ENHANCING THE READABILITY OF THE WIGNER DISTRIBUTION BY EXPLOITING ITS CROSS-TERMS GEOMETRY

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ABSTRACT

A simple scheme to improve the readability of the Wigner distribution is introduced in this paper. The method aims to suppress the interference terms by way of multiplying the original distribution with a two-dimensional mask. The design of the mask is based on the localization of the signal components in the time-frequency plane. This is made possible by exploiting the geometry of the cross-terms generated between the signal at hand and a circularly symmetric Gaussian atom. Experimentation has shown that the proposed approach can deliver a more readable representation as compared with other familiar methods of interference reduction.

***Index Terms*—** Wigner distribution, time-frequency analysis, cross-terms interference

1. INTRODUCTION

Joint time-frequency (TF) analysis has proved to be a useful signal processing area with a wide range of applications. TF representations are usually classified into linear and bilinear methods. Although linearity is an attractive property, it is desirable to have distributions behaving similarly to a TF energy density function. Since energy is a quadratic signal representation, bilinear methods can be interpreted in terms of signal energy.

A prominent member of the class of quadratic TF representations is the Wigner distribution (WD) [1], which satisfies an exceptionally large number of desirable mathematical properties and exhibits the least amount of spread in the TF plane [2]. While the bilinearity of the WD increases the sharpness of local signal structure, it also generates spurious values between separate signal components in the TF plane. This interference phenomenon can significantly reduce the readability of the WD in practical applications especially when multi-component or non-linear frequency modulated signals are concerned.

A practically useful TF representation must be accurate and easy to interpret. Therefore, it should exhibit high concentration of the signal components as well as maintaining the presence of any misleading interference to a minimum. Considerable amount of research has been carried out with the aim of developing modified versions of the WD with suppressed cross-terms. Most of these methods were concerned with the design of interference-attenuating kernels, and can be classified into signal-independent (e.g. [3], [4], [5]), and signal dependent schemes (e.g. [6]). The common shortcoming of the above methods is that the attenuation of interference generally comes at the cost of increasing the TF spread of the auto-terms, which reduces the accuracy of the representation.

An alternative approach [7] for decreasing the amount of interference is to post-process the WD with a masking operation. We draw on this strategy using a computationally efficient technique to estimate the supports of the signal components in the TF plane. Based on this information, we design and apply a TF mask such that any cross-terms outside the areas of actual signal energy are eliminated.

The paper is organized into four sections. Section 2 provides an overview of the theoretical background and describes the algorithm. In section 3, the proposed method is experimentally assessed and contrasted with some well-known TF representations. Concluding remarks are finally made in section 4.

2. CROSS-TERM SUPPRESSION

2.1. Theoretical Background

The WD of the signal $x(t)$ is defined as [1],

$$W_x(t,\omega) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau \,. \qquad (1)$$

Due to its quadratic structure in the signal $x(t)$, the WD is subject to the quadratic superposition principle. Thus, for the two-component signal $x(t) = x_1(t) + x_2(t)$ the WD can be expressed as:

$$W_x(t,\omega) = W_{x_1}(t,\omega) + W_{x_2}(t,\omega) + 2Re\{W_{x_1,x_2}(t,\omega)\}\,, \qquad (2)$$

where it can be seen that apart from the WDs of the individual signal elements there appears an additional component, i.e. the cross-term, which is equal to the real part of the cross-WD,

$$W_{x_1,x_2}(t,\omega) = \int_{-\infty}^{\infty} x_1\left(t+\frac{\tau}{2}\right) {x_2}^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau \,. \qquad (3)$$

In general, for L separate signal components there will exist L(L-1)/2 spurious terms which have an oscillatory nature and their amplitude can peak at a value twice as high as that of the actual signal terms [1].

The geometry of the cross-terms of the WD in the TF plane has been well-studied [8], [9]. It is known that a cross-term will appear midway between any two TF points of actual signal presence. Therefore, for every pair of signal points (t_1, f_1) and (t_2, f_2), there will be an interference element in the middle of the imaginary line connecting the two points. The coordinates $(t_{1,2}, f_{1,2})$ of the cross-term can then be obtained as:

$$t_{1,2} = \frac{t_1+t_2}{2} \quad and \quad f_{1,2} = \frac{f_1+f_2}{2} \ . \qquad (4)$$

The direction of oscillation of the cross-terms is perpendicular to the line connecting the two signal points whereas the rate of oscillation is proportional to the distance between the two points.

2.2. The Proposed Algorithm

Central to the presented method is the detection of the areas in the TF plane within which the actual signal components lie. We show that this can be achieved by exploiting the simple geometric law (4). Let $x_1(t)$ be the arbitrary signal whose WD we aim to enhance, and $x_2(t)$ be an auxiliary, user-defined signal.. Then, according to (2), by subtracting the corresponding WDs from the WD of the sum of the two signals we can isolate the cross-terms between $x_1(t)$ and $x_2(t)$:

$$2Re\{W_{x_1,x_2}(t,\omega)\} = W_x(t,\omega) - W_{x_1}(t,\omega) - W_{x_2}(t,\omega) \ . \qquad (5)$$

If the signal $x_2(t)$ possesses circular symmetry around a known centre in the TF plane, then due to (4) the shape of the isolated cross-terms will reflect the TF location of the signal $x_1(t)$. Consequently, it is possible to compute the TF location of $x_1(t)$ based on (4), since the coordinates of the interference and of $x_2(t)$ can both be determined. A detailed description of the overall process that we have followed to implement the above idea is provided next. The presentation is based on an illustrative example. (Fig. 1)

The WD of a sinusoidal frequency-modulated (FM) signal $x_1(t)$ is shown in Fig.1a whereas the WD of the sum $x(t)$ of the above signal and a Gaussian atom $x_2(t)$ is depicted in Fig.1b. The interference between the two signals is precisely detected after carrying out the subtractions in (5), as it can be seen in Fig.1c. The above generation of interference is also illustrated in Fig. 2. This oscillating interference is subsequently rectified, and smoothed using a 2-D averaging window. (Fig.1d). The binary image of Fig.1e is created by thresholding the image of Fig.1d.

Since the support of the Gaussian atom is not a single TF point, the size of the black area of Fig.1e is the result of the superposition of the individual interactions between $x_1(t)$ and all the points within the support of the Gaussian atom.

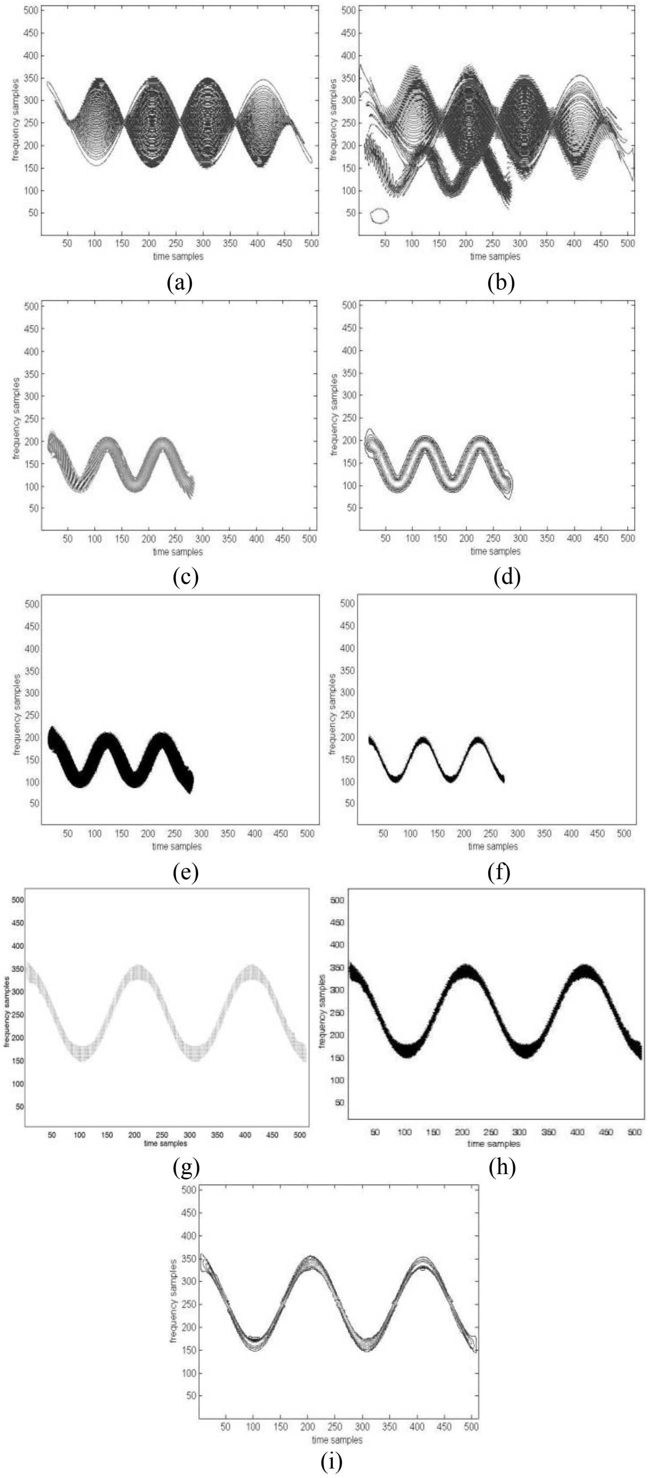

Fig. 1. Auto-term localization example: (a) The WD of an example signal; (b) the WD of the sum of the original signal and a Gaussian atom; (c) detection of the induced cross-terms (d) rectification and smoothing of the cross-terms; (e) generation of a uniform representation; (f) trimmed area; (g) projection to the estimated support of the example signal; (h) 'closing' and formation of mask; (i) the masked WD of (a).

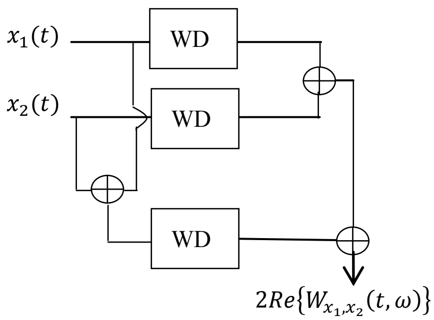

Fig. 2. Isolation of interference between an arbitrary signal $x_1(t)$ and the user-defined $x_2(t)$.

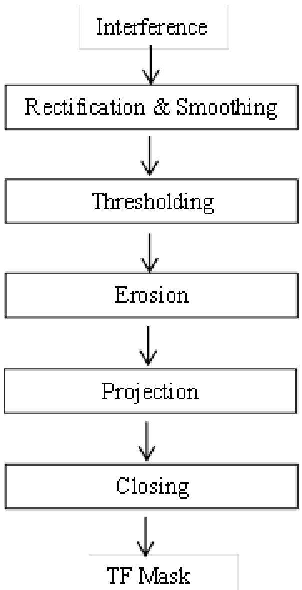

Fig. 3. A workflow diagram of the proposed algorithm for localizing the TF support of an arbitrary signal $x_1(t)$.

However, we aim to estimate the portion of this area which is only due to the interaction with the centre point of the Gaussian. From the geometry of (4), we can consider that this region must be expanded by *r/2* on all sides – where *r* is the radius of the atom. Therefore, we erode the edges of the area in Fig.1e by *r/2* to obtain the desirable region (Fig.1f). Now, using the coordinates of the centre of the Gaussian on one hand, and those of the points within the eroded area on the other, we can invert (4). This way, we construct an estimate of the original location of the sinusoidal FM signal by projecting the eroded area into that location (Fig.1g). The gaps due to the resulting magnification are subsequently removed by a 'closing' operation [10], and all non-zero values are set equal to one. The generated mask which is presented in Fig.1h is then multiplied with the original WD of Fig.1a to suppress the cross-terms and enhance its readability (Fig.1i). The workflow diagram of the proposed algorithm is shown in Fig. 3.

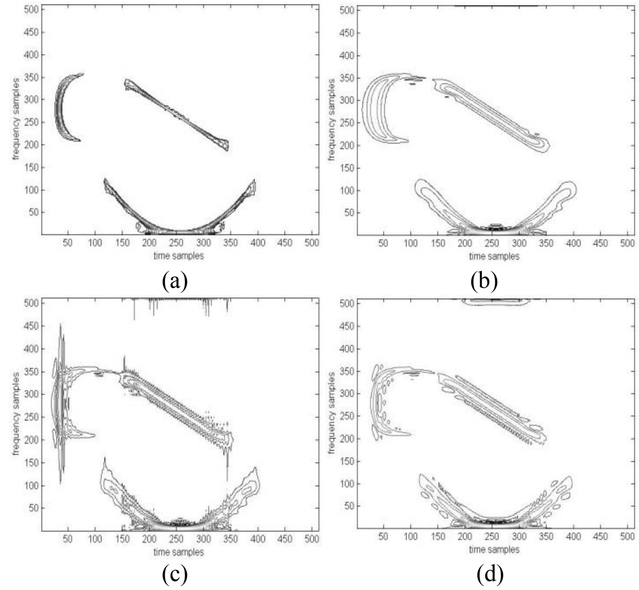

Fig. 4. TF representation of the signal in the first example according to: (a) the proposed method; (b) the SPWD; (c) the CWD; (d) the ZAMD.

3. EXPERIMENTAL RESULTS

In this Section, we showcase the proposed algorithm in a series of tutorial examples. The first example contains non-overlapping elements in the TF plane, whereas there is severe overlap of the signal components in the second example. The waveforms in the first two examples are computer simulations, while the third example involves a real-life recording.

We have also compared the proposed scheme with three well-known reduced-interference TF representations. In particular, we have considered the smoothed pseudo Wigner distribution (SPWD) [3], the Choi-Williams distribution (CWD) [4], and the Zhao-Atlas-Marks distribution (ZAMD) [5]. Hamming windows were employed for the above methods. The different kernel parameters required for their implementation were all determined empirically in order to achieve both high time-frequency concentration and substantial cross-terms suppression.

3.1. First Example: Non-overlapping Auto Terms

The signal in the first example is the sum of a quadratic chirp, a linear chirp and a third component which is obtained as the impulse response of a Butterworth band-pass filter. The three components have different orientations, and do not overlap in the TF plane. The representation of the above signal based on the proposed method is shown in Fig. 4a. It is clear that the signal appears well-concentrated, and

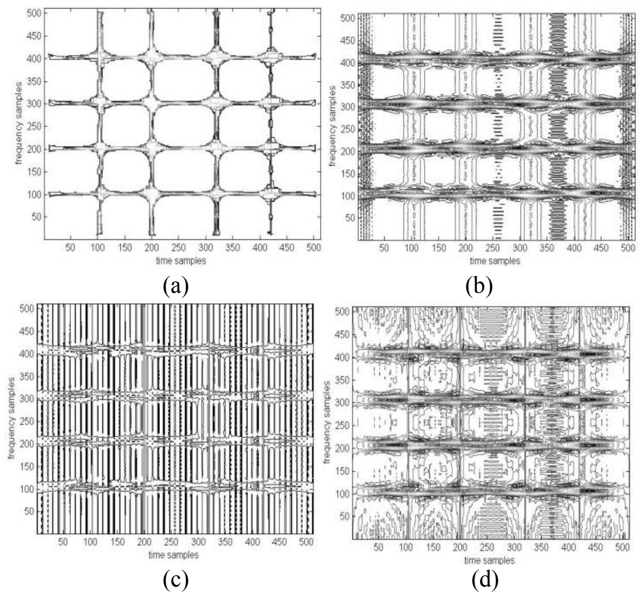

Fig. 5. TF representation of the signal in the second example according to: (a) the proposed method; (b) the SPWD; (c) the CWD; (d) the ZAMD.

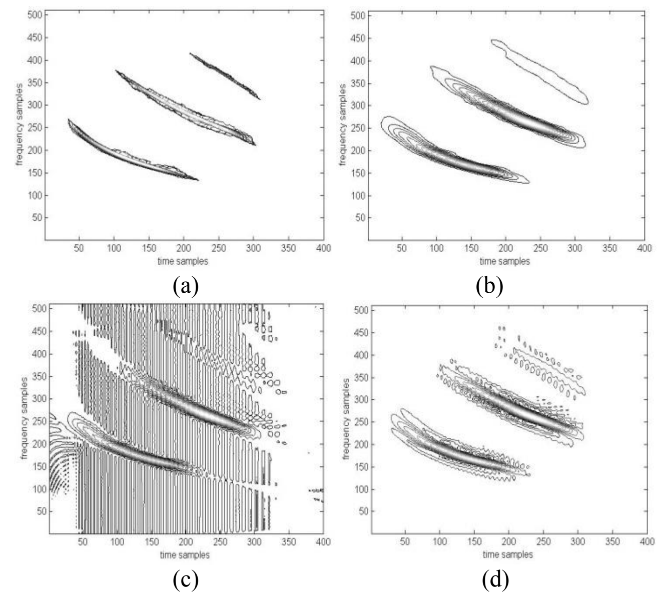

Fig. 6. TF representation of the signal in the third example according to: (a) the proposed method; (b) the SPWD; (c) the CWD; (d) the ZAMD.

the cross-terms have been largely suppressed. The SPWD, CWD, and ZAMD on the other hand significantly broaden the auto-terms and introduce various distortions as it can be observed in Fig.4b – Fig.4d.

3.2. Second Example: Overlapping Auto Terms

The second example signal is composed of four unit-pulses and four constant-frequency sinusoids designed to form a grid pattern. This is a challenging scenario for most TF representations as it can be appreciated by examining Fig.5b – Fig.5d. The CWD apparently suffers from heavy distortion due to the overlap in time of the four sinusoids, and therefore fails to resolve this signal. The SPWD enlarges the auto-terms without being able either to suppress interference equally across the TF plane. Although the grid structure can be recognized in the ZAMD there still remains a considerable amount of interference. Nevertheless, the proposed method yields a highly readable result with most of the WD cross-terms eliminated, and auto-terms that are relatively accurately represented in Fig.5a.

3.3. Third Example: Real-Life Signal

The final example involves a digitized echolocation pulse emitted by the Large Brown Bat, Eptesicus Fuscus, which is a common test signal in the literature related to TF analysis. The signal consists of 400 samples and the sampling period was 7 microseconds. The results for the different TF methods are shown in Fig.6. Again, the simple scheme proposed in this work outperforms the three alternative approaches since it can provide the highest resolution of the analysed signal.

4. CONCLUSION

A simple method for enhancing the readability of the WD by suppressing its cross-terms has been presented. The geometric law defining the positions of the interference in the TF plane has been exploited in order to find the location of the actual signal components. Subsequently, any values of the WD which fall outside the estimated support are reduced to zero. The result is a representation which considerably reduces misleading interference, and can preserve the concentration of the auto-terms of the original WD. Experimentation has shown that the proposed scheme can outperform other commonly used methods for the reduction of cross-terms, in the sense that it delivers a clearer picture of the TF behavior of the signal.

5. ACKNOWLEDGEMENTS

The authors would like to thank Curtis Condon, Ken White, and Al Feng of the Beckman Center at the University of Illinois for the bat signal used in the third example of Section 3.

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