OPTIMAL DETERMINISTIC COMPRESSED SENSING MATRICES

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ABSTRACT

We present the first deterministic measurement matrix construction with an order-optimal number of rows for sparse signal reconstruction. This improves the measurements required in prior constructions and addresses a known open problem in the theory of sparse signal recovery. Our construction uses adjacency matrices of bipartite graphs that have large girth. The main result is that girth (the length of the shortest cycle in the graph) can be used as a certificate that a measurement matrix can recover almost all sparse signals. Specifically, our matrices guarantee recovery "*for-each*" sparse signal under basis pursuit. Our techniques are coding theoretic and rely on a recent connection of compressed sensing to LP relaxations for channel decoding.

1. INTRODUCTION

Consider m linear measurements of an unknown vector $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y},$$

where $\mathbf{A}$ is a real-valued matrix of size $m \times n$, called the measurement matrix. When $m < n$ this is an under-determined system of linear equations and one fundamental problem involves recovering $\mathbf{x}$ assuming that it is also k-sparse, *i.e.* it has k or less non-zero entries. Recent breakthrough results [1–3] showed that it is possible to construct measurement matrices with $m = O(k \log(n/k))$ rows that recover k-sparse signals exactly in polynomial time. This scaling of m is also optimal, as discussed in [4, 5]. These results rely on randomized matrix constructions and establish that the optimal number of measurements will be sufficient with high probability over the choice of the matrix and/or the signal. Random matrices generated by various distributions are known to work well with high probability, but in a real system one needs to implement a *specific matrix realization*. The related issues of deterministically creating "good" matrices or efficiently checking the performance of a random matrix realization are therefore important. One can rely on simulations, but a theoretical guarantee for deterministic measurement matrices is naturally desired.

Unfortunately, the required properties of Restricted Isometry Property (RIP) [1], Nullspace [6, 7], and high expansion (expansion quality $\epsilon < 1/6$) have no known ways to be deterministically constructed or efficiently checked. There are several explicit constructions of measurement matrices (*e.g.* [8–10]) which, however, require a slightly sub-optimal number of measurements (e.g. m growing super-linearly as a function of n for $k = p \cdot n$). This is closely related to constructing optimal deterministic measurement matrices that have RIP. As is well known, RIP is sufficient to imply recovery of sparse signals, but not necessary. The construction of deterministic measurement matrices with $m = \Theta(n \log(n/k))$ rows that have RIP is a well-known open problem in sparse approximation theory. The current state of the art is from [10] which, however, remains slightly sub-optimal.

In our recent prior work [11] we constructed explicit matrices for the linear sparsity regime (when $k = \Theta(n)$). In this paper we extend this machinery for any k, constructing the first known family of deterministic matrices with $m = \Theta(k \log(n/k))$ measurements that recover k-sparse signals. Our result is a 'for-each' signal guarantee [4]. This means that we have a fixed deterministic matrix and show the recovery of the sparse signal with high probability over the support of the signal. The recovery algorithm used is the basis pursuit linear program. To the best of our knowledge, this is the first deterministic construction with an order-optimal number of rows.

Note that we do not show that our matrices have RIP and hence do not resolve the deterministic RIP open problem. Our matrices recover sparse signals because they have the Nullspace condition, a condition that is known to be *necessary and sufficient* for recovery. Our techniques are coding-theoretic and rely on recent developments that connect the channel decoding LP relaxation by Feldman *et al.* [12] to compressed sensing [13]. We rely on a primal-based density evolution technique initiated by Koetter and Vontobel [14] and analytically strengthened by Arora *et al.* [15]. This important work established the best-known finite-length threshold results for LDPC codes under LP decoding. It is the translation of this method to the real-valued compressed sensing recovery problem via [13] that allows us to obtain our result.

We note that even though our analysis involves a rigorous density evolution argument, our decoder is always the basis pursuit linear relaxation which is substantially different from the related work on message-passing algorithms for compressed sensing [16, 17].

1.1. Our contribution

Our main result is the construction of deterministic measurement matrices with an optimal number of rows.

Theorem 1 *For any sparsity k, construct an $m \times n$ zero-one measurement matrix with $m = \Theta(k \log(n/k))$ using the Progressive Edge Growth (PEG) algorithm [18] using the degrees of (5). Assume the signal support is chosen uniformly among the $\binom{n}{k}$ possible choices. Under basis pursuit decoding, these matrices recover the unknown signal with probability at least $1 - 1/n$.*

In the low-sparsity regime, it is possible to obtain recovery of all possible supports. Specifically, if $k = O(\log(n))$, our construction has a 'for-all' signal guarantee.

Theorem 2 *The optimal measurement matrices designed by our construction, recover all k-sparse signals under basis pursuit when $k = O(\log(n))$.*

Due to space constraints we omit several proofs which can be found in the full manuscript [19].

2. BACKGROUND AND PRELIMINARIES

We start with some background material. We introduce the noiseless compressed sensing problem and basis pursuit decoding. Then, we mention the finite field channel coding problem and its relaxation. Finally, we discuss the LP decoding performance guarantee of [15].

2.1. Compressed sensing preliminaries

The simplest noiseless compressed sensing (CS) problem for exactly sparse signals consists of recovering the sparsest real vector $\mathbf{x}'$ of a given length n, *i.e.*, the vector with minimum ℓ_0 norm, from a set of m real-valued measurements $\mathbf{y}$, given by $\mathbf{A} \cdot \mathbf{x}' = \mathbf{y}$; As is well-known, ℓ_0 minimization is NP-hard, and one can relax the minimization by replacing the ℓ_0 norm with ℓ_1. Specifically,

$$\textbf{CS-LPD:} \quad \begin{array}{ll} \text{minimize} & \|\mathbf{x}'\|_1 \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x}' = \mathbf{y}. \end{array}$$

This LP relaxation is also known as basis pursuit. A fundamental question in compressed sensing is under what conditions the solution given by **CS-LPD** equals (or is very close to, in the case of approximately sparse signals) the solution given by ℓ_0 norm minimization, *i.e.*, the LP relaxation is tight. There has been a substantial amount of work in this area, see *e.g.* [1–4, 6, 7].

One sufficient way to certify that a given measurement matrix is "good" is through the Restricted Isometry Property (RIP), which guarantees that the LP relaxation will be tight for all k-sparse vectors $\mathbf{x}$ and further the recovery will be robust to approximate sparsity [1, 2]. However, RIP condition is not a complete characterization of the LP relaxation of "good" measurement matrices (see, *e.g.*, [20]). In this paper we rely on the null-space characterization (see, *e.g.*, [7, 21]) instead, that gives a necessary and sufficient condition for a matrix to be "good".

Definition 1 *Let $\mathcal{S} \subset \{1, \dots, n\}$, and let $C > 0$. We say that $\mathbf{A}$ has the nullspace property $NSP_{\mathbb{R}}^{\leq}(\mathcal{S}, C)$, and write $\mathbf{A} \in NSP_{\mathbb{R}}^{\leq}(\mathcal{S}, C)$, if*

$$C \cdot \|\nu_{\mathcal{S}}\|_1 \leq \|\nu_{\overline{\mathcal{S}}}\|_1, \textit{ for all } \nu \in \mathcal{N}(\mathbf{A}).$$

where for a vector ν and index set $\mathcal{S}$, we denote by $\nu_{\mathcal{S}}$ the subvector obtained by extracting the components of ν indexed by S. Further, We say that $\mathbf{A}$ has the strict nullspace property $NSP_{\mathbb{R}}^{<}(\mathcal{S}, C)$ and write $\mathbf{A} \in NSP_{\mathbb{R}}^{<}(\mathcal{S}, C)$, if

$$C \cdot \|\nu_{\mathcal{S}}\|_1 < \|\nu_{\overline{\mathcal{S}}}\|_1, \textit{ for all } \nu \in \mathcal{N}(\mathbf{A}) \setminus \{\mathbf{0}\}.$$

For $|\mathcal{S}| \leq k$, observe that the requirement is for vectors in the nullspace of $\mathbf{A}$ to have their ℓ_1 mass spread in substantially more than k coordinates. In fact, it can be shown that for $C > 1$, at least $2k$ coordinates must be non-zero. In this paper, however, we focus on the case $C = 1$ and leave the case $C > 1$ as a future direction. The following theorem, adapted from [22] (Proposition 2) explicitly states the tightness of the LP-relaxation under nullspace property.

Theorem 3 *Let $\mathbf{A}$ be a measurement matrix. Further, assume that $\mathbf{A} \cdot \mathbf{x} = \mathbf{y}$ and that $\mathbf{x}$ has at most k nonzero elements,* i.e., *$\|\mathbf{x}\|_0 \leq k$. Then the estimate $\mathbf{x}'$ produced by* **CS-LPD** *will equal the estimate $\mathbf{x}'$ produced by ℓ_0 minimization if $\mathbf{A} \in NSP_{\mathbb{R}}^{<}(\mathcal{S}, C)$.*

2.2. Channel coding preliminaries

A linear binary code $\mathcal{C}$ of length n is defined by an $m \times n$ parity-check matrix $\mathbf{H}$, *i.e.*, $\mathcal{C} \triangleq \{\mathbf{c} \in \mathbb{F}_2^n | \mathbf{H} \cdot \mathbf{c} = \mathbf{0}\}$. We define the set of codeword indices $\mathcal{I} \triangleq \mathcal{I}(\mathbf{H}) \triangleq \{1, \dots, n\}$, the set of check indices $\mathcal{J} \triangleq \mathcal{J}(\mathbf{H}) \triangleq \{1, \dots, m\}$, the set of check indices that involves the i-th codeword position $\mathcal{J}_i \triangleq \mathcal{J}_i(\mathbf{H}) \triangleq \{j \in \mathcal{J} | [\mathbf{H}]_{j,i} = 1\}$, and the set of codeword positions that are involved in the j-th check $\mathcal{I}_j \triangleq \mathcal{I}_j(\mathbf{H}) \triangleq \{i \in \mathcal{I} | [\mathbf{H}]_{j,i} = 1\}$. The degree of a variable node i and a check node j are $|\mathcal{J}_i|$ and $|\mathcal{I}_j|$, respectively. For a regular code $|\mathcal{J}_i| = d_\ell$ and $|\mathcal{I}_j| = d_r$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. We denote such a code by (d_ℓ, d_r)-regular code. Throughout, we assume $d_r, d_\ell \geq 3$. The relation between the degrees and number of rows and columns is controlled by the so-called hand-shaking lemma:

$$\textbf{HS:} \qquad m \cdot d_r = n \cdot d_\ell.$$

If a codeword $\mathbf{c} \in \mathcal{C}$ is transmitted through a binary-input memoryless channel with transition probability measure $\mathbb{P}(r|c)$, and an output sequence $\mathbf{r}$ is received, then one can potentially decode $\mathbf{c}$ by solving for the maximum likelihood codeword in $\mathcal{C}$, namely

$$\begin{array}{lll}\textbf{CC-MLD:} & \text{minimize} & \lambda^T \mathbf{c}' \\ & \text{subject to} & \mathbf{c}' \in \text{conv}(\mathcal{C})),\end{array}$$

where λ is the likelihood vector with components $\lambda_i = \log(\frac{\mathbb{P}(r_i|c_i=0)}{\mathbb{P}(r_i|c_i=1)})$, and conv($\mathcal{C}$) is the convex hull of all codewords of $\mathcal{C}$ in $\mathbb{R}^n$. **CC-MLD** is NP-hard and therefore an efficient description of the exact codeword polytope is very unlikely to exist.

The channel decoding LP relaxation [12] is:

$$\begin{array}{lll}\textbf{CC-LPD:} & \text{minimize} & \lambda^T \mathbf{c}' \\ & \text{subject to} & \mathbf{c}' \in \mathcal{P}(\mathbf{H}),\end{array}$$

where $\mathcal{P} = \mathcal{P}(\mathbf{H})$ is known as the fundamental polytope [12, 14]. The fundamental polytope is compactly described as follows: If $\mathbf{h_j}^T$ is the j-th row of $\mathbf{H}$, then

$$\mathcal{P} = \cap_{1 \leq j \leq m} \text{conv}(\mathcal{C}_j), \tag{1}$$

where $\mathcal{C}_j = \{\mathbf{c} \in \mathbb{F}^n \mid \mathbf{h_j}^T \mathbf{c} = 0 \text{ mod } 2\}$. Due to the symmetries of the fundamental polytope [12] we can focus on the cone around the all-zeros codeword without loss of generality. Given the parity check matrix $\mathbf{H}$, its fundamental cone $\mathcal{K}(\mathbf{H})$ is defined as the smallest cone in $\mathbb{R}^n$ that contains $\mathcal{P}(\mathbf{H})$

Given the fundamental cone of a code $\mathcal{C}$, we define the following property.

Definition 2 *Let $S \subset \{1, 2, \cdots, n\}$ and $C \geq 1$ be fixed. A code $\mathcal{C}$ with parity check matrix $\mathbf{H}$ is said to have the fundamental cone property FCP$(\mathcal{S}, C)$ if for every $\mathbf{w} \in \mathcal{K}(\mathbf{H})$ the following holds:*

$$C \cdot \|\mathbf{w}_\mathcal{S}\|_1 < \|\mathbf{w}_{\overline{\mathcal{S}}}\|_1. \tag{2}$$

Lemma 1 *Let $\mathbf{H}$ be a parity-check matrix of a code $\mathcal{C}$ and let $\mathcal{S} \subset \mathcal{I}(\mathbf{H})$ be a particular set of coordinate indices that are flipped by a binary symmetric channel (BSC) with cross-over probability $p > 0$. The solution of* **CC-LPD** *equals the codeword that was sent if and only if $\mathbf{H}$ has the FCP$(\mathcal{S}, 1)$.*

The proof can be found in [23].

2.3. LP-decoding performance guarantee

Arora *et al.* [15] introduced the following best known performance guarantee for the channel decoding LP:

Theorem 4 *Let $\mathcal{C}$ be a regular (d_ℓ, d_r)-LDPC code whose corresponding bipartite graph has girth equal to g. Further, let $0 \leq p \leq 1/2$ be the probability of a bit flip in BSC, and S be the random set of flipped bits. If*

$$\gamma = \min_{t \geq 0} \Gamma(t, p, d_r, d_\ell) < 1,$$

where

$$\begin{aligned}\Gamma(t, p, d_r, d_\ell) = \;& \left((1-p)^{d_r-1} e^{-t} + (1-(1-p)^{d_r-1}) e^t\right) \\ & \cdot \left((d_r - 1) \cdot ((1-p)e^{-t} + pe^t)\right)^{1/(d_\ell - 2)}\end{aligned} \tag{3}$$

Then with probability at least

$$Prob(\mathbf{H} \in FCP(\mathcal{S}, 1)) \geq 1 - n\gamma^{d_\ell (d_\ell - 1)^{T-1} - d_\ell}, \tag{4}$$

where $T = \lfloor g/4 \rfloor$, the code $\mathcal{C}$ corrects the error pattern S, i.e.*, it has FCP$(S, 1)$, where T is any integer $T < g/4$.*

Note that the value γ can be derived for specific values of p, d_ℓ, and d_r. The proof can be found in [15].

Note that for the above probability to converge to one, it is required that $\gamma < 1$ and T to be an increasing function in n. For instance, for $T = \Theta(\log(n) < g/4$, we get the probability of error to be $O(\exp(-cn))$ for some positive constant c which depends on d_ℓ.

2.4. Establishing the Connection

In this section, through Lemma 2 (taken from [24]), we will establish a bridge between LP-decoding and compressed sensing via which we later import the guarantees for LP decoding into compressed sensing.

Lemma 2 *(Lemma 6 in [24]): Let $\mathbf{A}$ be a zero-one measurement matrix. Further, let $\mathbf{H}_A$ denote the same matrix $\mathbf{A}$ as the parity check matrix of a code over GF(2). Then*

$$\nu \in \mathcal{N}(\mathbf{A}) \Rightarrow |\nu| \in \mathcal{K}(\mathbf{H}_A),$$

where $|\nu|$ is the vector obtained by taking —·— of the components of ν. Note that the bridge established in [23] and through Lemma 2 connects **CC-LPD** of the binary linear channel code and **CS- LPD** based on a zero-one measurement matrix over reals by viewing this binary parity-check matrix as a measurement matrix. This connection allows the translation of performance guarantees from one setup to the other. Using this bridge, we can show that parity-check matrices of "good" channel codes can be used as provably "good" measurement matrices under basis pursuit.

To import the performance guarantee of [15] into compressed sensing setup, we take the following steps. First through Theorem 4, we show that the code $\mathcal{C}$ has the fundamental cone property FCP$(\mathcal{S}, 1)$ with probability at least $1 - n\gamma^{d_\ell(d_\ell-1)^{T-1}-d_\ell}$. Next, through Lemma 1, we demonstrate that $\mathcal{C}$ corrects the error configuration $\mathcal{S}$ at the output of the BSC. Finally, by Lemma 2, we establish a connection between the properties of $\mathbf{H}_A$ as parity check matrix (*i.e.* FCP condition) and its null space properties as a measurement matrix in compressed sensing. Consequently, with probability at least $1 - n\gamma^{d_\ell(d_\ell-1)^{T-1}-d_\ell}$ the solution of **CS-LPD** is equal to ℓ_0 minimization by Theorem 3.

3. ACHIEVABILITY

In this section we present our construction. That is, as mentioned in section 2, we build a parity-check matrix of a (d_ℓ, d_r)-regular LDPC code of length n using progressive edge growth method discussed in [18]. As mentioned, this code can correctly decode an output of a BSC channel with a flip probability p with high probability because of its girth properties. Then, based on Lemma 2, the parity-check matrix when used as a measurement matrix can recover k-sparse signals for $k = pn$ with high probability.

Proof of Theorem 1: We construct a (d_ℓ, d_r)-regular LDPC code with the following number of rows and degrees using the Progressive Edge Growth algorithm [18].

$$\begin{aligned} m &= c \cdot np\log(1/p) \\ d_r &= c_r \cdot 1/p \\ d_\ell &= c_\ell \cdot \log(1/p), \end{aligned} \tag{5}$$

where c, c_r, c_ℓ are constants which must satisfy **HS**, *i.e.*, $c_\ell = c \cdot c_r$.

Note that for the lower bound (4) to tend to one, we require $\gamma < 1$ and T or d_ℓ to be a growing function of n. That is, if we require to show that there exist constants c, c_r, c_ℓ, and t for which the function Γ defined in (3) becomes less than one for our construction.

Lemma 3 *There exists c, c_r, c_ℓ, and t for which the function Γ (3) with values (5) is less than one.*

The proof can be found in [19].

Note that, a reader might assume that our construction may fail to recover sparsity regimes p for which T becomes a constant. Let us find such instances and show that our construction is able to recover them.

As mentioned, the minimum number of measurements m_{min} required for recovering a k-sparse signal is $np\log(1/p)$. Inserting m_{min} into **HS** gives $\frac{d_r}{d_\ell} = \frac{1}{p\log(1/p)}$.

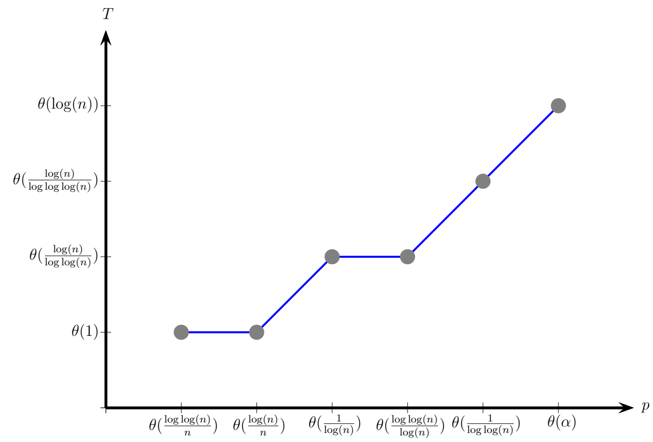

Fig. 1: The scale of the girth T against p if $d_r = d_r^{min}$ is used.

The progressive edge growth technique, requires the parity-check matrix to satisfy $1 + \sum_{i=1}^{T-1} d_\ell^i \cdot d_r^i < n$. Assuming that either $d_\ell \cdot d_r \gg 1$ or $T \to \infty$ we can neglect the smaller terms, and by taking logarithm from both sides we get $T\log(d_\ell \cdot d_r) < \log(n)$. Further, from **HS** we know that for fixed n and m, $d_r = \frac{n}{m}d_\ell$ is an increasing function of d_ℓ. Therefore, the minimum d_r can be achieved when d_ℓ is a constant. Let us denote this minimum d_r by d_r^{min}, which is

$$d_r^{min} = \Theta(\frac{1}{p\log(1/p)}) \tag{6}$$

Furthermore, increasing d_ℓ will also cause the product $d_\ell \cdot d_r$ to grow. As a result, the minimum of this product is attained at $d_r = d_r^{min}$ and d_ℓ equal a constant.

After inserting (6) into $T\log(d_\ell \cdot d_r) < \log(n)$, it is interesting to check for which sparsity regimes T grows in n as $n \to \infty$. As shown in Fig. 1, for classes $p = \log(n)/n$ and $\log\log(n)/n$, the depth T becomes a constant. Thus, a reader might assume that our construction cannot recover sparsities $p = O(\log(n)/n)$. Here, we discuss that this is not the case, and our construction is indeed valid for all sparsities.

As shown in (5), for our construction $d_\ell = \Theta(\log(1/p))$. That is, for sublinear sparsities, since p is a decreasing function of n, d_ℓ becomes a monotonically increasing function. Looking at the lower bound (4), it seems possible that for $\gamma < 1$ and large enough d_ℓ, the bound converges to one with speed higher than $1 - 1/n$. In the following lemma, we show that this indeed happens and our scheme works for all sparsity regimes.

Lemma 4 *For the codes generated by progressive edge growth technique and values (5), the lower bound on the probability of successful decoding (4) tends to one for all values of $p \in (0, 1/2)$ including those for which $T > 2$ is a constant.*

The proof can be found in [19] and thus the proof of the Theorem 1 is complete.

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