FRAMES FROM GROUPS: GENERALIZED BOUNDS AND DIHEDRAL GROUPS

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ABSTRACT

The problem of designing low coherence matrices and low-correlation frames arises in a variety of fields, including compressed sensing, MIMO communications and quantum measurements. The challenge is that one must control the $\binom{n}{2}$ pairwise inner products of the columns of the matrix. In this paper, we follow the group code approach of David Slepian [1], which constructs frames using unitary group representations and which in general reduces the number of distinct inner products to $n-1$. We examine representations of cyclic groups as well as generalized dihedral groups, and we expand upon previous results which bound the coherence of the resulting frames.

Index Terms— Coherence, frame, unit norm tight frame, dihedral group, unitary system.

1. INTRODUCTION AND PREVIOUS WORK

Let $\mathbf{M} \in \mathbb{C}^{m\times n}$ be a complex matrix with columns $\{f_i\}_{i=1}^n$ which form a frame. The frame is called *tight* if $\mathbf{M}\mathbf{M}^*$ is a scalar multiple of the identity $\mathbf{I_m}$, and *unit norm* if $||f_k||_2 = 1, \forall k$. We define the *coherence* μ of $\mathbf{M}$ to be the maximum correlation between any two distinct columns:

$$\mu = \max_{i\neq j} \frac{|\langle f_i, f_j\rangle|}{||f_i||_2 \cdot ||f_j||_2}. \quad (1)$$

Designing matrices and frames with low coherence is a problem that has applications arising in a wide range of fields, including compressive sensing [3–8], spherical codes [10,13], MIMO communications [11,12], quantum measurements [14, 15], etc.

Of particular interest is when a frame is *equiangular*, ie, the magnitude of the inner product between any two distinct frame elements is constant: $|\langle f_i, f_j\rangle| = \alpha$ for some α and all $i \neq j$. If a frame is both tight and equiangular, then it achieves the following lower bound on coherence, known as the *Welch bound* [13]:

Theorem 1 *Let $\mathbb{E}$ be a field, and $\{f_k\}_{k=1}^n$ be a frame for $\mathbb{E}^m$. Then*

$$\max_{i\neq j} \frac{|\langle f_i, f_j\rangle|}{||f_i||_2 \cdot ||f_j||_2} \geq \sqrt{\frac{n-m}{m(n-1)}}, \quad (2)$$

with equality if and only if $\{f_k\}_{k=1}^n$ is both tight and equiangular.

In general, frames that are both tight and equiangular do not exist for all values of m and n, but it can be shown that if there is a small number of inner product magnitudes between the elements of a tight frame, then it will tend to have low coherence. Thus, it is of interest to construct tight frames with few mutual inner products between the elements.

It should be noted that the study of frames is interesting in its own right and has received substantial attention in both engineering and applied math communities (see [17–19]). Much prior work has been done in studying structured frames, including some which are tight and/or equiangular [13, 20, 21] and several of these have employed group theoretic methods [1, 16, 22], some of which we will describe (see Sec. 2).

2. CYCLIC GROUP CODES

The challenge in designing a low-coherence frame is that we need to control $\binom{n}{2}$ inner products. A structure that reduces the number of inner products we need to consider to $n-1$ was first introduced by Slepian [1] and has since been generalized [16]. To this end, let $\mathcal{U} = \{\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_n\}$ be a (multiplicative) group of unitary matrices. Suppose that for each i, we have $\mathbf{U}_i \in \mathbb{C}^{m\times m}$. Let $\mathbf{v} = [v_1, ..., v_m]^T \in \mathbb{C}^{m\times 1}$ be any vector, and let $\mathbf{M}$ be the matrix whose i^{th} column is $\mathbf{U}_i\mathbf{v}$:

$$\mathbf{M} = \begin{bmatrix}\mathbf{U}_1\mathbf{v} & \mathbf{U}_2\mathbf{v} & \dots & \mathbf{U}_n\mathbf{v}\end{bmatrix}.$$

Since $\mathcal{U}$ is a unitary group, we have $\mathbf{U}_i^*\mathbf{U}_j = \mathbf{U}_i^{-1}\mathbf{U}_j = \mathbf{U}_k$, for some $k \in [n]$. Thus, the inner product between columns i and j of $\mathbf{M}$ is

$$\langle \mathbf{U}_i\mathbf{v}, \mathbf{U}_j\mathbf{v}\rangle = \mathbf{v}^*\mathbf{U}_i^*\mathbf{U}_j\mathbf{v} = \mathbf{v}^*\mathbf{U}_k\mathbf{v}. \quad (3)$$

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Thus, there is a distinct inner product for each $\mathbf{U}_k$, and it is not too difficult to see that each of these inner products occurs with the same multiplicity. As promised, this group structure of the frame reduced the number of distinct inner products from $\binom{n}{2}$ to $n-1$ (ignoring the inner product corresponding to the identity element).

In [23], we consider the case where $\mathcal{U}$ is a cyclic unitary group, which we represent as the powers of a single matrix $\mathbf{U} = diag(\omega^{k_1}, \omega^{k_2}, ..., \omega^{k_m})$ of order n, where $\omega = e^{\frac{2\pi i}{n}}$: $\mathcal{U} = \{\mathbf{U}, \mathbf{U}^2, ..., \mathbf{U}^{n-1}, \mathbf{U}^n = \mathbf{I_m}\}$. We choose n to be an odd prime, and m to be a divisor of $n-1$. Then we choose the set $K := \{k_1, ..., k_m\}$ to be the unique subgroup of $G := (\mathbb{Z}/n\mathbb{Z})^\times$ of order m (where $(\mathbb{Z}/n\mathbb{Z})^\times$ is the multiplicative group of nonzero integers modulo n). If we set $\mathbf{v} = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T$, then the inner products take the form

$$\frac{|\mathbf{v}^*\mathbf{U}^\ell \mathbf{v}|}{||\mathbf{v}||_2^2} = \frac{1}{m}\left|\sum_{i=1}^{m} \omega^{\ell \cdot k_i}\right|. \tag{4}$$

Here we can see that there is a distinct inner product for each *coset* of K in G. A coset is a set in the form $\ell K = \{\ell k_1, ..., \ell k_m\}$, and the number of cosets of K in G is simply the quotient of their sizes, so the number of inner products becomes only $r := \frac{n-1}{m}$. In Figure 1, we illustrate how this formulation reduces both the number of distinct inner products and the overall coherence compared to choosing the k_i randomly.

We note that with the forms of $\mathbf{U}$ and $\mathbf{v}$ chosen above, the rows of $\mathbf{M} = \begin{bmatrix}\mathbf{v} & \mathbf{Uv} & \mathbf{U}^2\mathbf{v} & \dots & \mathbf{U}^{n-1}\mathbf{v}\end{bmatrix}$ will correspond to distinct scaled rows of the $n \times n$ Fourier matrix (a so-called *harmonic* frame), and thus our frame will be tight.

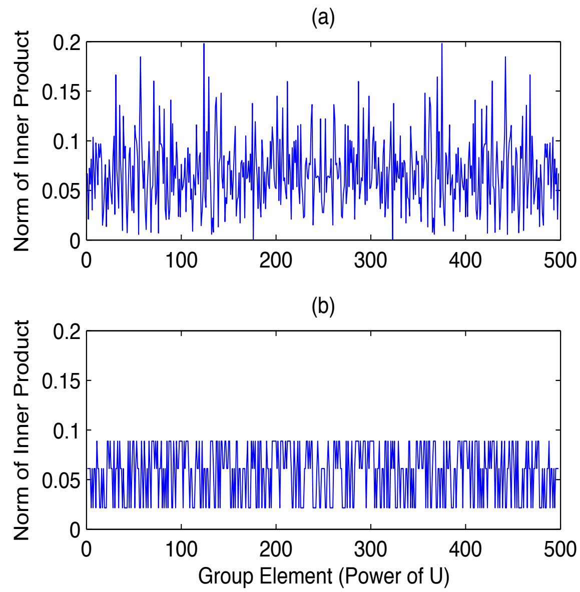

Fig. 1. The norms of the inner products associated to each group element for (a) randomly-chosen K, and (b) K selected to be a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ of index 3. Here, $n = p = 499, m = 166$. In (b), as expected, there are only three distinct values of the inner products between distinct, normalized columns.

3. GENERALIZING BOUNDS ON COHERENCE

In [23], we prove the following bounds on the coherence for when $r = 2$ and $r = 3$:

Theorem 2 *In the above framework when $r = 2$, all of our inner products are real. If m is even, our two inner products are $\frac{-1 \pm \sqrt{1+2m}}{2m}$. In this case, our frame has coherence $\sqrt{\frac{n-m-\frac{1}{2}}{m(n-1)}} + \frac{1}{2m}$.*

If m is odd, then our frame is equiangular, so we achieve the Welch bound. Our two inner products are $\pm\sqrt{\frac{1}{m}\left(\frac{1}{2} + \frac{1}{2m}\right)}$, and our coherence is $\sqrt{\frac{n-m}{m(n-1)}}$.

Theorem 3 *If $r = 3$ in our construction, then the coherence of our matrices will satisfy*

$$\mu \leq \frac{1}{3}\left(2\sqrt{\frac{1}{m}\left(3 + \frac{1}{m}\right)} + \frac{1}{m}\right) \approx \sqrt{\frac{4}{3m}}, \tag{5}$$

and for large enough m, we will asymptotically have the following lower bound on coherence:

$$\mu \geq \frac{1}{\sqrt{m}} \text{ (asymptotically)}, \tag{6}$$

which is strictly greater than the Welch bound.

We now generalize these theorems to find upper bounds on the coherence of our frames for all possible values of r and m.

Theorem 4 *In our frames constructed above, the coherence is upper-bounded by*

$$\mu \leq \frac{1}{r}\left((r-1)\sqrt{\frac{1}{m}\left(r - \frac{1}{m}\right)} + \frac{1}{m}\right). \tag{7}$$

Theorem 5 *If m is odd, then the coherence of our frames is upper-bounded by*

$$\mu \leq \frac{1}{r}\sqrt{\left(\frac{1}{m} + \left(\frac{r}{2} - 1\right)\beta\right)^2 + \left(\frac{r}{2}\right)^2 \beta^2}, \tag{8}$$

where $\beta = \sqrt{\frac{1}{m}\left(r + \frac{1}{m}\right)}$.

While we must omit the proofs of these theorems due to space constraints, we remark that they rely on rather involved extensions of a connection between harmonic frames and difference sets posed by Xia, Zhou and Giannakis [2].

We can use similar methods to obtain lower bounds on the coherence for certain values of r, some of which are strictly greater than the Welch bound, but we have yet to completely generalize them. When $r = 4$, for instance, we find that $\mu \geq \frac{1+\sqrt{2}}{4}\sqrt{\frac{1}{m}\left(4+\frac{1}{m}\right)}$ when m is even, and μ can be as low as the Welch bound when m is odd. We plot these bounds for $r = 4$ in Figure 2.

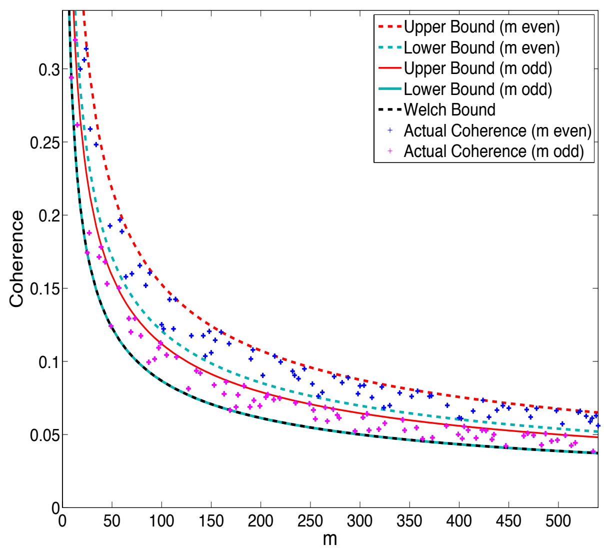

Fig. 2. The upper and lower bounds on coherence for $r = 4$.

4. GENERALIZED DIHEDRAL GROUPS

We now venture beyond abelian groups to see what we can gain. The simplest class of nonabelian groups are semidirect products of cyclic groups. On this note, consider the following group presentation (which arises in [25]):

$$G_{n,r} = \langle \sigma, \tau \mid \sigma^n = 1, \tau^D = 1, \tau\sigma\tau^{-1} = \sigma^r \rangle. \qquad (9)$$

Here, D is the order of r modulo n, and $r - 1$ is chosen to be relatively prime to n. This last condition is automatically satisfied when we take n to be an odd prime as in the cyclic case. This is precisely a semidirect product in the form $\frac{\mathbb{Z}}{n\mathbb{Z}} \rtimes \frac{\mathbb{Z}}{D\mathbb{Z}}$, and if we take $D = 2$ and $r = -1$, we see that we obtain the familiar dihedral group D_{2n}.

$G_{n,r}$ has an irreducible representation in the form

$$\sigma \mapsto \mathbf{S} := diag(\omega, \omega^r, ..., \omega^{r^{D-1}}), \qquad (10)$$

$$\tau \mapsto \mathbf{T} := \begin{bmatrix} & \mathbf{I_{D-1}} \\ 1 & \end{bmatrix}, \qquad (11)$$

where $\omega = e^{\frac{2\pi i}{n}}$ and $\mathbf{I_{D-1}}$ is the $(D-1) \times (D-1)$ identity matrix (see again [25]). In order to construct our frames, we naturally would like to select a representation in the form

$$\sigma \mapsto [\sigma] := diag(\mathbf{S}^{k_1}, ..., \mathbf{S}^{k_m}), \qquad (12)$$

$$\tau \mapsto [\tau] := diag(\mathbf{T}, ..., \mathbf{T}) \qquad (13)$$

where the k_i are cleverly chosen integers. Note that in our above notation, this will be a Dm-dimensional representation of $G_{n,r}$, so our resulting frame matrices will have dimensions $Dm \times Dn$ (provided that the greatest common divisor between the k_i is relatively prime to n).

At this point, we can see that in order to minimize coherence we must deviate from our original construction, for if we were to set $\mathbf{v}$ to the vector of all ones $\mathbf{1}$, then it would be fixed by $[\tau]^b$ for any b, and the inner product corresponding to $[\tau]^b$ would be 1:

$$\frac{\mathbf{v}^*[\tau]^b\mathbf{v}}{||\mathbf{v}||_2^2} = \frac{1}{||\mathbf{1}||_2^2}\mathbf{1}^*[\tau]^b\mathbf{1} = \frac{1}{||\mathbf{1}||_2^2}\mathbf{1}^*\mathbf{1} = 1.$$

We address this problem as follows: in order to preserve as much of the structure from our previous construction as possible, we would like each entry of $\mathbf{v}$ to have the same norm. This will ensure that the inner products corresponding to the elements $[\sigma]^a$ will have the same values as those in our previous construction corresponding to the elements of the cyclic group generated by $[\sigma]$. A natural form for $\mathbf{v}$ would be to find some D-dimensional vector $\mathbf{w} = [w_1, ..., w_D]^T$ and set $\mathbf{v}$ equal to the periodic vector $\mathbf{v} = \begin{bmatrix}\mathbf{w}^T & \mathbf{w}^T & \dots & \mathbf{w}^T\end{bmatrix}^T$. So the question now becomes how to choose $\mathbf{w}$?

Let us require that w_d be unit norm for each d, and consider attempting to force $\mathbf{w}$ to satisfy the constraint that

$$\mathbf{w}^*\mathbf{T}^b\mathbf{w} = \sum_d w_d^* w_{d+b}^* = 0, \ \forall b \qquad (14)$$

where the indices are taken modulo D. It turns out that we can satisfy all our requirements on $\mathbf{w}$ by selecting its indices to form a *Zadoff-Chu* (ZC) sequence:

$$w_d = e^{\frac{i\pi d^2}{D}} \text{ if } D \text{ is even}$$

$$w_d = e^{\frac{i\pi d(d+1)}{D}} \text{ if } D \text{ is odd}$$

There are $n \cdot D$ group elements in $G_{n,r}$, each of which can be written in the form $\sigma^a\tau^b$ for some integers $0 \leq a < n$ and $0 \leq b < D$. Thus, our frame elements will take the form

$$[\sigma]^a[\tau]^b\mathbf{v} = \begin{bmatrix} \mathbf{S}^{ak_1}\mathbf{w}_{d+b} \\ \vdots \\ \mathbf{S}^{ak_m}\mathbf{w}_{d+b} \end{bmatrix}, \qquad (15)$$

where $\mathbf{w}_{d+b} = \mathbf{T}^b\mathbf{w}$ denotes the vector obtained by cyclically shifting the entries of $\mathbf{w}$ by b positions. (Note that by this notation, $\mathbf{w} = \mathbf{w}_d$). Our inner products will take the form

$$\frac{\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}}{||\mathbf{v}||_2^2} = \frac{1}{m \cdot D}\sum_{j=1}^{m} \mathbf{w}_d^* \mathbf{S}^{ak_j}\mathbf{w}_{d+b}. \quad (16)$$

Our new frames maintain the desirable quality of being tight:

Theorem 6 *If* $\mathbf{w} = [w_1, ..., w_D]^T$ *is a ZC-sequence, and* $\mathbf{v} = \begin{bmatrix}\mathbf{w}^T & \ldots & \mathbf{w}^T\end{bmatrix}^T$, *then the the columns of* $\mathbf{M} = \begin{bmatrix}\ldots & [\sigma]^a[\tau]^b\mathbf{v} & \ldots\end{bmatrix}$ *form a tight frame.*

Proof: It is not too difficult to see that $\mathbf{M}$ will have $D \cdot m$ rows which can be indexed by a pair of numbers (d, j), where $1 \le d \le D$ and $1 \le j \le m$. Row (d, j) will be given by

$$\begin{bmatrix}\mathbf{z}_1^{(d,j)} & \mathbf{z}_2^{(d,j)} & \ldots & \mathbf{z}_D^{(d,j)}\end{bmatrix},$$

where $\mathbf{z}_{b+1}^{(d,j)} = \begin{bmatrix}\ldots & \omega^{r^{d-1}k_j a} w_{d-b} & \ldots\end{bmatrix}_{0 \le a < n}$.

Now we can see that the inner product between row (d, j) and row (d', j') will be

$$\sum_{b=0}^{D-1}\sum_{a=0}^{n-1} \omega^{(-r^{d-1}k_j + r^{d'-1}k_{j'})a} w_{d+b}^* w_{d'+b} \quad (17)$$

$$= \left[\sum_{a=0}^{n-1} \omega^{(-r^{d-1}k_j + r^{d'-1}k_{j'})a}\right] \cdot \left[\sum_{b=0}^{D-1} w_{d+b}^* w_{d'+b}\right] \quad (18)$$

Now, since the entries of $\mathbf{w}$ form a ZC-sequence, then $\sum_{b=0}^{D-1} w_{d+b}^* w_{d'+b}$ is zero unless $d = d'$, in which case it is D. In this latter case,

$$\sum_{a=0}^{n-1} \omega^{(-r^{d-1}k_j + r^{d'-1}k_{j'})a} = \sum_{a=0}^{n-1} \omega^{r^{d-1}(k_{j'} - k_j)a}, \quad (19)$$

which is zero unless $j = j'$. Thus, the rows of $\mathbf{M}$ are orthogonal, so the frame is indeed tight. □

We now come to the main result of this section. For a set of integers $K = \{k_1, ..., k_m\}$ in $\mathbb{Z}/n\mathbb{Z}$, let μ_K^{cyc} be the coherence of the frame arising from our original representation of the cyclic group $\mathbb{Z}/n\mathbb{Z}$, and let μ_K^D be the coherence of the frame arising from our generalized dihedral group $G_{n,r}$, where D is the order of r modulo n.

Theorem 7 *Let n be a prime and m a divisor of $n - 1$. Let K be the unique subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$. Then the number of distinct inner products in the generalized dihedral frame associated to $G_{n,r}$ is at most $D \cdot \frac{n-1}{m}$, and $\mu_K^D \le \mu_K^{cyc}$.*

Proof: From (16), we see that the inner products for the generalized dihedral representation will take the form

$$\frac{\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}}{||\mathbf{v}||_2^2} = \frac{1}{m \cdot D}\sum_{d} w_d^* w_{d+b} \sum_{k \in K} \omega^{kar^{d-1}} \quad (20)$$

$$= \frac{1}{m \cdot D}\sum_{d} w_d^* w_{d+b} \sum_{k \in K} \omega^{ka'}, \quad (21)$$

where $a' = ar^{d-1}$. In this form, we see that for each value of d in the summation, there are $\frac{n-1}{m}$ possible distinct inner products associated to the different cosets $a'K$, so there $D \cdot \frac{n-1}{m}$ possible values. Furthermore, since the entries of $\mathbf{w}$ are unit norm,

$$\frac{|\mathbf{v}^*[\sigma]^a[\tau]^b\mathbf{v}|}{||\mathbf{v}||_2^2} \le \frac{1}{m \cdot D}\sum_{d}\left|\sum_{k \in K} \omega^{ka'}\right| \quad (22)$$

$$\le \frac{1}{m \cdot D}\sum_{d} m\mu_K^{cyc} \quad (23)$$

$$= \mu_K^{cyc}, \quad (24)$$

so $\mu_K^D \le \mu_K^{cyc}$. □

In the case of regular dihedral groups ($D = 2$), our $\mathbf{w}$ becomes $[1, i]^T$, and we can readily calculate our inner products to be

$$\frac{\mathbf{v}^*[\sigma]^\ell\mathbf{v}}{||\mathbf{v}||_2^2} = Re\left(\frac{1}{m}\sum_{j=1}^{m} \omega^{\ell k_j}\right), \quad (25)$$

$$\frac{\mathbf{v}^*[\sigma]^\ell[\tau]\mathbf{v}}{||\mathbf{v}||_2^2} = Im\left(-\frac{1}{m}\sum_{j=1}^{m} \omega^{\ell k_j}\right). \quad (26)$$

As we can clearly see, each of these has magnitude bounded that of the corresponding inner product in the cyclic counterpart, $\frac{1}{m}\left|\sum_{j=1}^{m} \omega^{\ell k_j}\right|$ (see Fig. 3). In general, the dihedral coherence could be substantially smaller than the corresponding cyclic coherence. Most importantly, by extending to generalized dihedral groups, we allow for frame matrices $\mathbf{M}$ with a greater variety of dimensions. In particular, the number of columns (nD) no longer need be prime.

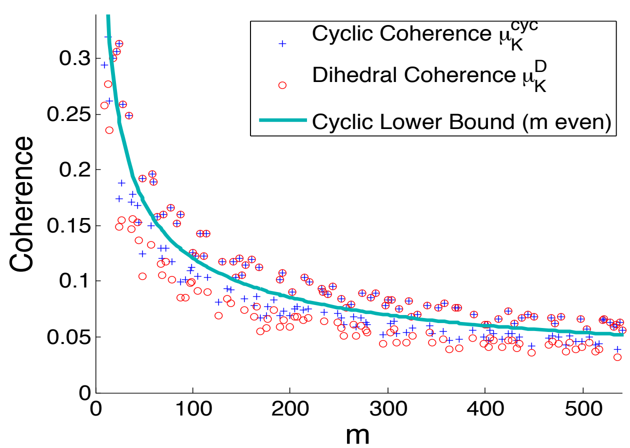

Fig. 3. Coherences arising from cyclic and dihedral representations for $r = 4$. When m is even, both coherences are the same because the cyclic inner products are real.

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