THE DIRECT BATCH GENERATION OF HERMITE-GAUSSIAN-LIKE EIGENVECTORS OF THE DFT MATRIX USING THE NOTION OF MATRIX PSEUDOINVERSE

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Abstract - The discrete fractional Fourier transform (DFRFT) depends heavily on the availability of Hermite-Gaussian-like (HGL) orthonormal eigenvectors of the DFT matrix F. The direct batch evaluation by constrained optimization algorithm (DBEOA) was developed under the assumption that the orthogonal projection of the approximate eigenvectors on the corresponding eigenspaces results in linearly independent vectors. The present paper handles the case when those vectors are not linearly independent which happens when the order N of matrix F is large. A more general treatment of the batch generation of HGL eigenvectors is presented and the notion of matrix pseudoinverse is used for solving the linear system which arises in the solution of the constrained optimization problem. The contributed technique is termed the Direct Batch Evaluation by constrained Optimization Algorithm using the notion of matrix Pseudoinverse (DBEOAP). The simulation results show that the DBEOAP results in smaller values of the norms of the approximation error vectors than those obtained when applying the DBEOA.

***Index Terms*: Discrete fractional Fourier transform (DFRFT), Hermite-Gaussian-like (HGL) eigenvectors, DFT matrix, singular value decomposition (SVD), matrix pseudoinverse.**

I. INTRODUCTION

The discrete fractional Fourier transform (DFRFT) has emerged as a discrete counterpart of the fractional Fourier transform (FRFT). The latter is a generalization of the continuous-time Fourier transform and can be viewed as a representation of a signal along an arbitrary axis in the time-frequency plane making an angle α with the time axis where α is related to the order a of the fractional transform by

$$\alpha = 0.5\,\pi\,a\,. \tag{1}$$

The kernel matrix $\mathbf{F^a}$ of the DFRFT is defined by[1] [1]:

$$\mathbf{F^a} = \mathbf{\hat{U}D^a\hat{U}^H}\,. \tag{2}$$

This definition is based on having a modal decomposition of the DFT matrix **F** expressed as:

$$\mathbf{F} = \mathbf{\hat{U}D\hat{U}^H} \tag{3}$$

where $\mathbf{\hat{U}}$ is a unitary modal matrix of **F** and **D** is a diagonal matrix having the eigenvalues on its main diagonal. Since the DFRFT has been developed as a computational machinery for the FRFT, it should approximate its analog counterpart. Since the Hermite-Gaussian functions are the eigenfunctions of the FRFT, it is highly desirable that the eigenvectors of matrix **F** - which are also eigenvectors of matrix $\mathbf{F^a}$ - be as close as possible to samples of the Hermite-Gaussian functions. Candan et al. [2] developed orthonormal eigenvectors of matrix **F** by a block diagonalization of a nearly tridiagonal matrix **S** which commutes with matrix **F**. Their work has been put on a more rigorous foundation by Hanna, Seif and Ahmed [3]. Pei, Yeh and Tseng [4] looked at the eigenvectors of matrix **S** as only initial ones and looked for superior ones - in the sense of better approximating samples of the Hermite-Gaussian functions - by applying either the gram-Schmidt algorithm (GSA) or the orthogonal procrustes algorithm (OPA). Hanna, Seif and Ahmed developed a third technique, namely the sequential orthogonal procrustes algorithm (SOPA) for the same purpose [5]. Moreover they arrived at an implementation of the GSA, OPA and SOPA based only on the orthogonal projection matrices on the eigenspaces of matrix **F** without having to first generate initial eigenvectors [6].

A direct attack on the problem of the *sequential* generation of Hermite-Gaussian-like eigenvectors of matrix **F** resulted in developing the Direct *Sequential* Evaluation by constrained Optimization Algorithm (DSEOA) [7]. Similarly a direct attack on the problem of the *batch* generation of the desirable optimal eigenvectors resulted in developing the Direct *Batch* Evaluation by constrained Optimization Algorithm (DBEOA) [8]. In the development of the DBEOA, a square Hermitian matrix $\mathbf{W^2}$ emerged - as will be delineated in Section II - and matrix **W** was needed in order to generate the orthonormal matrix $\mathbf{\hat{U}_k}$ whose columns form the desired optimal basis of the k^{th} eigenspace of matrix **F**. Matrix $\mathbf{W^2}$ was taken for granted to be nonsingular and the DBEOA depended on first finding **W** and next finding its inverse.

The main objective of the present paper is to deal with the degenerate case when $\mathbf{W^2}$ is singular since for large values of the

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[1] The superscripts ^{T} , * , ^{H} respectively denote the transpose, the complex conjugate and the Hermitian transpose (i.e. the complex conjugate transpose).

order N of the DFT matrix **F**, matrix $\mathbf{W^2}$ becomes at least algorithmically singular. The notion of the Moore-Penrose matrix pseudoinverse will be utilized for finding the optimal Hermite-Gaussian-like (HGL) eigenvectors of **F**.

In section II the batch generation of optimal HGL eigenvectors will be delineated and in section III matrix **W** will be evaluated and the notion of pseudoinverse will be resorted to for the sake of finding the target orthonormal matrix of eigenvectors for each eigenspace separately. In section IV some simulation results will be presented demonstrating the need for the contributed Direct Batch Evaluation by constrained Optimization Algorithm using the notion of matrix Pseudoinverse (DBEOAP).

II. Batch Generation of Optimal Eigenvectors

Let r_k be the dimension of the k[th] eigenspace E_k of the DFT matrix **F** of order N pertaining to the eigenvalue λ_k , $k = 1,\cdots,4$. (Matrix **F** has only 4 distinct eigenvalues). Let $\hat{\mathbf{U}}_\mathbf{k}$ be the target optimal $N \text{ x } r_k$ matrix whose columns are orthonormal basis of E_k and let $\mathbf{U_k}$ be the corresponding matrix whose columns are approximate (since they are not exact) but desirable (since they have the key feature of being samples of the Hermite-Gaussian functions) eigenvectors of **F** pertaining to λ_k. The unitarity of **F** implies the orthogonality of the eigenspaces E_k , $k = 1,\cdots,4$ and consequently the problem of generating optimal orthonormal eigenvectors of **F** decouples into 4 separate problems.

Given matrix $\mathbf{U_k}$, matrix $\hat{\mathbf{U}}_\mathbf{k}$ will be evaluated by minimizing the square Frobenius norm:

$$J_a = \left\| \mathbf{U_k} - \hat{\mathbf{U}}_\mathbf{k} \right\|_F^2 \tag{4}$$

subject to the constraints:

$$\left(\mathbf{F} - \lambda_k \mathbf{I}\right)\hat{\mathbf{U}}_\mathbf{k} = \mathbf{0} \tag{5}$$

and

$$\hat{\mathbf{U}}_\mathbf{k}^\mathbf{H} \hat{\mathbf{U}}_\mathbf{k} - \mathbf{I}_{\mathbf{r_k}} = \mathbf{0} \tag{6}$$

Constraint (5) guarantees that the columns of $\hat{\mathbf{U}}_\mathbf{k}$ are eigenvectors of **F** and constraint (6) guarantees their orthonormality. It was shown that the solution of this constrained optimization problem is given by [8]:

$$\hat{\mathbf{U}}_\mathbf{k} \mathbf{W} = \widetilde{\mathbf{U}}_\mathbf{k} \tag{7}$$

where

$$\widetilde{\mathbf{U}}_\mathbf{k} = \mathbf{P_k} \mathbf{U_k} \tag{8}$$

with $\mathbf{P_k}$ being the orthogonal projection matrix on the k[th] eigenspace of **F** given explicitly in [5]. It was also shown that the square matrix **W** of order r_k appearing in (7) is Hermitian and satisfies:

$$\mathbf{W^2} = \widetilde{\mathbf{U}}_\mathbf{k}^\mathbf{H} \widetilde{\mathbf{U}}_\mathbf{k} \,. \tag{9}$$

In order to solve the linear system (7), one has to first evaluate matrix **W** based on the availability of $\mathbf{W^2}$.

III. MATRIX W AND ITS PSEUDOINVERSE

The singular value decomposition (SVD) of the known matrix $\mathbf{W^2}$ is given by:

$$\mathbf{W^2} = \mathbf{T \Lambda T^H} \tag{10}$$

where **T** is a unitary matrix and Λ is a diagonal matrix that can be expressed as:

$$\Lambda = \begin{pmatrix} \Lambda_{p_k} & 0 \\ 0 & 0 \end{pmatrix}. \tag{11}$$

In the above equation the diagonal matrix Λ_{p_k} is given by:

$$\Lambda_{p_k} = Diag\left\{d_1,\cdots,d_{p_k}\right\} \tag{12}$$

where $d_1 \geq d_2 \geq \cdots \geq d_{p_k} > 0$. For the sake of generality it has been assumed that the $N \text{ x } r_k$ matrix $\widetilde{\mathbf{U}}_\mathbf{k}$ has rank p_k where $p_k \leq r_k$. Since matrix $\mathbf{W^2}$ defined by (9) has the same rank p_k of matrix $\widetilde{\mathbf{U}}_\mathbf{k}$, its SVD has the peculiar form portrayed by (10)-(12).

It should be mentioned that although (10) can be viewed as an eigenvalue decomposition of the Hermitian matrix $\mathbf{W^2}$, one is advised to apply the SVD because if $\mathbf{W^2}$ has a repeated eigenvalue, an eigendecomposition routine available in a general software package is not guaranteed to generate orthogonal eigenvectors corresponding to a repeated eigenvalue. The family of square roots of (10) is given by:

$$\mathbf{W} = \mathbf{TST^H} \tag{13}$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{S_{p_k}} & 0 \\ 0 & 0 \end{pmatrix} \tag{14}$$

and

$$\mathbf{S_{p_k}} = Diag\left\{\pm\sqrt{d_1},\cdots,\pm\sqrt{d_{p_k}}\right\}. \tag{15}$$

The minimum-norm solution of the linear system (7) is given by:

$$\hat{\mathbf{U}}_\mathbf{k} = \widetilde{\mathbf{U}}_\mathbf{k} \mathbf{W}^\dagger \tag{16}$$

where $\mathbf{W}^\dagger$ is the Moore-Penrose pseudoinverse of **W** given by [9]:

$$\mathbf{W}^\dagger = \mathbf{T} \begin{pmatrix} \mathbf{S_{p_k}^{-1}} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{T^H}. \tag{17}$$

Equation (15) implies the existence of 2^{p_k} solutions. In order to single out the unique solution which minimizes criterion (4), one starts by expressing the latter in the form:

$$J_a = tr\left(\mathbf{U_k^H U_k}\right) + r_k - J_b \tag{18}$$

where

$$J_b = 2\,\mathrm{Real}\left\{tr\left(\hat{\mathbf{U}}_\mathbf{k}^\mathbf{H}\mathbf{U_k}\right)\right\}. \tag{19}$$

In the above equation $tr(\;)$ is the trace of a matrix and $\mathrm{Real}\{\;\}$ denotes the real part. The fact that **W** is Hermitian implies that $\mathbf{W}^\dagger$ is Hermitian and consequently (16) and (8) result in:

$$\hat{\mathbf{U}}_\mathbf{k}^\mathbf{H}\mathbf{U_k} = \mathbf{W}^\dagger \tilde{\mathbf{U}}_\mathbf{k}^\mathbf{H}\mathbf{U_k} = \mathbf{W}^\dagger \mathbf{U_k^H P_k^H U_k}. \tag{20}$$

Since the orthogonal projection matrix $\mathbf{P_k}$ is both Hermitian and idempotent, (8) and (9) result in:

$$\mathbf{U_k^H P_k^H U_k} = \mathbf{U_k^H P_k U_k} = \mathbf{U_k^H P_k^2 U_k} = \mathbf{U_k^H P_k^H P_k U_k} = \mathbf{W^2} \tag{21}$$

Substituting (21) in (20) and utilizing (10), (11) and (17), one gets:

$$\hat{\mathbf{U}}_\mathbf{k}^\mathbf{H}\mathbf{U_k} = \mathbf{W}^\dagger\mathbf{W^2} = \mathbf{T}\begin{pmatrix} \mathbf{S}_{\mathbf{p_k}}^{-1}\Lambda_{\mathbf{p_k}} & 0 \\ 0 & 0 \end{pmatrix}\mathbf{T^H}. \tag{22}$$

Utilizing the properties of the trace of a matrix and using (12) and (15), one gets:

$$tr\left(\hat{\mathbf{U}}_\mathbf{k}^\mathbf{H}\mathbf{U_k}\right) = tr\left(\mathbf{S}_{\mathbf{p_k}}^{-1}\Lambda_{\mathbf{p_k}}\right) = \sum_{i=1}^{p_k}\left(\pm\sqrt{d_i}\right). \tag{23}$$

The above equation implies that the unique matrix $\mathbf{S}_{\mathbf{p_k}}$ out of the family of (15) which maximizes criterion (19) - and consequently minimizes criterion (4) - is given by:

$$\mathbf{S}_{\mathbf{p_k}} = Diag\left\{\sqrt{d_1},\cdots,\sqrt{d_{p_k}}\right\}. \tag{24}$$

Based on the above findings, the Direct Batch Evaluation by constrained Optimization Algorithm using the notion of matrix Pseudoinverse (DBEOAP) can be summarized in the following steps:

1. Form matrix $\mathbf{W^2}$ as $\mathbf{W^2} = \mathbf{U_k^H}\left(\mathbf{P_k U_k}\right)$.
2. Find the SVD of $\mathbf{W^2}$ as given by (10) and (11).
3. Compute $\mathbf{W}^\dagger$ according to (17) where $\mathbf{S}_{\mathbf{p_k}}$ is given by (24).
4. Evaluate $\hat{\mathbf{U}}_\mathbf{k}$ as $\hat{\mathbf{U}}_\mathbf{k} = \left(\mathbf{P_k U_k}\right)\mathbf{W}^\dagger$.

It remains to discuss the numerical evaluation of the rank p_k of matrix **W**. Upon finding the SVD of matrix $\mathbf{W^2}$ as given by (10)-(12), the diagonal elements of Λ are arranged in descending order as $d_1 \geq d_2 \geq \cdots \geq d_{r_k} \geq 0$. A singular value which is theoretically zero, will numerically be found to be negligibly small rather than being zero. One should set a threshold such that singular values equal to or below it will be regarded zeros. This threshold is given by:

$$threshold = r_k * d_1 * eps * mtol \tag{25}$$

where eps is the built-in MATLAB constant called the floating-point relative accuracy defined as the distance from 1.0 to the next larger double-precision number[2] ,i.e. $eps = 2\wedge(-52) \cong 2.22e-016$. The arbitrary multiplying factor $mtol$ appearing in (25) has been introduced in order to allow the user more flexibility in coping with the numerically challenging rank determination problem. As a rule of thumb it has been found after tedious experimentation that the value $mtol = 1e006$ suffices in most cases.

The contributed DBEOAP is general since it applies to the case of a rank-deficient $N \text{ x } r_k$ matrix $\tilde{\mathbf{U}}_\mathbf{k}$ having rank p_k where $p_k < r_k$. The previously published DBEOA [8] corresponds to the full-rank case characterized by $p_k = r_k$ where there is no need for the notion of the pseudoinverse. Consequently the contributed DBEOAP is a generalization of the DBEOA.

IV. SIMULATION RESULTS

The contributed DBEOAP has been applied for generating HGL eigenvectors of matrix **F** of orders $N = 16, 32, 64, 128$. In each case the rank of matrix $\mathbf{W^2}$ for each of the four eigenspaces has been found to equal the dimension of that space, i.e. $p_k = r_k$, $k = 1,\cdots,4$. However for larger values of N, it has been found that $p_k < r_k$. Table 1 shows the discrepancy between r_k and p_k for the four eigenspaces for $N = 256, 512, 1024, 2048$. This testifies to the need for contributing the DBEOAP for the rank-deficient case in contrast to the DBEOA which should be only used for the full-rank case. Figure 1 shows a plot of the norm of the approximation error vectors $\mathbf{e_m} \equiv \left(\hat{\mathbf{u}}_\mathbf{m} - \mathbf{u_m}\right), m = 1,\cdots,N$ between the exact and approximate eigenvectors of matrix **F** of order $N = 1024$ when the DBEOAP and DBEOA are applied. The spiky appearance of the plot of the DBEOA is due to treating matrix $\mathbf{W^2}$ as nonsingular although it is at least algorithmically singular.

V. CONCLUSION

The direct batch evaluation technique of optimal HGL eigenvectors of the DFT matrix **F** has been generalized to handle the case of a rank-deficient matrix $\tilde{\mathbf{U}}_\mathbf{k}$ whose columns are the orthogonal projection of the approximate eigenvectors on the corresponding eigenspace. The notion of matrix pseudoinverse has been used in evaluating the target HGL eigenvectors. The simulation results testify to the numerical superiority of the contributed DBEOAP to the DBEOA which should be applied solely in the full-rank case.

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[2] Here the MATLAB notation is used where $e-016$ stands for $10\wedge(-16)$.

Table 1: The discrepancy between the dimensions of the eigenspaces of the DFT matrix and the rank of matrix W^2.

N	Serial number k of the eigenspace	Dimension r_k of the k^{th} eigenspace	Rank p_k of matrix W^2
256	1	65	63
	2	64	62
	3	64	62
	4	63	61
512	1	129	121
	2	128	120
	3	128	121
	4	127	120
1024	1	257	239
	2	256	238
	3	256	238
	4	255	237
2048	1	513	458
	2	512	458
	3	512	458
	4	511	457

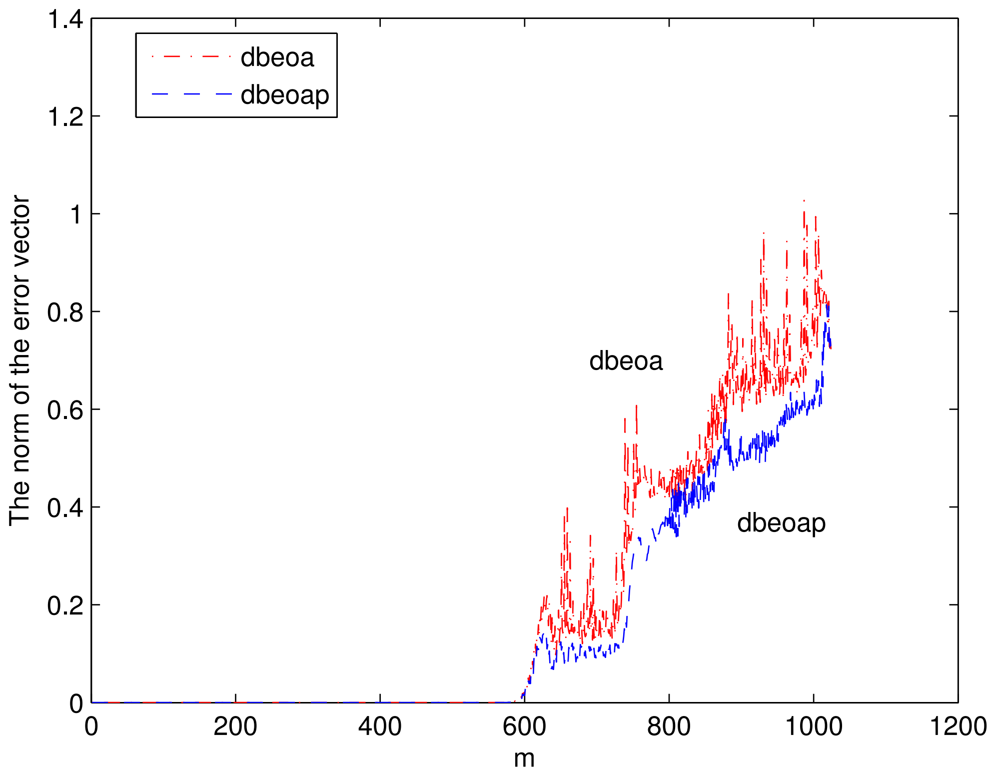

Fig. 1: The norm of the approximation error vectors $\mathbf{e}_\mathbf{m} \equiv (\hat{\mathbf{u}}_\mathbf{m} - \mathbf{u}_\mathbf{m}), m = 1, \cdots, N$ between the exact and approximate eigenvectors of the DFT matrix F using the DBEOA and DBEOAP for N = 1024.

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