A PROCEDURE FOR N-D FORNASINI-MARCHESINI STATE-SPACE MODEL REALIZATION BASED ON RIGHT MATRIX FRACTION DESCRIPTION

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ABSTRACT

Since the duality between an n-D transfer matrix and its transpose does not hold for the n-D Fornasini-Marchesini (F-M) (local) state-space model, the existing realization method based on a left matrix fraction description (MFD) is not applicable to an n-D system described by a right MFD. The purpose of this paper is to propose a new constructive procedure that can generate an F-M state-space model realization for an n-D system given by a right MFD. The effectiveness of the proposed procedure will be demonstrated by a numerical example.

Index Terms— Multidimensional systems, Fornasini-Marchesini state-space model, state-space realization, matrix fraction description.

1. INTRODUCTION

The fundamental issue in the multidimensional (n-D) system theory to realize a given rational transfer function or transfer matrix by means of the Roesser state-space model or the Fornasini-Marchesini (F-M) model has attracted considerable research attention during the last two decades [1–4, 6–8, 12, 13], and some recent results have been successfully applied to the LFR (linear fractional representation) uncertainty modeling [8, 12] and the implementation of distributed grid sensor networks [9]. It is well known that, different to the one-dimensional (1-D) case, it is difficult in general to obtain a minimal state-space realization for the n-D ($n \geq 2$) cases [7]. Therefore, it is of great importance to establish realization procedures that can generate low-order n-D state-space realizations.

In particular, for the problem of realizing an n-D system or filter in the F-M model, a constructive method has been proposed by Alpay and Dubi [4]. Since the structural properties of the given rational transfer function was not taken into account, the method of [4] usually generates an F-M model realization with rather high order, and an alternative procedure has been recently established by Cheng et al.[13] which can produce F-M model realizations with much lower realization order than the method of [4]. However, all these methods are based on a left MFD of the given n-D system, and cannot be applied to the case where an n-D system is given by a right MFD as the duality between a transfer matrix and its transpose does not hold for the F-M model. Moreover, it is known that, for a certain system, the realization order obtained for its right MFD is in general different to the one obtained for its left MFD, and in practice the the lower one should be chosen.

The purpose of this paper is to propose a new constructive procedure that can generate an F-M model realization for an n-D system given by a right MFD. Therefore, it can be in general used to obtain an F-M model realization with lower order than the method of [13] for the case where the sum of the column degrees of the given transfer matrix is larger than the sum of its row degrees. The paper is organized as follows. In the next section, some preliminaries for the F-M model realization will be presented. In Section 3, the main results of this paper will be shown. In Section 4, a numerical example will be presented to show the effectiveness of the proposed procedure. Finally, conclusions will be given in Section 5.

2. PRELIMINARIES

The n-D F-M local state-space model [1–4] is described by

$$\begin{aligned} & x(i_1+1, i_2+1, \ldots, i_n+1) \\ & = A_1 x(i_1, i_2+1, \ldots, i_n+1) + \ldots + A_n x(i_1+1, \ldots, i_{n-1}+1, i_n) \\ & \quad + B_1 u(i_1, i_2+1, \ldots, i_n+1) + \ldots + B_n u(i_1+1, \ldots, i_{n-1}+1, i_n) \\ & y(i_1, \ldots, i_n) = C x(i_1, \ldots, i_n) + D u(i_1, \ldots, i_n) \end{aligned} \tag{1}$$

where $x(i_1, \ldots, i_n) \in \mathbb{R}^r, u(i_1, \ldots, i_n) \in \mathbb{R}^l, y(i_1, \ldots, i_n) \in \mathbb{R}^m$ are the (local) state, input and output vectors, respectively; and $A_1, \ldots, A_n \in \mathbb{R}^{r \times r}, B_1, \ldots, B_n \in \mathbb{R}^{r \times l}, C \in \mathbb{R}^{m \times r}, D \in \mathbb{R}^{m \times l}$. r is called the order or dimension of the F-M model. The n-D system (1) is also simply denoted by $(\boldsymbol{A}, \boldsymbol{B}, C, D)$ with $\boldsymbol{A} \triangleq (A_1, \cdots, A_n)$ and $\boldsymbol{B} \triangleq (B_1, \cdots, B_n)$.

The $m \times l$ transfer matrix of (1) is given by

$$H(z_1, \ldots, z_n) = D + C \left(I_r - \sum_{i=1}^{n} z_i A_i \right)^{-1} \left(\sum_{i=1}^{n} z_i B_i \right). \tag{2}$$

Note that $z_1, \ldots, z_n$ can be viewed as the unit backward-shift or delay operators here (see, e.g., [10]).

Let $z = (z_1, \ldots, z_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and denote an n-D monomial (or power product)

$z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ using the multi-index notation $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$. Then, an n-D polynomial $p(z_1,\ldots,z_n)$ can be expressed as

$$p(z) = \sum_{0\le|\alpha|\le k} p_\alpha\, z^\alpha \tag{3}$$

where $k = \deg p(z) \triangleq \max\{|\alpha| \,|\, \forall\alpha \text{ s.t. } p_\alpha \neq 0\}$ [4].

An n-D rational function $h(z) = q(z)/p(z)$, with $q(z)$, $p(z)$ being respectively the numerator and denominator polynomials, is said to be causal if $p(0,\ldots,0)\neq 0$, while an n-D rational matrix $H(z)$ is causal if its every entry is causal [7].

For a given n-D rational transfer matrix $H(z)$, if there is an n-D system described by (1), or simply, $(\boldsymbol{A},\boldsymbol{B},C,D)$ such that (2) holds true, then $(\boldsymbol{A},\boldsymbol{B},C,D)$ is called an F-M (state-space) model realization of $H(z)$. The necessary and sufficient condition for a given $H(z)$ to admit an F-M model realization is that $H(z)$ is causal [8, 13].

In this paper, we will consider the F-M model realization problem for an $m\times l$ n-D transfer matrix given by

$$H(z) = \begin{bmatrix} \frac{q_{11}(z)}{p_1(z)} & \cdots & \frac{q_{1l}(z)}{p_l(z)} \\ \vdots & \ddots & \vdots \\ \frac{q_{m1}(z)}{p_1(z)} & \cdots & \frac{q_{ml}(z)}{p_l(z)} \end{bmatrix} \tag{4}$$

which has the same denominator polynomials $p_j(z)$ $(j = 1,\ldots,l)$ in each column, respectively, and is in fact equivalent to the right MFD

$$H(z) = N_r(z)D_r(z)^{-1} \tag{5}$$

with $D_r(z) = \mathrm{diag}\{p_1(z),\ldots,p_l(z)\}$ and

$$N_r(z) = \begin{bmatrix} q_{11}(z) & q_{12}(z) & \cdots & q_{1l}(z) \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1}(z) & q_{m2}(z) & \cdots & q_{ml}(z) \end{bmatrix}.$$

It should be emphasized that differing from the n-D Roesser model [7], the duality between $H(z)$ and its transpose $H(z)^T$ does not hold for the F-M model. Namely, for

$$\begin{aligned} H(z) &= N_r(z)D_r(z)^{-1} \\ &= D + C\left(I - \sum_{j=1}^n A_j z_j\right)^{-1} \sum_{j=1}^n B_j z_j, \end{aligned} \tag{6}$$

the corresponding transpose is

$$\begin{aligned} H(z)^T &= (D_r(z)^{-1})^T N_r(z)^T \\ &= D^T + \left(\sum_{j=1}^n B_j^T z_j\right)\left(I - \sum_{j=1}^n A_j^T z_j\right)^{-1} C^T \end{aligned} \tag{7}$$

which obviously lose the structure of the F-M model. This fact means that we cannot obtain an F-M model realization for an n-D system specified by a right MFD, say $H(z) = N_r(z)D_r(z)^{-1}$ via the realization for the left MFD $H(z)^T = (D_r(z)^T)^{-1}(N_r(z)^T)$ by using the known methods of [4, 13]. Therefore, we have to consider the realization problem for a right MFD or a system given in the form of (4), separately.

3. THE MAIN RESULTS

In this section, we first consider the special case of an n-D system with 1 input and m outputs, whose transfer matrix is accordingly given by a rational column vector, and then show that the realization for a general $m\times l$ transfer matrix can be obtained by constructing the realization of each column of it. Specifically, let the rational column vector $\boldsymbol{h}(z)$ be given by

$$\boldsymbol{h}(z) = [h_1(z),\ldots,h_m(z)]^T$$

where $h_j(z) = q_j(z)/p(z) (j = 1,\ldots,m)$. The column degree of $\boldsymbol{h}(z)$ is defined as $k = \max\{\deg p(z), \deg q_i(z)\ (i = 1,\ldots,l)\}$. Moreover, let $\boldsymbol{h}(z)$ be causal, i.e., $p(0,\ldots,0)\neq 0$ and assume, without loss of generality, that $p(0,\ldots,0) = 1$.

The basic idea adopted in this paper can be briefly stated as follows. Since $\boldsymbol{h}(z)$ is causal, there exists an F-M model realization $(\boldsymbol{A},\boldsymbol{B},C,D)$ such that

$$\boldsymbol{h}(z) = D + C\left(I_r - \sum_{i=1}^n z_i A_i\right)^{-1} \sum_{i=1}^n z_i B_i. \tag{8}$$

It is easy to see that $D = \boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{h}(0,\ldots,0)$. Then, letting

$$\tilde{G}(z) = \left(I_r - \sum_{i=1}^n z_i A_i\right)^{-1} \sum_{i=1}^n z_i B_i, \tag{9}$$

we have that

$$\boldsymbol{h}(z) - \boldsymbol{h}(0) = C\tilde{G}(z). \tag{10}$$

Equation (9) can also be expressed as

$$\tilde{G}(z) = \sum_{i=1}^n z_i A_i \tilde{G}(z) + \sum_{i=1}^n z_i B_i. \tag{11}$$

It can be seen from the above arguments that if we can find a suitable rational column vector $\tilde{G}(z)$ for the given $\boldsymbol{h}(z)$ such that (10) and (11) hold true, then it would be easy to construct $A_1,\ldots,A_n,B_1,\ldots,B_n$ from (11) and C from (10).

Now, we have the questions what are the conditions such a $\tilde{G}(z)$ should satisfy and how it can be constructed. We first consider the first question. Let

$$\tilde{G}(z) = [\beta_1(z),\ldots,\beta_{\tilde{r}}(z)]^T. \tag{12}$$

where $\tilde{r}$ is a positive integer, $\beta_i(z) = z^{\alpha^{(i)}}/p(z)$, $\alpha^{(i)} = (\alpha_1^{(i)},\ldots,\alpha_n^{(i)}) \in \mathbb{Z}^n (i = 1,\ldots,\tilde{r})$ with $\max\{|\alpha^{(1)}|,\ldots,|\alpha^{(\tilde{r})}|\} = k$. Then, the following result can be given.

Theorem 3.1. *If the n-D rational column vector $\tilde{G}(z)$ described by (12) satisfies the following conditions, then the relations of (10) and (11) will hold true, and thus there exists an F-M model realization for the given $\boldsymbol{h}(z)$.*

(a) *All the n-D monomials occurring in $\boldsymbol{h}(z)$ are included as numerators in some entries of $\tilde{G}(z)$;*

(b) *Each entry of $\tilde{G}(z)$, except the ones having the form $\frac{z_i}{p(z)}, i \in \{1, \ldots, n\}$, can be obtained from another entry of $\tilde{G}(z)$ by multiplying some z_j, $j \in \{1, \ldots, n\}$.*

(**Proof:** Omitted.)

Based on the conditions obtained in Theorem 1, we can now give a procedure for the construction of such $\tilde{G}(z)$ and the corresponding F-M model realization, which gives an answer to the second question raised previously. Since the realization for the general transfer matrix can be obtained by constructing the realizations for all the columns respectively and then combining them into the overall realization for the transfer matrix, we directly give the matrix version of the realization procedure here.

Let

$$H(z)=\begin{bmatrix} \frac{q_{11}(z)}{p_1(z)} & \cdots & \frac{q_{1l}(z)}{p_l(z)} \\ \frac{q_{21}(z)}{p_1(z)} & \cdots & \frac{q_{2l}(z)}{p_l(z)} \\ \vdots & & \vdots \\ \frac{q_{m1}(z)}{p_1(z)} & \cdots & \frac{q_{ml}(z)}{p_l(z)} \end{bmatrix} \triangleq \left[\boldsymbol{h}_1(z) \cdots \boldsymbol{h}_l(z)\right] \qquad (13)$$

where $p_j(\mathbf{0}) = 1 \neq 0$ $(j = 1, \ldots, l)$. Denote the column degree of the jth column of $H(z)$ by k_j, i.e., $k_j = \max\{\deg p_j(z), \deg q_{ij}(z), i = 1, \ldots, m\}$.

Realization Procedure:

Step 0: $j = 0$.

Step 1: $j = j + 1$. If $j > l$, go to Step 6.

Otherwise, execute the following operations: Collect all the monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \alpha_1, \ldots, \alpha_n \in \{0, 1, \ldots, k_j\}$, $|\alpha| \leq k_j$, occurring in the entries of $\boldsymbol{h}_j(z)$ with non-zero coefficients, and denote by Γ_j the collected monomials and by $\tilde{r}_j$ the size of Γ_j. Construct a $\tilde{r}_j \times 1$ column vector $\tilde{G}_j(z)$ as

$$\tilde{G}_j(z) = [\beta_{1j}(z) \cdots \beta_{\tilde{r}_j j}(z)]^T$$

where each $\beta_{sj}(z), s \in \{1, \ldots, \tilde{r}_j\}$, is in the form $\frac{z^\alpha}{p_j(z)}$ with $z^\alpha \in \Gamma_j$.

Note that condition (a) is satisfied for $\boldsymbol{h}_j(z)$ and $\tilde{G}_j(z)$.

Step 2: Check whether $\tilde{G}_j(z)$ satisfies condition (b). If some entry, say $\beta_{ij}(z) = \frac{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}{p_j(z)}, i \in \{1, \ldots, \tilde{r}_j\}$, does not satisfy condition (b), then insert a new entry $\beta_{(\tilde{r}+1)j}(z) = \frac{z_1^{\alpha_1} \cdots z_t^{\alpha_t - 1} \cdots z_n^{\alpha_n}}{p_j(z)}, \alpha_t \geq 1, t \in \{1, \ldots, n\}$ into $\tilde{G}_j(z)$ and then set $\tilde{r}_j = \tilde{r}_j + 1$. It is easy to see that $\beta_{ij}(z) = z_t \beta_{(\tilde{r}+1)j}(z)$. Repeat the operation until $\tilde{G}_j(z)$ satisfies condition (b). Redefine $\tilde{r}_j$ as the (row) dimension of the finally updated $\tilde{G}_j(z)$.

Step 3: Express $\boldsymbol{h}_j(z)$ as

$$\boldsymbol{h}_j(z) - \boldsymbol{h}_j(\mathbf{0}) = C_j \tilde{G}_j(z) \qquad (14)$$

where C_j can be determined by the coefficients of the numerator polynomials in $\boldsymbol{h}_j(z)$. This operation is always possible as condition (a) is satisfied.

Step 4: Construct the $\tilde{r}_j \times 1$ column vector B_{ij} for $i = 1, \ldots, n$ in the following way: Set initially all the entries of B_{ij} to zero. If the sth entry of $\tilde{G}_j(z)$, i.e., $\beta_{sj}(z)$ satisfies $\beta_{sj}(z) = \frac{z_i}{p_j(z)}$, then change the sth entry of B_{ij} to 1, i.e., $B_{ij}(s) = 1$, and keep all the other entries of B_{ij} to be zero.

Step 5: Let

$$\hat{G}_j(z) = \tilde{G}_j(z) - \sum_{i=1}^{n} B_{ij} z_i = [\hat{\beta}_{1j}, \ldots, \hat{\beta}_{\tilde{r}_j j}(z)]^T.$$

For $v = 1, \ldots, \tilde{r}_j$, if $\hat{\beta}_{vj}(z) = \tilde{\beta}_{vj}(z)$ and $\tilde{\beta}_{vj}(z) = z_i \tilde{\beta}_{sj}(z), i \in \{1, \ldots, n\}, s \in \{1, \ldots, \tilde{r}_j\}, s \neq j$ express $\tilde{\beta}_{vj}(z)$

$$\tilde{\beta}_{vj}(z) = (A_{1vj} z_1 + \cdots + A_{nvj} z_n) \tilde{G}_j(z)$$

with A_{ivj} being a $1 \times \tilde{r}_j$ vector having 1 at the sth position and 0 at all the other position, and $A_{tvj} = \mathbf{0}_{1 \times \tilde{r}_j}, t = 1, \ldots, n, t \neq i$, if $\hat{\beta}_{vj}(z) \neq \tilde{\beta}_{vj}$, the $\hat{\beta}_{vj}(z)$ must have the from $\tilde{\beta}_{vj}(z) = \frac{z_i \tilde{q}_j}{p_j(z)}$ and it can be expressed as

$$\hat{\beta}_{vj}(z) = (A_{1vj} z_1 + \cdots + A_{nvj} z_n) \tilde{G}(z)$$

with A_{ivj} being determined by the coefficients of $\tilde{q}_j(z)$ and $A_{tvj} = \mathbf{0}_{1 \times \tilde{r}_j}, t = 1, \ldots, n, t \neq i$.

Then, it is easy to see that

$$\tilde{G}_j(z) - \sum_{i=1}^{n} B_{ij} z_i = (A_{1j} z_1 + \cdots + A_{nj} z_n) \tilde{G}_j(z), \quad (15)$$

where $A_{ij} = [A_{i1j}^T \ \cdots \ A_{i\tilde{r}_j j}^T]^T$.

Return to Step 1.

Step 6: For $i = 1, \ldots, n$, set

$$A_i = \text{diag}\{A_{i1} \cdots A_{il}\} \in \mathbb{R}^{\tilde{r} \times \tilde{r}},$$
$$B_i = \text{diag}\{B_{i1} \cdots B_{il}\} \in \mathbb{R}^{\tilde{r} \times l}$$

where $\tilde{r} = \tilde{r}_1 + \cdots + \tilde{r}_l$. Let

$$\tilde{G}(z) = \text{diag}\{\tilde{G}_1(z), \ldots, \tilde{G}_l(z)\} \qquad (16)$$

and it is easy to see that the row dimension of $\tilde{G}(z)$ is just $\tilde{r}$. It then follows from (14) that

$$\tilde{G}(z) - \sum_{i=1}^{n} z_i B_i = \sum_{i=1}^{n} z_i A_i \tilde{G}(z)$$

or equivalently,

$$\tilde{G}(z) = \left(I_{\tilde{r}} - \sum_{i=1}^{n} z_i A_i\right)^{-1} \sum_{i=1}^{n} z_i B_i. \qquad (17)$$

Set $C = [C_1 \ \cdots \ C_l] \in \mathbb{R}^{\tilde{r}\times m}$ and $D = H(\mathbf{0})$. It can be seen from (14), (16) and (17) that

$$H(z) = C\left(I_{\tilde{r}} - \sum_{i=1}^{n} z_i A_i\right)^{-1} \sum_{i=1}^{n} z_i B_i + D. \qquad (18)$$

By (18), we see that the constructed $(\boldsymbol{A}, \boldsymbol{B}, C, D)$ gives an F-M realization of $H(z)$ with $r = \tilde{r}$, $\boldsymbol{A} = (A_1 \cdots A_n)$, $\boldsymbol{B} = (B_1 \cdots B_n)$.

4. EXAMPLE

A simple example is presented here to show the effectiveness of the proposed realization procedure. Consider the following 3×2 transfer matrix of a 3-D system:

$$H(z) = \begin{bmatrix} \frac{n_{11}z_2+n_{12}z_2z_3}{p_1(z)} & \frac{n_{41}z_1z_2+n_{42}z_2z_3}{p_2(z)} \\ \frac{n_{21}z_1+n_{22}z_3}{p_1(z)} & \frac{n_{51}z_1z_2z_3}{p_2(z)} \\ \frac{n_{31}z_1z_2}{p_1(z)} & \frac{n_{61}z_2z_3}{p_2(z)} \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{h}_1(z) & \boldsymbol{h}_2(z) \end{bmatrix}.$$

where $z = (z_1, z_2, z_3)$,

$$p_1(z) = 1 + d_{11}z_2 + d_{12}z_3 - d_{13}z_1z_2,$$
$$p_2(z) = 1 - d_{21}z_1 - d_{22}z_1z_2 - d_{23}z_1z_2z_3.$$

The column degrees of $H(z)$ are $k_1 = 2,\ k_2 = 3$.

Then, by the proposed procedure, we can obtain the following results for $\boldsymbol{h}_1(z)$.

$$\tilde{G}_1(z) = \begin{bmatrix} \frac{z_3z_2}{p_1(z)} & \frac{z_2z_1}{p_1(z)} & \frac{z_3}{p_1(z)} & \frac{z_2}{p_1(z)} & \frac{z_1}{p_1(z)} \end{bmatrix}^T,$$

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_{13} & -d_{12} & -d_{11} & 0 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_{13} & -d_{12} & -d_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_{13} & -d_{12} & -d_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B_{13} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} n_{12} & 0 & 0 & n_{11} & 0 \\ 0 & 0 & n_{22} & 0 & n_{21} \\ 0 & n_{31} & 0 & 0 & 0 \end{bmatrix}, \tilde{r}_1 = 5. \qquad (19)$$

Similarly, the following results can be obtained for $\boldsymbol{h}_2(z)$.

$$G_2(z) = \begin{bmatrix} \frac{z_3z_2z_1}{p_2(z)} & \frac{z_3z_2}{p_2(z)} & \frac{z_2z_1}{p_2(z)} & \frac{z_3}{p_2(z)} & \frac{z_1}{p_2(z)} \end{bmatrix}^T,$$

$$A_{21} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d_{23} & 0 & d_{22} & 0 & d_{21} \end{bmatrix},\ B_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d_{23} & 0 & d_{22} & 0 & d_{21} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B_{23} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & n_{42} & n_{41} & 0 & 0 \\ n_{51} & 0 & 0 & 0 & 0 \\ 0 & n_{61} & 0 & 0 & 0 \end{bmatrix}, \tilde{r}_2 = 5.$$

By using the above results, we can now construct an F-M realization for the given transfer matrix $H(z)$ as

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{21} \end{bmatrix}, A_2 = \begin{bmatrix} A_{12} & 0 \\ 0 & A_{22} \end{bmatrix}, A_3 = \begin{bmatrix} A_{13} & 0 \\ 0 & A_{23} \end{bmatrix},$$
$$B_1 = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{21} \end{bmatrix}, B_2 = \begin{bmatrix} B_{12} & 0 \\ 0 & B_{22} \end{bmatrix}, B_3 = \begin{bmatrix} B_{13} & 0 \\ 0 & B_{23} \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D = H(0,0,0) = \mathbf{0}_{2\times 2}.$$

The realization order obtained here by the proposed procedure is $\tilde{r} = \tilde{r}_1 + \tilde{r}_2 = 5 + 5 = 10$. In contrast, the order of the realization obtained by first expressing $H(z)$ as a left MFD and then applying the method of [13] is 48. That is, for a certain system, the realization order obtained for its right MFD is in general different to the one obtained for its left MFD, and thus the realization with the lower order should be chosen.

5. CONCLUSIONS

A constructive procedure has been proposed for the F-M model realization of an n-D system described by a right MFD. In particular, we have first clarified the conditions that the rational function matrix $\tilde{G}(z)$ defined in (12) has to satisfy, and then based on these conditions, shown a procedure for constructing $\tilde{G}(z)$ and the corresponding F-M model realization for an n-D system given by a right MFD. Finally, a numerical example has been presented to illustrate the effectiveness of the proposed procedure.

6. REFERENCES

[1] Fornasini, E., & Marchesini, G. (1976). "State-space realization theory of two-dimensional filters," *IEEE Transactions on Automatic Control*, vol. AC-21, no. 4, pp. 484–492, 1976.

[2] E. Fornasini and G. Marchesini, "Computation of Reachable and Observable Realization of Spatial Filters," *Int. J. Control*, vol. 25, no. 4, pp. 621–635, 1977.

[3] M. Bisiacco, E. Fornasini and G. Marchesini, "Dynamic regulation of 2-D systems: a state-space approach," *Linear Algebra and Its Applications*, vol. 122/124, pp. 195–218, 1989.

[4] D. Alpay and C. Dubi, "A realization theorem for rational functions of several complex variables," *Systems & Control Letters*, vol. 49, pp. 225–229, 2003.

[5] L. Xu, L. Wu, Q. Wu, Z. Lin and Y. Xiao, "On realization of 2D discrete systems by Fornasini-Marchesini model," *International Journal of Control, Automation, and Systems*, vol. 3, no. 4, pp. 631–639, 2005.

[6] L. Xu, Q. Wu, Z. Lin and Y. Xiao, "A New Constructive Procedure for 2-D Coprime Realization in Fornasini-Marchesini Model," *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 54, no. 9, pp. 2061–2069, 2007.

[7] L. Xu, H. Fan, Z. Lin and N. K. Bose, "A Direct-Construction Approach to Multidimensional Realization and LFR Uncertainty Modeling," *Multidimensional Systems and Signal Processing*, vol. 19, no. 3-4, pp. 323–359, 2008.

[8] L. Xu, H. Fan, Z. Lin, Y. Xiao, "Coefficient-dependent direct-construction approach to realization of multidimensional systems in Roesser model," *Multidimensional Systems and Signal Processing*, vol. 22, no. 1-3, pp. 97–129, 2011.

[9] M. G. B. Sumanasena and P. H. Bauer, "Realization Using the Fornasini- Marchesini Model for Implementations in Distributed Grid Sensor Networks," IEEE Transactions on Circuits and Systems I, vol. 58, issue 11, pp. 2708–2717, 2011.

[10] Z. Lin, "Feedback stabilization of MIMO n-D linear systems," *Multidimensional Systems and Signal Processing*, vol. 9, no. 2, pp. 149–172, 1998.

[11] L. Xu and S. Yan, "A New Elementary Operation Approach to Multidimensional Realization and LFR Uncertainty Modeling: The SISO Case," *Multidimensional Systems and Signal Processing*, vol. 21, no. 4, pp. 343–372, 2010.

[12] L. Xu, S. Yan, Z. Lin, and S. Matsushita, "A New Elementary Operation Approach to Multidimensional Realization and LFR Uncertainty Modeling: the MIMO Case," *IEEE Trans. Circuits and Systems I*, vol. 59, issue 3, pp. 638–651, 2012.

[13] H. Cheng, T. Saito, S. Matushita, L. Xu, "Realization of Multidimensional Systems in Fornasini-Marchesini State-space Model," *Multidimensional Systems and Signal Processing*, vol. 22, no. 4, pp. 319–333, 2011.