# AN OPERATOR-BASED AND SPARSITY-BASED APPROACH TO ADAPTIVE SIGNAL SEPARATION

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## ABSTRACT

An operator-based and sparsity-based approach is proposed to adaptively separate a signal into additive subcomponents. The proposed approach can be formulated as an optimization problem. Since the design of the operator can be adaptively customized to the target signal, we can propose different types of operators for different types of signals. The subcomponents are a kind of local narrow band signals in the null space of an adaptive operator and a residual signal which is a sparse signal in some sense. Our experiments, including simulated signals and a real-life signal, demonstrate the efficacy and accuracy of the proposed approach.

***Index Terms***— Signal separation, adaptive operator, the null space, $\ell_1$ constraint, sparse signal

## 1. INTRODUCTION

Recently, single-channel signal separation has attracted a lot of interests since it has affected many applications. Many approaches have been proposed to decompose a single-channel signal into a mixture of several additive coherent subcomponents. The different definitions of subcomponents lead to different kinds of decomposition methods. For example, Empirical Mode Decomposition (EMD) [1, 2] decomposes an oscillatory signal into a summation of intrinsic mode functions (IMFs); Matching Pursuit (MP) [3] decomposes a signal into a summation of time-frequency atoms; Null Space Pursuit (NSP) [4], which is an operator-based approach, decomposes a signal into some local narrow band signals which are defined in the null space of adaptive operators. Among those approaches, we have a particular interest in NSP.

The NSP approach [4], uses an adaptive operator $\Gamma_S$ to separate a signal $S$ into additive subcomponents: $R$ and $U(U = S - R)$. It can be formulated as an optimization problem:

$$\min_{R} \{\|\Gamma_S(S-R)\|^2+\lambda_1(\|R\|^2+\gamma\|S-R\|^2)+F(\Gamma_S)\}, \quad (1)$$

where the operator $\Gamma_S$ can be adaptively estimated from the signal $S$, and the last term is the Lagrange term for the parameters of the operator $\Gamma_S$. Minimizing the term $\|\Gamma_S(S-R)\|^2$ can ensure that $S - R$ is in the null space of the operator $\Gamma_S$. NSP can be used to decompose the residual signal $R$ repeatedly. Hence, $S$ can be represented as the summation of subcomponents in the null spaces of a sequence of operators derived from the corresponding sequence of residual signals. Peng *et al.* proposed two types of singular operator: an integral operator and a differential operator [5]. The charming features of NSP are that the design of the operator $\Gamma_S$ can be customized to the target signal $S$, and that the operator's parameters and the Lagrangian multipliers can be adaptively estimated [4]. Although NSP has a solid mathematical foundation, it cannot decompose some signals successfully either. For example, if one subcomponent of a signal $S$ is a piecewise smooth signal, $S$ is difficult to be separated by NSP effectively. This is because the $\ell_2-$ norm, which is very sensitive to singular points of a piecewise smooth signal, has been used in NSP.

Here, we are interested in the type of signals:$s(t) = s_1(t) + s_2(t)$, where $s_1(t)$ is a narrow band signal in the null space of an operator, and $s_2(t)$ is a sparse signal in some sense. In order to decompose this type of signals, we consider measures of sparsity rather than measures of energy used in NSP. In fact, a very simple and intuitive measure of sparsity of a vector $x \in \mathrm{R}^m$ is the $\ell_0$ norm: $\|x\|_0 = \{i : x_i \neq 0\}$, if $\|x\|_0 \ll m$, $x$ is sparse. But the minimization problem:

$$(P_0): \quad \min_{x} \|x\|_0 \quad subject\ to\ \ b = Ax. \quad (2)$$

is a non-convex combinatorial optimization problem, and indeed, it has been proved that $(P_0)$ is, in general, NP-Hard [6]. According to [7], in most cases, $(P_0)$ can be equivalent to the $(P_1)$ problem,

$$(P_1): \quad \min_{x} \|x\|_1 \quad subject\ to\ \ b = Ax. \quad (3)$$

where $\|x\|_1 = \sum_i |x_i|$. The problem $(P_1)$ can be cast as a standard linear programming (LP) problem, and solved using simplex methods, modern interor-point methods, or other techniques, such as homotopy methods [8].

In this paper, we adopt $\ell_1-$ constraint instead of $\ell_2-$ constraint in NSP, and propose an operator-based and sparsity-based approach to adaptive signal separation. We demonstrate

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the efficacy and accuracy of the proposed approach in decomposing simulated signals and a real-life signal.

More precisely, this paper is organized as follows. In Section 2, the proposed operator-based and sparsity-based algorithm is presented in detail. In Section 3, experiments on simulated signals and a real-life signal are reported respectively to address the efficiency and performance of the proposed approach. We summarize our conclusions and some issues in Section 4.

## 2. PROPOSED OPERATOR-BASED AND SPARSITY-BASED APPROACH

In the previous section, we reviewed the operator-based approach proposed in [4, 5], and analyzed the rationality of $\ell_1-$ constraint leading to sparse solution. Now we turn to discuss adaptive decomposition of signals which we focus on, and propose our operator-based and sparsity-based adaptive signal separation approach.

### 2.1. Signal Model

We assume that a signal $S$ can be separated into two additive subcomponents: $U$ and $R$. Let $U$ be a narrow band signal which is in the null space of the operator $\Gamma_s$, and $R$ be a sparse signal in some sense, such as an impulse signal, or a piecewise smooth signal. The signal model assumes that:

$$S = U + R, \tag{4}$$

where $\Gamma_s U = 0$, and $DR$ is a sparse signal. $D$ is the identity operator, or the first order differential operator, or the second order differential operator.

### 2.2. The Operator-based and Sparsity-based Approach

Here, we propose an operator-based and sparsity-based approach to adaptively decompose $S$. We introduce the following optimization problem:

$$\{R, \alpha\} = arg\min_{R,\alpha}\{\|\Gamma_s(S-R)\|^2 + \lambda_1\|R\|_1 + \lambda_2\|D_2\alpha\|^2\} \tag{5}$$

where $\Gamma_s$ can be adaptively estimated from the target signal, and we choose the differential operator: $\Gamma_S = \frac{d^2}{dt^2} + \alpha(t)$ to demonstrate the proposed approach; $\alpha$ is the square of the instantaneous frequency of the signal, and $D_2$ is the second order differential operator to ensure that $\alpha$ is a smooth function; $\lambda_1, \lambda_2$ are the Lagrangian multipliers, and $D$ is the identity operator ($R$ itself is a sparse signal ), or the first order differential operator ($R$ is a square wave ), or the second order differential operator ($R$ is a triangular wave) to make sure that $DR$ is a sparse signal. In fact, $D$ can be always the second order differential operator in the above cases, since the differential of a sparse signal is still a sparse signal.

In a discrete case, the optimization problem (5) can be rewritten as:

$$\begin{aligned}\{R, \alpha\} = arg\min_{R,\alpha,\lambda_1,\lambda_2}\{&\|(D_2 + P_\alpha)(S-R)\|^2 \\ &+ \lambda_1\|DR\|_1 + \lambda_2\|D_2\alpha\|^2\}\end{aligned} \tag{6}$$

where $S, R, \alpha$ are column vectors, $P_\alpha$ is a diagonal matrix whose diagonal elements are equal to $\alpha$, and $D_2$ is a matrix of the second order difference, $D$ is also the corresponding matrix. Let $\hat{\alpha}$, $\hat{R}$ be the solution of Equation (6). Given $\hat{\lambda}_1$ and $\hat{\lambda}_2$, Equation (6) becomes

$$\begin{aligned}\mathrm{F}(\alpha, R, \hat{\lambda}_1, \hat{\lambda}_2) =& \|(D_2 + P_\alpha)(S-R)\|^2 \\ &+ \hat{\lambda}_1\|DR\|_1 + \hat{\lambda}_2\|D_2\alpha\|^2\end{aligned} \tag{7}$$

Then, we have

$$\frac{\partial \mathrm{F}}{\partial \alpha} = 2P_{S-R}^T(P_{S-R}\alpha + D_2(S-R)) + 2\hat{\lambda}_2 D_2^T D_2\alpha, \tag{8}$$

where $P_{S-R}$ is a diagonal matrix whose diagonal elements are equal to $(S-R)$. Let $\frac{\partial \mathrm{F}}{\partial \alpha} = 0$, we obtain the estimation of $\alpha$:

$$\hat{\alpha} = -(P_{S-R}^T P_{S-R} + \hat{\lambda}_2 D_2^T D_2)^{-1} P_{S-R}^T D_2(S-R). \tag{9}$$

Then, we rewrite Equation (7) as:

$$\mathrm{F}(\hat{\alpha}, R, \hat{\lambda}_1, \hat{\lambda}_2) = \|b - AR\|^2 + \hat{\lambda}_1\|DR\|_1 + \hat{\lambda}_2\|D_2\hat{\alpha}\|^2 \tag{10}$$

where $b = (D_2 + P_{\hat{\alpha}})S$ and $A = (D_2 + P_{\hat{\alpha}})$. To estimate $R$, we solve the following unconstrained optimization problem:

$$\min_R \|b - AR\|^2 + \hat{\lambda}_1\|DR\|_1. \tag{11}$$

Here, the iteratively reweighted least squares(IRLS) algorithm [9, 10, 11] is used to solve the optimization problem (11). Setting $P_R = diag(|R|)$, i.e. a diagonal matrix whose diagonal elements are equal to $|R|$, then $\|R\|_1 = R^T P_R^{-1} R$. In this way, we may view the $\ell_1$-norm as an adaptively-weighted version of the squared $\ell_2$-norm. Thus the optimization problem (11) can be rewritten:

$$\min_R \|b - AR\|^2 + \hat{\lambda}_1 R^T D^T P_{DR}^{-1} DR, \tag{12}$$

where $P_{DR} = diag(|DR|)$. This is a quadratic optimization problem, which is solvable using standard linear algebra.

We summarize the above derivations to solve the optimization problem (6) in the following algorithm 1.

## 3. EXPERIMENT RESULTS AND DISCUSSIONS

In this section, we demonstrate the performance of the proposed algorithm through decomposing several simulated signals and a real-life signal. We compare our experiment results with which achieved by EMD and NSP.

**Algorithm 1** the operator-based and sparsity-based algorithm

1: Input: the signal $S$, the parameter $\hat{\lambda}_1$, $\hat{\lambda}_2$, and the stopping threshold $\epsilon$.

2: Let $j = 0$, $\hat{R}_j = 0$.

3: Compute $\hat{\alpha}_j$ according to Equation (9) as follows:

$$\hat{\alpha}_j = -(P^T_{S-\hat{R}_j} P_{S-\hat{R}_j} + \hat{\lambda}_2 D_2^T D_2)^{-1} P^T_{S-\hat{R}_j} D_2(S-\hat{R}_j),$$

where $P_{S-\hat{R}_j}$ is a diagonal matrix whose diagonal elements are equal to $S-\hat{R}_j$, and $D_2$ is the matrix of the second order difference.

4: Compute $\hat{R}_{j+1}$ by using IRLS: approximately solve the linear system

$$(\hat{\lambda}_1 D^T P^{-1}_{D\hat{R}_j} D + A^T A)\hat{R}_{j+1} = A^T b,$$

where $D$ is the corresponding matrix: an identity matrix or the matrix of the first (second) order difference, $b = (D_2 + P_{\hat{\alpha}_j})S$ and $A = (D_2 + P_{\hat{\alpha}_j})$.

5: If $\|\hat{R}_{j+1} - \hat{R}_j\| > \epsilon\|S\|$, then set $j = j+1$ and go to Step 3.

6: Output: the optimal solution $\hat{R} = \hat{R}_{j+1}$ and the operator parameter $\hat{\alpha} = \hat{\alpha}_j$.

In the first example, we show that the proposed algorithm can separate square wave signals and amplitude modulation and frequency modulation (AM-FM) signals. We decompose the signal $S(t)$, where $S(t) = s_1(t)+s_2(t)$, $s_1(t) = 0.25(2+0.5cos(\pi t))cos(2\pi t^2)$ and $s_2(t)$ is a square wave. Since the first order difference of a square wave is a sparse signal, $D$ is selected as the matrix of the first order difference in Equation (6). Figure 1 shows the extracted subcomponents and the error signals by applying our proposed algorithm and NSP respectively. In Figure 1, (a): the input signal; (b): the logarithmic of Fourier spectrum of the input signal; (c): the first extracted component by the proposed algorithm ; (d): the error signal obtained by subtracting our first extracted component from $s_1(t)$; (e): the second extracted component by the proposed algorithm; (f): the error signal obtained by subtracting our second extracted component from $s_2(t)$; (g): the extracted component by NSP; (h): the error signal by NSP. EMD decomposes $S(t)$ into several harmonic signals. Neither NSP nor EMD can successfully extracted the square wave from $S(t)$.

The second example shows that the proposed algorithm can separate triangular wave signals and AM-FM signals. We decompose the signal $S(t)$, where $S(t) = s_1(t) + s_2(t)$, $s_1(t) = 0.25(1.2 + sin(2\pi t))cos(2\pi(t^2 + 12t))$ and $s_2(t)$ is a triangular wave. Since the second order difference of a triangular wave is a sparse signal, $D$ is selected as the matrix of the second order difference in Equation (6). Figure 2 shows the extracted subcomponents and the error signals

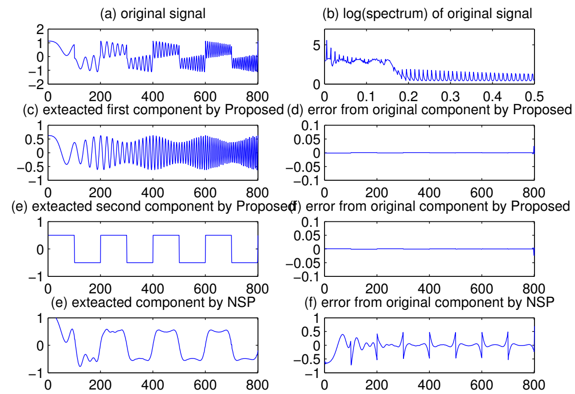


**Fig. 1**. Decompose $0.25(2 + 0.5cos(\pi t))cos(2\pi t^2) + a\ square\ wave$ by applying the proposed approach and NSP respectively.

achieved by our proposed algorithm. By contrast, the corresponding triangular wave components, which are extracted by EMD and NSP respectively, are shown in Figure 3. Both of them are replaced by harmonic signals. Thus the errors of EMD and NSP are much larger than ours. Figure 3(a) shows the original triangular wave and the subcomponents extracted by different methods. Figure 3(b) is a detailed view of Figure 3(a). And through it, we can see more clearly and intuitively the performance of the proposed algorithm.

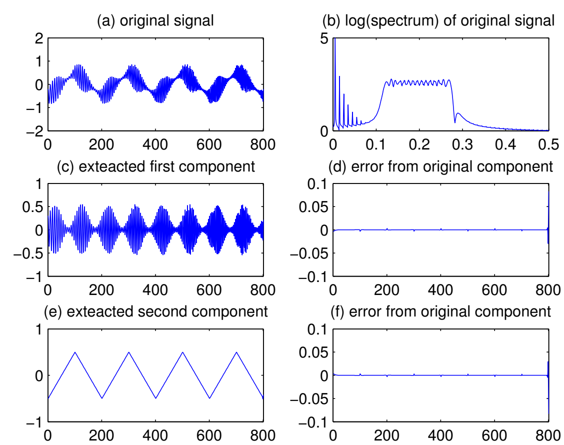


**Fig. 2**. Using the proposed approach to decompose $0.25(1.2+sin(2\pi t))cos(2\pi(t^2 + 12t)) + a\ triangular\ wave.$

We show that the proposed algorithm can separate impulse signals and AM-FM signals in the third example. We decompose the signal $S(t)$, where $S(t) = s_1(t) + s_2(t)$, $s_1(t) = 0.2(1.5 + cos(0.05\pi t))cos(0.01\pi t^2)$ and $s_2(t)$ is an impulse signal. Since an impulse signal itself is a sparse sig-

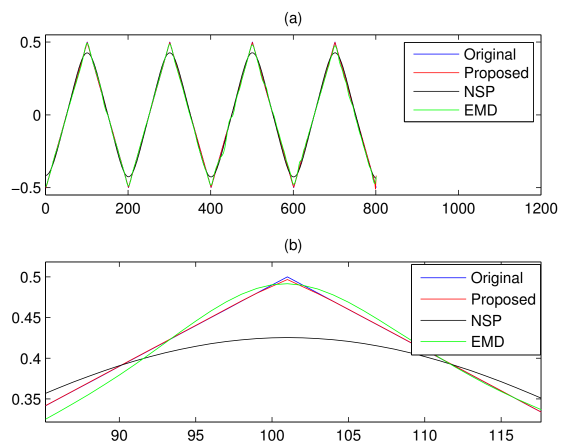


**Fig. 3**. The original triangular wave and the subcomponents extracted by different methods.

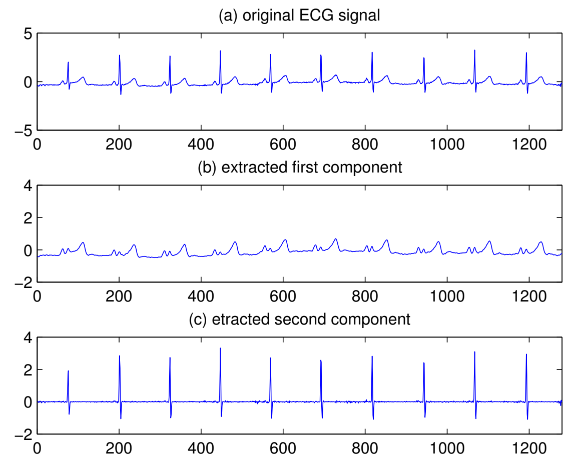


**Fig. 5**. The ECG signal and the components extracted from ECG by applying the proposed approach.

nal, $D$ is selected as an identity matrix in Equation (6). Figure 4 shows the extracted subcomponents and the error signals by using our proposed algorithm. By contrast, neither NSP nor EMD can decompose $S(t)$ effectively.

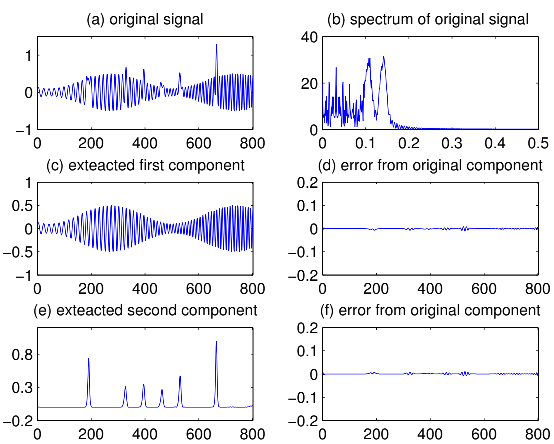


**Fig. 4**. Using the proposed approach to decompose $S(t) = 0.2(1.5 + cos(0.05\pi t))cos(0.01\pi t^2) + an\ impulse\ signal.$

Since in the ECG signals, the QRS complex is a spike-type signal, which satisfies the sparse property described above, we apply our proposed algorithm to separate a real ECG signal [1] [12] in the fourth example, where $D$ is selected as the identity matrix. The separation result is shown in Figure 5, from which, we can find that PT-waves and QRS-waves can be clearly separated from the recordings. This may suggest that our proposed algorithm might be an effective and useful tool for ECG analysis and diagnosis [13, 14].

Finally, we come to make a description. Although we choose $D$ as an identity matrix or the matrix of the first (second) order difference respectively in our implementation, these experiment results almost can be achieved when $D$ is always the matrix of the second order difference.

## 4. CONCLUSION

We propose an operator-based and sparsity-based approach for extracting a sparse signal in some sense from a given target signal. And we provide several examples, including simulated signals and an ECG signal, to demonstrate the performance of our algorithm. We compare the proposed approach with NSP and EMD. The experiment results show that our approach is more effective for this type of signals. In our future work, we shall develop procedures to estimate the regularization parameter in our algorithm.

## 5. RELATION TO PRIOR WORK

The work has focused on the operator-based and sparsity-based approach [8, 7] to adaptively decompose signals. It takes advantage of the adaptive estimation of operator and measures of sparsity by the $\ell_1$ norm. The work by Peng [4, 5, 15] adopted $\ell_2$ norm, which is a measure of energy and smoothness, and is not fit for the type of signals which we are interested in. We apply our proposed algorithm to separate a real ECG signal, and extract PT-waves and QRS-waves from the recordings successfully. Therefore, the approach might be an effective and useful tool for ECG analysis and diagnosis.

[1] `http://physionet.org/physiobank/database/nsrdb/`

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