n+\tau} \epsilon_k \epsilon_{k+\tau} \} x_w^{(1)}[n] \\ &\quad \times x_w^{(2)}[n+\tau] x_w^{(1)}[k] x_w^{(2)}[k+\tau].\end{aligned}$$

Simple calculations show that

$$\mathbb{E} \{ \epsilon_n \epsilon_{n+\tau} \epsilon_k \epsilon_{k+\tau} \} = \begin{cases} \rho & \text{when } n = k \text{ and } \tau = 0 \\ \rho^2 & \text{when } n \neq k \text{ and } \tau = 0 \\ \rho^2 & \text{when } n = k \text{ and } \tau \neq 0 \\ \rho^4 & \text{when } n \neq k \text{ and } \tau \neq 0. \end{cases}$$

Plugging this back and using the expression for the expectation in Section 5.1, we obtain two cases. When $\tau = 0$, we have

$$\begin{aligned}\text{Var} \left(\tilde{R}_{x_w^{(1)} x_w^{(2)}}(0) \right) &= \frac{1}{(N - |\tau|)^2} \left(\frac{1}{\rho} - 1 \right) \\ &\quad \times \sum_n \left(x_w^{(1)}[n] \right)^2 \left(x_w^{(2)}[n] \right)^2\end{aligned}$$

When $\tau \neq 0$, we have

$$\begin{aligned}\text{Var} \left(\tilde{R}_{x_w^{(1)} x_w^{(2)}}(\tau) \right) &= \frac{1}{(N - |\tau|)^2} \left(\frac{1}{\rho^2} - 1 \right) \\ &\quad \times \sum_n \left(x_w^{(1)}[n] \right)^2 \left(x_w^{(2)}[n+\tau] \right)^2\end{aligned}$$

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