COMPRESSIVE SHIFT RETRIEVAL

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ABSTRACT

The classical shift retrieval problem considers two signals in vector form that are related by a cyclic shift. In this paper, we develop a compressive variant where the measurement of the signals is undersampled. While the standard procedure to shift retrieval is to maximize the real part of their dot product, we show that the shift can be exactly recovered from the corresponding compressed measurements if the sensing matrix satisfies certain conditions. A special case is the partial Fourier matrix. In this setting we show that the true shift can be found by as low as two measurements. We further show that the shift can often be recovered when the measurements are perturbed by noise.

Index Terms— Compressed sensing, shift retrieval, signal reconstruction, signal registration.

1. INTRODUCTION

Compressive sensing has received growing interest in signal processing. The theory mainly concerns recovery of a high-dimensional signal from its downsampled measurement when the source signal is sufficiently sparse [1, 2]. In this paper, we show how the basic premise of compressive sensing can be used in the context of shift retrieval. In particular, we consider two signals of the same dimension that are related by a shift transform, and we are interested to know under what conditions such a transform can be exactly recovered from measurement of each of the signals in a compressed low-dimensional space. We refer to this approach as *compressive shift retrieval* (CSR).

Concretely, let $\boldsymbol{y}$ and $\boldsymbol{x}$ be vectors of length n and let $\boldsymbol{D}^\ell$ denote a cyclic-shift by ℓ. Assume that $\boldsymbol{y}$ and $\boldsymbol{x}$ are related by

$$\boldsymbol{y} = \boldsymbol{D}^\ell \boldsymbol{x}. \tag{1}$$

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We assume throughout the paper that the shift is unique (up to a multiple of n). Let $\boldsymbol{A}$ be a $m \times n$ sensing matrix where $m \le n$, and define the compressed measurement signals $\boldsymbol{z} = \boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{x}$. Our goal is to determine the shift ℓ from $\boldsymbol{z}$ and $\boldsymbol{v}$.

In shift retrieval and other more general signal alignment problems, a standard procedure is to seek the shift that maximizes the real part of the inner product:

$$\max_s \Re\{\langle \boldsymbol{x}, \boldsymbol{D}^s \boldsymbol{y}\rangle\} = \max_s \Re\{\boldsymbol{x}^* \boldsymbol{D}^s \boldsymbol{y}\}. \tag{2}$$

In this paper we consider the related test:

$$\max_s \Re\{\langle \boldsymbol{z}, \bar{\boldsymbol{D}}^s \boldsymbol{v}\rangle\} = \max_s \Re\{\boldsymbol{z}^* \bar{\boldsymbol{D}}^s \boldsymbol{v}\}, \tag{3}$$

with $\bar{\boldsymbol{D}}^s = \boldsymbol{A}\boldsymbol{D}^s\boldsymbol{A}^*$. We show that when the sensing matrix $\boldsymbol{A}$ is taken to be a partial Fourier matrix, then under suitable conditions the true shift can be recovered from both noise free and noisy measurements using (3). Furthermore, (3) reduces both the computational load and the number of samples needed. This is of particular interest since recent developments in sampling [3, 4, 5] have shown that Fourier coefficients can efficiently be obtained from space (or time) measurements by the use of an appropriate filter and by subsampling the output. Remarkably, our results also show that in some cases sampling one Fourier coefficient given by $\boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{A}\boldsymbol{y}$, respectively, is enough to perfectly recover the true shift.

2. COMPRESSIVE SHIFT RETRIEVAL

Assume that $\boldsymbol{z}$ and $\boldsymbol{v}$ are given and that these are related to $\boldsymbol{x}$ and $\boldsymbol{y}$ via the sensing matrix $\boldsymbol{A}$. Our goal is to find the shift relating $\boldsymbol{x}$ and $\boldsymbol{y}$. To this end, we propose to use the test (3). As we will see in the following, this simple test has several desirable properties and can be guaranteed to recover the true shift under certain conditions.

We first prove the following main theorem of CSR assuming the measurement is not affected by noise:

Theorem 1 (Shift Recovery from Low Rate Data)**.** *Let $\boldsymbol{X}$ be an $n \times n$ matrix with the ith column equal to $\boldsymbol{D}^i\boldsymbol{x}$, $i = 1, \dots, n$, and define $\bar{\boldsymbol{D}}^s = \boldsymbol{A}\boldsymbol{D}^s\boldsymbol{A}^*$. If the sensing matrix $\boldsymbol{A}$ satisfies the following conditions:*

1) $\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{D}^s = \boldsymbol{D}^s\boldsymbol{A}^*\boldsymbol{A}$*,*

2) $\exists\alpha \in \mathbb{R}, \alpha\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{I}$ *and*

3) all columns of $\boldsymbol{AX}$ *are different,*

then

$$\max_s \Re\{\langle \boldsymbol{z}, \bar{\boldsymbol{D}}^s\boldsymbol{v}\rangle\} \tag{4}$$

recovers the true shift.

The conditions of Theorem 1 may seem restrictive. However, as we will show in the following, if $\boldsymbol{A}$ is chosen as a partial Fourier matrix, then the two first conditions of Theorem 1 are trivially satisfied. The last condition is the only one that needs to be checked and will lead to a condition on the Fourier coefficients sampled.

Before proving the theorem, we state two lemmas.

Lemma 1 (Recovery of Shift using Projections)**.** *Let* $\boldsymbol{X}$ *be the* $n \times n$*-matrix made up of cyclically shifted versions of* $\boldsymbol{x}$ *as columns. If the columns of* $\boldsymbol{AX}$ *are distinct, then the true shift can be recovered by*

$$\min_{\boldsymbol{q}\in\{0,1\}^n} \|\boldsymbol{Ay} - \boldsymbol{AXq}\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{q}\|_0 = 1. \tag{5}$$

Proof of Lemma 1. Since the shift relating $\boldsymbol{x}$ and $\boldsymbol{y}$ is assumed unique, it is clear that the true shift is recovered by

$$\min_{\boldsymbol{q}\in\{0,1\}^n} \|\boldsymbol{y} - \boldsymbol{Xq}\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{q}\|_0 = 1. \tag{6}$$

Assume that the solution of (5) is not equivalent to that of (6). Namely, assume that (6) gives $\boldsymbol{q}$, (5) gives $\tilde{\boldsymbol{q}}$ and $\boldsymbol{q} \neq \tilde{\boldsymbol{q}}$. Since $\boldsymbol{q}$ will give a zero objective value in (5), so must $\tilde{\boldsymbol{q}}$. We therefore have that $\boldsymbol{Ay} = \boldsymbol{AX}\tilde{\boldsymbol{q}} = \boldsymbol{AXq}$ and hence

$$\boldsymbol{AX}\tilde{\boldsymbol{q}} - \boldsymbol{AXq} = \boldsymbol{AX}(\tilde{\boldsymbol{q}} - \boldsymbol{q}) = 0. \tag{7}$$

Since $\boldsymbol{q}$, $\tilde{\boldsymbol{q}} \in \{0,1\}^n$, $\|\tilde{\boldsymbol{q}}\|_0 = \|\boldsymbol{q}\|_0 = 1$, and $\boldsymbol{q} \neq \tilde{\boldsymbol{q}}$, $\boldsymbol{AX}(\tilde{\boldsymbol{q}} - \boldsymbol{q}) = 0$ implies that two columns of $\boldsymbol{AX}$ are identical. This is a contradiction and we therefore conclude that both (5) and (6) recover the true shift. □

Lemma 2 (From (5) to (4))**.** *Under conditions 1) and 2) of Theorem 1, the shifts recovered by* (5) *and* (4) *are the same.*

Proof of Lemma 2. Consider the objective of (5):

$$\begin{aligned}\|\boldsymbol{Ay} - \boldsymbol{AXq}\|_2^2 =& (\boldsymbol{Ay})^*\boldsymbol{Ay} + (\boldsymbol{AXq})^*\boldsymbol{AXq} \\ &- (\boldsymbol{Ay})^*\boldsymbol{AXq} - (\boldsymbol{AXq})^*\boldsymbol{Ay}.\end{aligned} \tag{8}$$

Writing $\boldsymbol{Xq} = \boldsymbol{D}^s\boldsymbol{x}$, problem (5) is equal to

$$\max_s 2\Re\{(\boldsymbol{Ay})^*\boldsymbol{AD}^s\boldsymbol{x}\} - (\boldsymbol{AD}^s\boldsymbol{x})^*\boldsymbol{AD}^s\boldsymbol{x}. \tag{9}$$

Now, if $\boldsymbol{A}^*\boldsymbol{AD}^s = \boldsymbol{D}^s\boldsymbol{A}^*\boldsymbol{A}$ and using that $(\boldsymbol{D}^s)^*\boldsymbol{D}^s = \boldsymbol{I}$ for a shift matrix, then

$$(\boldsymbol{AD}^s\boldsymbol{x})^*\boldsymbol{AD}^s\boldsymbol{x} = \boldsymbol{x}^*(\boldsymbol{D}^s)^*\boldsymbol{A}^*\boldsymbol{AD}^s\boldsymbol{x} = \|\boldsymbol{Ax}\|_2^2, \tag{10}$$

which is independent of s. Therefore, the shift recovered by (9) is the same as that of

$$\max_s \Re\{(\boldsymbol{Ay})^*\boldsymbol{AD}^s\boldsymbol{x}\}. \tag{11}$$

Lastly, if we again use that $\boldsymbol{A}^*\boldsymbol{AD}^s = \boldsymbol{D}^s\boldsymbol{A}^*\boldsymbol{A}$ and $\alpha\boldsymbol{AA}^* = \boldsymbol{I}$, then (4) follows from

$$\begin{aligned}\Re\{(\boldsymbol{Ay})^*\boldsymbol{AD}^s\boldsymbol{x}\} &= \Re\{\boldsymbol{y}^*\boldsymbol{A}^*\boldsymbol{AD}^s\boldsymbol{x}\} && (12)\\ &= \alpha\Re\{\boldsymbol{y}^*\boldsymbol{A}^*\boldsymbol{AA}^*\boldsymbol{AD}^s\boldsymbol{x}\} && (13)\\ &= \alpha\Re\{\boldsymbol{y}^*\boldsymbol{A}^*\boldsymbol{AD}^s\boldsymbol{A}^*\boldsymbol{Ax}\} && (14)\\ &= \alpha\Re\{\langle \boldsymbol{z}, \bar{\boldsymbol{D}}^s\boldsymbol{v}\rangle\} && (15)\end{aligned}$$

where $\boldsymbol{z} = \boldsymbol{Ay}$ and $\boldsymbol{v} = \boldsymbol{Ax}$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. The assumptions of Theorem 1 imply that requirements of both Lemmas 1 and 2 are satisfied. The theorem therefore follows trivially. □

The conditions of Theorem 1 can be checked prior to actually computing the estimate of the shift. However, knowing the estimate of the shift, it is easy to see from the proof of Lemma 1 that it is enough to check if the column of $\boldsymbol{AX}$ associated with the nonzero element of $\tilde{\boldsymbol{q}}$ is different than all other columns of $\boldsymbol{AX}$. We hence do not need to check if all columns of $\boldsymbol{AX}$ are different.

Of particular interest is the case where $\boldsymbol{A}$ is made up of a partial Fourier basis. That is, $\boldsymbol{A}$ takes the form

$$\boldsymbol{A} = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & e^{-\frac{2j\pi k_1}{n}} & e^{-\frac{4j\pi k_1}{n}} & \cdots & e^{-\frac{2(n-1)j\pi k_1}{n}} \\ 1 & e^{-\frac{2j\pi k_2}{n}} & & \ddots & e^{-\frac{2(n-1)j\pi k_2}{n}} \\ \vdots & \vdots & & & \\ 1 & e^{-\frac{2j\pi k_m}{n}} & e^{-\frac{4j\pi k_m}{n}} & \cdots & e^{-\frac{2(n-1)j\pi k_m}{n}} \end{bmatrix}$$

where $k_1, \dots, k_m \in \{0, 1, 2, \dots n-1\}, m \leq n$. For this specific choice,

$$\boldsymbol{AX} = \frac{1}{\sqrt{n}}\begin{bmatrix} X_{k_1} & X_{k_1}e^{\frac{2k_1\pi j}{n}} & \cdots & X_{k_1}e^{\frac{2(n-1)k_1\pi j}{n}} \\ X_{k_2} & \ddots & & X_{k_2}e^{\frac{2(n-1)k_2\pi j}{n}} \\ \vdots & & & \\ X_{k_m} & X_{k_m}e^{\frac{2k_m\pi j}{n}} & \cdots & X_{k_m}e^{\frac{2(n-1)k_m\pi j}{n}} \end{bmatrix}$$

where X_r denotes the rth Fourier coefficient of the Fourier transform of $\boldsymbol{x}$. Using this result in Theorem 1 gives the following corollary:

Corollary 1 (Shift Recovery from Low Rate Fourier Data)**.** *With* $\boldsymbol{A}$ *denoting a partial Fourier matrix and* z_i *and* v_i *the* i*th element of* $\boldsymbol{z}$ *and* $\boldsymbol{v}$*,*

$$\max_s \Re\left\{\sum_{i=1}^m z_i v_i e^{\frac{-2\pi j k_i s}{n}}\right\} \tag{16}$$

recovers the true shift if there exists a $p \in \{1, \dots, m\}$ *such that* $X_{k_p} \neq 0$ *and* $\{1, \dots, n-1\}\frac{k_p}{n}$ *contains no integers.*

Remarkably the corollary states that all we need is two scalar measurements, $\boldsymbol{z}$ and $\boldsymbol{v}$, to perfectly recover the true shift. The scalar measurements can be any nonzero Fourier coefficient of $\boldsymbol{x}$ and $\boldsymbol{y}$ as long as all elements in $\{1,\dots,n-1\}\frac{k_1}{n}$ contains no integers. Also note that only $2mn$ multiplications are required to evaluate the test.

We need the following lemma to prove Corollary 1.

Lemma 3. *Let $\boldsymbol{A}$ be a partial Fourier matrix. Then $\boldsymbol{D}^s\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A}^*\boldsymbol{A}\boldsymbol{D}^s$ for all $s = 0, 1, \dots, n-1$.*

Proof of Lemma 3. Let $\boldsymbol{M} = \boldsymbol{A}\boldsymbol{D}^s$ and $\boldsymbol{Q} = \boldsymbol{A}(\boldsymbol{D}^s)^*$. By the definition of $\boldsymbol{D}^s$, $\boldsymbol{M}$ is a column permutation of $\boldsymbol{A}$ where the columns are shifted s to the right. Thus, the rth column of $\boldsymbol{M}$ is equal to the tth column of $\boldsymbol{A}$ where $t = (r-s)\text{mod } n$. It is also easy to see that $(\boldsymbol{D}^s)^*$ permutes the columns of $\boldsymbol{A}$ by s to the left so that the rth column of $\boldsymbol{Q}$ is equal to the qth column of $\boldsymbol{A}$ where $q = (r+s)\text{mod } n$. Now, the prth element of $\boldsymbol{A}^*\boldsymbol{M} = \boldsymbol{A}^*\boldsymbol{A}\boldsymbol{D}^s$ is given by

$$(\boldsymbol{A}_{:,p})^*\boldsymbol{M}_{:,r} = (\boldsymbol{A}_{:,p})^*\boldsymbol{A}_{:,r-s} = \frac{1}{n}\sum_{i=1}^{m} e^{2j\pi k_i(p-r+s)}, \tag{17}$$

where $\boldsymbol{A}_{:,p}$ is used to denote the pth column of $\boldsymbol{A}$ and $\boldsymbol{M}_{:,r}$ the rth column or $\boldsymbol{M}$. On the other hand, the prth element of $\boldsymbol{Q}^*\boldsymbol{A} = \boldsymbol{D}^s\boldsymbol{A}^*\boldsymbol{A}$ is given by

$$(\boldsymbol{Q}_{:,p})^*\boldsymbol{A}_{:,r} = (\boldsymbol{A}_{:,p+s})^*\boldsymbol{A}_{:,r} = \frac{1}{n}\sum_{i=1}^{m} e^{2j\pi k_i(p+s-r)}. \tag{18}$$

Clearly, the two are equivalent. □

We are now ready to prove Corollary 1.

Proof of Corollary 1. Lemma 3 gives that Condition 1) of Theorem 1 is satisfied. Since a full Fourier matrix is orthonormal, a matrix made up of a selection of rows of a Fourier matrix satisfies Condition 2). The last condition of Theorem 1 requires columns of $\boldsymbol{A}\boldsymbol{X}$ to be distinct. A sufficient condition is that there exists a row with all distinct elements. As shown previously, the prth element of $\boldsymbol{A}\boldsymbol{X}$ is $X_{k_p}e^{\frac{2j\pi k_p(r-1)}{n}}$. If X_{k_p} is assumed nonzero, a sufficient condition for $\boldsymbol{A}\boldsymbol{X}$ to have distinct columns is hence that $e^{\frac{2j\pi k_p r_1}{n}} \neq e^{\frac{2j\pi k_p r_2}{n}}$, $r_1, r_2 \in \{0,\dots,n-1\}, r_1 \neq r_2$. This condition can be simplified to $\frac{k_p r_1}{n} \neq \frac{k_p r_2}{n} + \gamma$, $\gamma \in \mathbb{Z}$. By realizing that $r_1 - r_2$ takes values in $\{-n+1,\dots,-1,1,\dots,n-1\}$ we get that the condition is equivalent to requiring that there is no integers in $\{-n+1,\dots,-1,1,\dots,n-1\}\frac{k_p}{n}$. Due to symmetry, a sufficient condition for distinct columns is that there exists a $p \in \{1,\dots,m\}$ such that $X_{k_p} \neq 0$ and $\{1,\dots,n-1\}\frac{k_p}{n}$ contains no integers. Last, if we write out $\boldsymbol{A}\boldsymbol{D}^s\boldsymbol{A}^*$ we get that the prth element is equal to $\delta_{p,r}e^{-\frac{2j\pi k_p s}{n}}/n$ and hence the simplified test proposed in (16). □

Example 1 (A Monte Carlo Simulation). *In this example we carry out a Monte Carlo simulation. In each trial we let m and ℓ be random integers between 1 and 9 and generate $\boldsymbol{x}$ by sampling from a n-dimensional uniform distribution. We let $n = 10$ and make sure that $\boldsymbol{A}$ in each trial is a partial Fourier basis satisfying the assumptions of Corollary 1. We carry out 10000 trials. The true shift was recovered in each run by the simplified test* (16) *of Corollary 1. This is quite remarkable since when $m = 1$, we recover the true shift using only two scalar measurement $\boldsymbol{z}$ and $\boldsymbol{v}$ and $1/5$ of the multiplications that maximizing the inner product between the unprojected signals* (2) *would have needed.*

3. NOISY COMPRESSIVE SHIFT RETRIEVAL

Now we consider the noisy version of CSR, where the measurements $\boldsymbol{z}$ and $\boldsymbol{v}$ are perturbed by noise:

$$\tilde{\boldsymbol{z}} = \boldsymbol{z} + \boldsymbol{e}_z, \quad \tilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{e}_v. \tag{19}$$

Similar to the noise free case, here we can also guarantee the recovery of the true shift. Our main result is given in the following theorem:

Theorem 2 (Noisy Recovery). *Let $\tilde{\boldsymbol{x}}$ be such that $\tilde{\boldsymbol{v}} = \boldsymbol{A}\tilde{\boldsymbol{x}}$, let the ith column of $\tilde{\boldsymbol{X}}$ be shifted versions of $\tilde{\boldsymbol{x}}$, assume that $\boldsymbol{A}$ is a partial Fourier matrix and that the noisy measurements are used in* (16) *to estimate the shift. If the ℓ_2-norm difference between any two columns of $\boldsymbol{A}\tilde{\boldsymbol{X}}$ is greater than*

$$\Delta_{\boldsymbol{zv}} \triangleq \|\boldsymbol{e}_z\|_2 + \|\boldsymbol{e}_v\|_2 + \sqrt{\|\tilde{\boldsymbol{v}}\|_2^2 + \|\tilde{\boldsymbol{z}}\|_2^2 - 2\max_s \Re\{\langle\tilde{\boldsymbol{z}}, \bar{\boldsymbol{D}}^s\tilde{\boldsymbol{v}}\rangle\}},$$

then the estimate of the shift is not affected by the noise.

Proof of Theorem 2. From Lemma 2 we can see that seeking s that maximizes $\Re\{\langle\tilde{\boldsymbol{z}}, \bar{\boldsymbol{D}}^s\tilde{\boldsymbol{v}}\rangle\}$ is equivalent to seeking $\boldsymbol{q}$ that solves

$$\min_{\boldsymbol{q}\in\{0,1\}^n} \|\tilde{\boldsymbol{z}} - \boldsymbol{A}\tilde{\boldsymbol{X}}\boldsymbol{q}\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{q}\|_0 = 1, \tag{20}$$

where the first column of $\boldsymbol{A}\tilde{\boldsymbol{X}}$ equals $\tilde{\boldsymbol{v}}$ (which defines the first column of $\tilde{\boldsymbol{X}}$) and the ith column of $\tilde{\boldsymbol{X}}$ a circular shift of the first column $i-1$ steps. Assume that $\hat{\boldsymbol{q}}$ solves (20). Since our measurements are noisy, we can not expect a zero loss. The loss can be shown given by

$$\|\tilde{\boldsymbol{z}} - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2^2 = \|\tilde{\boldsymbol{v}}\|_2^2 + \|\tilde{\boldsymbol{z}}\|_2^2 - \max_s 2\Re\{\tilde{\boldsymbol{z}}^*\bar{\boldsymbol{D}}^s\tilde{\boldsymbol{v}}\}. \tag{21}$$

Now, consider $\|\tilde{\boldsymbol{z}} - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2$. Assume that $\boldsymbol{q}_0$ solves the noise free version of (20) and let $\tilde{\boldsymbol{X}} = \boldsymbol{X} + \boldsymbol{H}$. We have the following inequality

$$\begin{aligned}
\|\tilde{\boldsymbol{z}} - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2 &= \|\boldsymbol{z} + \boldsymbol{e}_z - \boldsymbol{z} + \boldsymbol{A}\boldsymbol{X}\boldsymbol{q}_0 - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2 \\
&= \|\boldsymbol{e}_z + \boldsymbol{A}\boldsymbol{X}\boldsymbol{q}_0 - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2 \\
&= \|\boldsymbol{e}_z + \boldsymbol{A}(\tilde{\boldsymbol{X}} - \boldsymbol{H})\boldsymbol{q}_0 - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2 \\
&\geq \|\boldsymbol{A}\tilde{\boldsymbol{X}}\boldsymbol{q}_0 - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2 - \|\boldsymbol{e}_z\|_2 - \|\boldsymbol{e}_v\|_2
\end{aligned}$$

where we used the fact that $\boldsymbol{A}\boldsymbol{H}\boldsymbol{q}_0 = \boldsymbol{e}_v$. Therefore

$$\|\boldsymbol{A}\tilde{\boldsymbol{X}}\boldsymbol{q}_0 - \boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}}\|_2 \leq \Delta_{\boldsymbol{z}\boldsymbol{v}}. \tag{22}$$

Since $\|\hat{\boldsymbol{q}}\|_0 = \|\boldsymbol{q}_0\|_0 = 1$, we get that if the ℓ_2 difference between any two columns of $\boldsymbol{A}\tilde{\boldsymbol{X}}$ is greater than $\Delta_{\boldsymbol{v}\boldsymbol{z}}$, then $\boldsymbol{q}_0 = \hat{\boldsymbol{q}}$. □

Example 2 (Recovery of Shift from Noisy Data)**.** *In this experiment we run a Monte Carlo simulation consisting of 10000 trials for each* $m = 1, \ldots, 10$*, and for two different SNR. In Figure 1, 10 histograms are shown (corresponding to* $m = 1, \ldots, 10$*) for*

$$SNR = \frac{\|\boldsymbol{z}\|_2^2}{\|\tilde{\boldsymbol{z}} - \boldsymbol{z}\|_2^2} \tag{23}$$

being 2 (low SNR) and in Figure 2, 10 (high SNR). $\boldsymbol{e}_z$ *and* $\boldsymbol{e}_x$ *were both generate by sampling from*

$$\mathcal{N}(0, \sigma^2) + j\mathcal{N}(0, \sigma^2). \tag{24}$$

We further used $n = 10$*,* $\ell = 5$ *and sampled* $\boldsymbol{x}$ *from a uniform (0,1)-distribution. The conclusion from the simulations is that the smaller the* m*, the more noise sensitive estimate of the shift. 12 % of the correctly estimated shifts could using Theorem 2 be predicted to be the same as the noise free estimate for* $m = 1$ *and the high SNR case while 40 % for* $m = 2$*.*

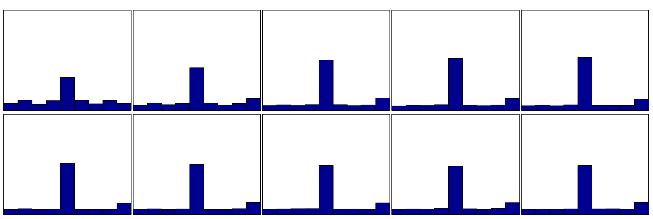

Fig. 1. Histogram plots for the estimated shift and low SNR. From left to right, top to bottom, $m = 1, \ldots, 10$. The true shift was set to 5 in all trials.

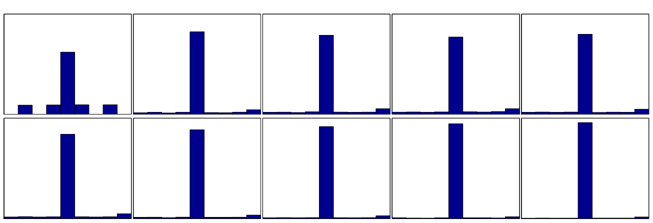

Fig. 2. Histogram plots for the estimated shift and high SNR. From left to right, top to bottom, $m = 1, \ldots, 10$. The true shift was set to 5 in all trials.

Theorem 2 gives conditions for when the noise does not affect the estimate of the shift. This is a good property but even better would be if the recovery of the true shift could be guaranteed. This is given by the following corollary.

Corollary 2 (Recovery of the True Shift from Noisy Data)**.** *If the* ℓ_2 *difference between any two columns of* $\boldsymbol{A}\tilde{\boldsymbol{X}}$ *is greater than* $2\|\boldsymbol{e}_v\|_2$ *and the conditions of Theorem 2 are fulfilled, then* (16) *recovers the true shift.*

Proof of Corollary 2. Let $\tilde{\boldsymbol{q}}$ and $\hat{\boldsymbol{q}}$ be any vectors such that $\|\hat{\boldsymbol{q}}\|_0 = \|\tilde{\boldsymbol{q}}\|_0 = 1$, $\hat{\boldsymbol{q}} \neq \tilde{\boldsymbol{q}}$ and $\hat{\boldsymbol{q}}, \tilde{\boldsymbol{q}} \in \{0, 1\}^n$. Using the triangle inequality we have that

$$\|\boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}} - \boldsymbol{A}\tilde{\boldsymbol{X}}\tilde{\boldsymbol{q}}\|_2 = \|\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{H})(\hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\|_2 \tag{25}$$

$$\leq \|\boldsymbol{A}\boldsymbol{X}(\hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\|_2 + \|\boldsymbol{A}\boldsymbol{H}(\hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\|_2 \tag{26}$$

$$\leq \|\boldsymbol{A}\boldsymbol{X}(\hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\|_2 + 2\|\boldsymbol{e}_v\|_2. \tag{27}$$

Hence, if $\|\boldsymbol{A}\tilde{\boldsymbol{X}}\hat{\boldsymbol{q}} - \boldsymbol{A}\tilde{\boldsymbol{X}}\tilde{\boldsymbol{q}}\|_2 - 2\|\boldsymbol{e}_v\|_2 > 0$ then $\|\boldsymbol{A}\boldsymbol{X}(\hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\|_2$ is greater than zero. Now since Theorem 2 gives that (16) recovers the same shift as if the measurements would have been noise-free, and since Corollary 1 gives that the noise-free estimate is equal to the true shift if $\|\boldsymbol{A}\boldsymbol{X}(\hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}})\|_2$ is greater than zero, we can guarantee the recovery of the true shift also in the noisy case. □

4. RELATION TO PRIOR WORK

Optimizing a shift retrieval function such as (2) can appear in a broad category of signal alignment problems, which have a wide range of applications in signal processing and image processing. In signal processing, shift retrieval has been used to align acoustic signals for segmentation or averaging [6, 7]. In image processing, studies related to image registration and target tracking typically are more focused on recovering more general 2D image transforms, including 2D shifts, scaling, and affine transform, that aligns a target image to a reference image [8, 9, 10]. In addition to the objective function (2) widely used in the literature, another popular dynamic program for signal alignment is known as *dynamic time warping* [11].

In the compressive sensing framework, compressive signal alignment problems have been addressed in only a few publications. In [12], the authors considered alignment of images under random projection. The work was based on the Johnson-Lindenstrauss property of random projection, and proposed an objective function that can be solved efficiently using *difference of two convex* programming algorithms. In this paper, we instead focus on proving theoretical guarantees of exact shift recovery when the signal is subsampled by a partial Fourier basis.

A more recent work called RASL addresses alignment of an ensemble of correlated signals [10]. The premise of the RASL method is that the ensemble of the signals in vector form can be concatenated in a low-rank matrix if the misalignment between them can be compensated. However, the method has not provided an analysis of the performance in a downsampled feature space, and does not deal with the pairwise alignment problem, e.g., the shift retrieval problem (2).

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