ALGEBRAIC PHASE UNWRAPPING FOR FUNCTIONAL DATA ANALYTIC ESTIMATIONS — EXTENSIONS AND STABILIZATIONS

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ABSTRACT

The phase unwrapping, which is a problem to reconstruct the continuous phase function of an unknown complex function from its finite observed samples, has been a key for estimating useful physical quantity in many signal and image processing applications. In the light of the functional data analysis, it is natural to estimate first the unknown complex function by a certain piecewise complex polynomial and then to compute the exact unwrapped phase of the piecewise complex polynomial with the algebraic phase unwrapping algorithms. In this paper, we propose several useful extensions and numerical stabilization of the algebraic phase unwrapping along the real axis. The proposed extensions include (i) removal of a certain critical assumption premised in the original algebraic phase unwrapping, and (ii) algebraic phase unwrapping for a pair of bivariate polynomials. Moreover, in order to resolve certain numerical instabilities caused by the coefficient growth in an inductive step in the original algorithm, we propose to compute directly a certain subresultant sequence without passing through the inductive step.

Index Terms— Algebraic phase unwrapping, Functional data analysis, Two-dimensional phase unwrapping, Path independence condition, Numerical stabilization

1. INTRODUCTION

In many signal and image processing problems, the phase unwrapping has been a key for estimating some physical quantity [1, 2], for example, surface topography in synthetic aperture radar (SAR) (and synthetic aperture sonar (SAS)) interferometry [3, 4, 5, 6, 7], wavefront distortion in adaptive optics [8, 9, 10], the degree of magnetic field inhomogeneity in the water/fat separation problem of magnetic resonance imaging (MRI) [11, 12, 13], the relationship between the object phase and the bispectrum phase in astronomical imaging [14, 15] and the accurate profiling of mechanical parts by x-ray [16, 17]. Recently the phase unwrapping has been applied to a frequency estimation problem [18] and a DOA estimation problem [19].

Suppose that

$$\big(d_{(0)}(\gamma(\zeta_k)), d_{(1)}(\gamma(\zeta_k))\big) = \big(f_{(0)}(\gamma(\zeta_k))+\varepsilon_{(0)}(\gamma(\zeta_k)), f_{(1)}(\gamma(\zeta_k))+\varepsilon_{(1)}(\gamma(\zeta_k))\big)\in\mathbb{R}^2 \quad (1)$$

$(k = 1, 2, \ldots, s)$ are given as a finite sequence of 2-D noisy real vectors, where $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ are unknown functions, $\varepsilon_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ are additive random noise functions, and $\gamma : [a, b] \to \mathbb{R}^2$ is a known piecewise C^1 function which defines a path along the sample points $\gamma(\zeta_k) \in \mathbb{R}^2$ $(a \le \zeta_1 < \zeta_2 < \cdots < \zeta_s \le b)$.

For simplicity, denote by $F : [a, b] \ni t \mapsto F_{(0)}(t) + jF_{(1)}(t) \in \mathbb{C}$ a univariate complex valued function defined as

$$F_{(i)}(t) := f_{(i)}(\gamma(t)) \text{ for all } t \in [a, b] \quad (i = 0, 1).$$

The two-dimensional phase unwrapping of $(f_{(0)}, f_{(1)})$ along γ at $(x^*, y^*) := \gamma(t^*) \in \mathbb{R}^2$, or the phase unwrapping of F at $t^* \in [a, b]$ along the real axis, is a problem of estimating the unwrapped phase

$$\theta_f^{[\gamma]}(x^*, y^*) := \theta_F(t^*) := \theta_F(a) + \int_a^{t^*} \Im\left\{\frac{F'_{(0)}(t) + jF'_{(1)}(t)}{F_{(0)}(t) + jF_{(1)}(t)}\right\} dt \quad (2)$$

by using the data $\big(d_{(0)}(\gamma(\zeta_k)), d_{(1)}(\gamma(\zeta_k))\big)$, where $\theta_F(a) \in (-\pi, \pi]$ satisfies $F(a) = |F(a)|e^{j\theta_F(a)}$.

Despite the tremendous effort made so far, a technically reliable phase unwrapping has not yet been established for its practical use in wide range of signal and image processing. This is mainly because $\theta_F(t)$ $(a \le t \le b)$ is continuously defined along the arc $\gamma([a, b])$ as in (2) while most existing phase unwrapping algorithms, e.g., path-following methods [4, 20, 21, 22], minimum-norm methods [23, 24, 25] and network flow methods [26, 27] estimate the unwrapped phase θ_F only at ζ_k $(k = 1, 2, \ldots, s)$ without checking the consistency with θ_F at $t \in (\zeta_k, \zeta_{k+1})$.

In this paper, in the spirit of functional data analysis [28, 29, 30, 31, 32, 33], we consider the situation where the functions $F_{(i)} : [a, b] \to \mathbb{R}$ $(i = 0, 1)$ have been approximated respectively by piecewise polynomials (i.e., Spline functions) $\widetilde{F}_{(i)} : [a, b] \to \mathbb{R}$ $(i = 0, 1)$ through some smoothing techniques. In such a case, it is natural to estimate $\theta_F(t^*)$ in (2) by

$$\theta_{\widetilde{F}}(t^*) := \theta_{\widetilde{F}}(a) + \int_a^{t^*} \Im\left\{\frac{\widetilde{F}'_{(0)}(t) + j\widetilde{F}'_{(1)}(t)}{\widetilde{F}_{(0)}(t) + j\widetilde{F}_{(1)}(t)}\right\} dt. \quad (3)$$

By dividing the interval $[a, b]$ into finite subintervals, the unwrapped phase $\theta_{\widetilde{F}}(t^*)$ in (3) can be computed [34] by the algebraic phase unwrapping along the real axis [19, 35] without requiring any numerical root finding or numerical integration technique.

However, in a direct computer implementation of the algorithms in the algebraic phase unwrapping [19, 36, 37][1] as well as in a direct implementation of Algorithm 1 (Sturm-$\mathcal{R}$) in Sec. 2.2, we encounter numerical instabilities, especially for polynomials of relatively large degree, due to the unavoidable gap between theoretical value and numerical value computed by digital computer using finite digit number systems. Therefore, thoughtless direct implementation of the algebraic phase unwrapping algorithms for polynomials of large degree, sometimes results in the failure of the phase unwrapping.

The goal of this paper is to present several extensions and numerical stabilization of the algebraic phase unwrapping along the real axis [19]. In Sec. 2, we present a new algorithm (Algorithm 1) to define a new *Sturm sequence*, unlike [19, SGA 2]. Theorem 1 based on Algorithm 1 can deal with a special case $A_{(0)}(a) = 0$ which is excluded in [19, Theorem 1]. In Sec. 3, we consider the two-dimensional phase unwrapping and elucidate the condition for

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[1] The algebraic phase unwrapping for complex polynomials along the unit circle was established first in [36]. As its continuations, the algebraic phase unwrapping along the real axis [19] and that along the imaginary axis [37] have been developed.

the path independence of the two-dimensional phase unwrapping. In particular, if bivariate polynomial functions $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i=0,1)$ satisfy $f(x,y) := f_{(0)}(x,y) + jf_{(1)}(x,y) \neq 0$ for all (x,y) in a simply connected domain $D \subset \mathbb{R}^2$, the two-dimensional unwrapped phase $\theta_f \in C^2(D)$ can be computed uniquely with Theorem 1. In Sec. 4, in order to stabilize the computation of $\theta_A(t^*)$ in Theorem 1, we propose to replace the inductive computation of the polynomials $\Psi_k(t)$ $(k=0,1,\ldots,q)$ in Algorithm 1, followed by their numerical evaluation at $t^* \in [a,b]$, with the direct numerical computation of the subresultant sequence [38, 39] at t^*. For this purpose, we present relation between the sign of the Sturm sequence and that of the subresultant sequence (Proposition 1). By the proposed replacement, the sign of the Sturm sequence can be computed without suffering from the propagation of errors caused by the coefficient growth in the process of Algorithm 1, and then the algebraic phase unwrapping is stabilized greatly even for polynomials of relatively large degree. The extensive numerical experiments exemplify the notable performance improvement made by the proposed numerical stabilization.

Relation to Prior Work

The work presented here focuses on the extension and stabilization of the algebraic phase unwrapping along the real axis [19]. The work by Yamada and Oguchi [19] does not consider the path independence of the two-dimensional phase unwrapping, and the original algorithm [19, SGA 2] sometimes causes certain numerical instabilities in the computer implementation. Therefore in this paper, we elucidate the condition for the path independence of the two-dimensional phase unwrapping, and extend the algebraic phase unwrapping for a pair of bivariate polynomials. Moreover we stabilize the algebraic phase unwrapping with the subresultant sequence.

2. ALGEBRAIC PHASE UNWRAPPING FOR POLYNOMIALS

2.1. Notations

Let $\mathbb{N}^*$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the set of all positive integers, real numbers and complex numbers. We use $j \in \mathbb{C}$ to denote the imaginary unit satisfying $j^2 = -1$. For any $c \in \mathbb{C}$, $\Re(c)$ and $\Im(c)$ stand respectively for the real and imaginary parts of c. For any $C(t) = \sum_{k=0}^{m} c_k t^k \in \mathbb{C}[t]$ (s.t. $c_m \neq 0$ and $m \geq 0$), we define $\deg(C) := m$ and $\mathrm{lc}(C) := c_m$. For any $C(t) = \sum_{k=0}^{m} c_k t^k \in \mathbb{C}[t]$, we use the expression $C(t) = C_{(0)}(t) + jC_{(1)}(t)$, where $C_{(0)}(t) := \sum_{k=0}^{m} \Re(c_k) t^k \in \mathbb{R}[t]$ and $C_{(1)}(t) := \sum_{k=0}^{m} \Im(c_k) t^k \in \mathbb{R}[t]$. For any $x \in \mathbb{R}$, its sign is defined by

$$\mathrm{sgn}(x) := \begin{cases} x/|x| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and $\arctan$ denotes the principle value inverse tangent satisfying $\tan(\arctan(x)) = x$ and $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$.

2.2. Algebraic Phase Unwrapping

The next theorem presents an exact solution of the phase unwrapping problem for complex polynomials along the real axis. This theorem is a relaxation of [19, Theorem 1]. Indeed, Theorem 1 can deal with a special case $A_{(0)}(a) = 0$ which is excluded in [19, Theorem 1].

Theorem 1 (Algebraic phase unwrapping for a univariate complex polynomial along the real axis) *Let $\{\Psi_k(t)\}_{k=0}^{q}$ be the Sturm sequence generated by applying Algorithm 1 (Sturm-$\mathcal{R}$) to $A_{(0)}(t) \in \mathbb{R}[t]$ and $A_{(1)}(t) \in \mathbb{R}[t]$ under the assumptions $A(t) := A_{(0)}(t) + jA_{(1)}(t) \neq 0$ $(t \in [a,b])$, $A_{(0)}(t) \not\equiv 0$ and $A_{(1)}(t) \not\equiv 0$. Define at each $t \in [a,b]$ the number of variations in the sign of $\{\Psi_k(t)\}_{k=0}^{q}$ by*

$$\begin{aligned} V\{\Psi(t)\} &:= V\{\Psi_0(t), \Psi_1(t), \ldots, \Psi_q(t)\} \\ &:= \left|\{i \mid 0 \leq i < q \text{ and } \Psi_i(t)\Psi_{i+\varrho(i)}(t) < 0\}\right|, \end{aligned}$$

where $\varrho(i) := \min\{k \in \mathbb{N}^* \mid \Psi_{i+k}(t) \neq 0\}$. Then, for every $t^* \in (a,b]$, we have

$$\begin{aligned} \theta_A(t^*) &= \theta_A(a) + \int_a^{t^*} \frac{A'_{(1)}(t)A_{(0)}(t) - A_{(1)}(t)A'_{(0)}(t)}{\{A_{(0)}(t)\}^2 + \{A_{(1)}(t)\}^2} dt \\ &= \theta_A(a) - \left\{ \begin{array}{ll} \arctan\{\mathcal{Q}_A(a)\} & \text{if } A_{(0)}(a) \neq 0, \\ \mathrm{sgn}(\Psi_0(a)\Psi_1(a))\pi/2 & \text{if } A_{(0)}(a) = 0, \end{array} \right\} \\ &+ \begin{cases} \arctan\{\mathcal{Q}_A(t^*)\} + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi & \text{if } A_{(0)}(t^*) \neq 0, \\ \pi/2 + [V\{\Psi(t^*)\} - V\{\Psi(a)\}]\pi & \text{if } A_{(0)}(t^*) = 0, \end{cases} \end{aligned} \quad (4)$$

where $\mathcal{Q}_A(t) := \frac{A_{(1)}(t)}{A_{(0)}(t)}$ and $\theta_A(a) \in (-\pi, \pi]$ s.t. $A(a) = |A(a)|e^{j\theta_A(a)}$.

Algorithm 1 Sturm generating algorithm (Sturm-$\mathcal{R}$)

Input: $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$ and $a \in \mathbb{R}$

$\Psi_0(t) \leftarrow \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$, $\Psi_1(t) \leftarrow \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$ (where e_i denotes the order of $t = a$ as a zero of polynomial $A_{(i)}(t)$ $(i = 0,1)$)
$k \leftarrow 1$
while $\deg(\Psi_k) \neq 0$ **do**
  $\Psi_{k+1}(t) \leftarrow -\Psi_{k-1}(t) + H_k(t)\Psi_k(t)$ (where $H_k(t) \in \mathbb{R}[t]$ and $\deg(\Psi_{k+1}) < \deg(\Psi_k)$)
  $k \leftarrow k+1$
end while
$q \leftarrow \begin{cases} k & \text{if } \Psi_k(t) \not\equiv 0 \\ k-1 & \text{if } \Psi_k(t) \equiv 0 \end{cases}$

Output: $\{\Psi_k(t)\}_{k=0}^{q}$

Example 1 *(Expression of the exact unwrapped phase by Theorem 1) Let us construct the unwrapped phase $\theta_A(t)$ $(0 \leq t \leq 1)$ of the univariate complex polynomial*

$$\begin{aligned} A(t) &:= A_{(0)}(t) + jA_{(1)}(t) \\ &= (t^4 - 1.11t^3 + 0.356t^2 - 0.0255t) \\ &\quad + j(t^4 - 2.525t^3 + 2.29995t^2 - 0.906172t + 0.131222) \end{aligned}$$

without using any root finding or numerical integration technique.

Applying Algorithm 1 to $A_{(0)}(t)$ and $A_{(1)}(t)$ for $a = 0$ and $b = 1$, we obtain the Sturm sequence $\{\Psi_k(t)\}_{k=0}^{5}$ as

$$\begin{aligned} \Psi_0(t) &= t^3 - \frac{111}{100}t^2 + \frac{89}{250}t - \frac{51}{2000}, \\ \Psi_1(t) &= t^4 - \frac{101}{40}t^3 + \frac{45999}{20000}t^2 - \frac{226543}{250000}t + \frac{65611}{500000}, \\ \Psi_2(t) &= -t^3 + \frac{111}{100}t^2 - \frac{89}{250}t + \frac{51}{2000}, \\ \Psi_3(t) &= -\frac{3733}{10000}t^2 + \frac{94233}{250000}t - \frac{190279}{2000000}, \\ \Psi_4(t) &= -\frac{27788829033}{260102169185000}t + \frac{15335859}{278705780000}, \\ \Psi_5(t) &= \frac{339145264784010639558466646077921181 1}{119967177270575015975354069525774695200000}. \end{aligned}$$

From $A_{(0)}(0) = 0$ and $A_{(1)}(0) = \frac{65611}{500000}$, we have $\theta_A(0) = \pi/2$. Moreover, from $\mathrm{sgn}(\Psi_0(0)\Psi_1(0)) = \mathrm{sgn}\left(-\frac{3346161}{1000000000}\right) = -1$

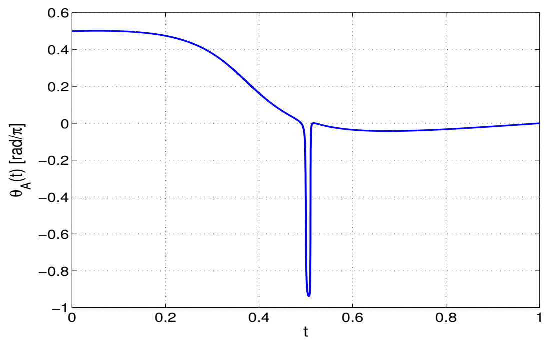

Fig. 1. Exact unwrapped phase by Theorem 1

and $V\{\Psi(0)\} = V\{-\frac{51}{2000}, \frac{65611}{500000}, \frac{51}{2000}, -\frac{190279}{2000000}, \frac{15335859}{278705780000}, \frac{33914526478401063955846664607792118 11}{1199671772705750159753540695257746952000 00}\} = 3$, *the unwrapped phase* $\theta_A(t)$ $(0 < t \leq 1)$ *in (4) is expressed as*

$$\theta_A(t) = \pi + \begin{cases} \arctan\{\mathcal{Q}_A(t)\} + [V\{\Psi(t)\} - 3]\pi & \textit{if } A_{(0)}(t) \neq 0, \\ \pi/2 + [V\{\Psi(t)\} - 3]\pi & \textit{if } A_{(0)}(t) = 0, \end{cases}$$

which is depicted in Fig. 1.

From Fig. 1, we observe that the unwrapped phase function θ_A *can vary rapidly even if* $\deg(A)$ *is small, which suggests the inherent difficulty in phase unwrapping problem. Obviously, this notable feature is hardly detectable by most exiting phase unwrapping algorithms, i.e., [2, 4, 20, 21, 22, 23, 24, 25, 26, 27], essentially based on discrete approximations. Moreover, we also observe that the necessary number of digits to express the coefficients of* $\{\Psi_k(t)\}_{k=0}^{q}$ *grows quickly. This phenomenon is called the coefficient growth, which causes numerical instabilities in the direct computer implementation of Algorithm 1 (Sturm-*$\mathcal{R}$*) (see Sec. 4.1).*

3. EXTENSION OF THE ALGEBRAIC PHASE UNWRAPPING ALONG THE REAL AXIS

The two-dimensional unwrapped phase generally depends on the path of integral.

The next theorem presents a condition which guarantees (i) the unique existence of the two-dimensional unwrapped phase as a C^2 function and (ii) the path independence of the unwrapped phase.

Theorem 2 (Path independence of two-dimensional phase unwrapping) *Let* $D \subset \mathbb{R}^2$ *be a simply connected domain. Suppose that* $f_{(i)} : \mathbb{R}^2 \to \mathbb{R}$ $(i = 0, 1)$ *are* $C^2(D)$ *functions satisfying* $f(x, y) := f_{(0)}(x, y) + j f_{(1)}(x, y) \neq 0$ *for all* $(x, y) \in D$*. Then the followings hold.*

(a) (Unique existence of two-dimensional unwrapped phase) *Suppose that* $\theta_0 \in (-\pi, \pi]$ *satisfying* $f(x_0, y_0) = |f(x_0, y_0)|e^{j\theta_0}$ *is given at some* $(x_0, y_0) \in D$*, then there exist a unique function* $\theta_f \in C^2(D)$ *satisfying* $\theta_f(x_0, y_0) = \theta_0$ *and for all* $(x, y) \in D$

$$\left.\begin{array}{l} \dfrac{\partial \theta_f}{\partial x}(x, y) = \Im\left\{ \dfrac{\frac{\partial f_{(0)}}{\partial x}(x, y) + j\frac{\partial f_{(1)}}{\partial x}(x, y)}{f_{(0)}(x, y) + j f_{(1)}(x, y)} \right\} \\ \dfrac{\partial \theta_f}{\partial y}(x, y) = \Im\left\{ \dfrac{\frac{\partial f_{(0)}}{\partial y}(x, y) + j\frac{\partial f_{(1)}}{\partial y}(x, y)}{f_{(0)}(x, y) + j f_{(1)}(x, y)} \right\} \end{array}\right\}.$$

(b) (Path independence of two-dimensional unwrapped phase) *Suppose* $\gamma^{\mathrm{I}} : [a, b] \to D$ *and* $\gamma^{\mathrm{II}} : [c, d] \to D$ *satisfy* $\gamma^{\mathrm{I}}(a) = \gamma^{\mathrm{II}}(c) = (x_0, y_0) \in D$ *and* $\gamma^{\mathrm{I}}(b) = \gamma^{\mathrm{II}}(d) = (x_1, y_1) \in D$*. Then*

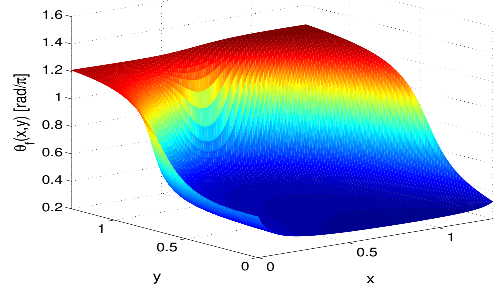

Fig. 2. Exact two-dimensional unwrapped phase

$$\begin{aligned} &\theta_f(x_1, y_1) \\ &= \theta_f(x_0, y_0) + \int_a^b \Im\left\{ \frac{(f_{(0)}(\gamma^{\mathrm{I}}(t)))' + j\,(f_{(1)}(\gamma^{\mathrm{I}}(t)))'}{f_{(0)}(\gamma^{\mathrm{I}}(t)) + j f_{(1)}(\gamma^{\mathrm{I}}(t))} \right\} dt \\ &= \theta_f(x_0, y_0) + \int_c^d \Im\left\{ \frac{(f_{(0)}(\gamma^{\mathrm{II}}(\tau)))' + j\,(f_{(1)}(\gamma^{\mathrm{II}}(\tau)))'}{f_{(0)}(\gamma^{\mathrm{II}}(\tau)) + j f_{(1)}(\gamma^{\mathrm{II}}(\tau))} \right\} d\tau. \end{aligned}$$

Example 2 *Let us construct the unwrapped phase of the bivariate complex polynomial* $f(x, y) := f_{(0)}(x, y) + j f_{(1)}(x, y)$ *over* $[0, 1.3] \times [0, 1.3]$ *by using Theorem 1 repeatedly, where*

$$\begin{aligned} f_{(0)}(x, y) &:= x^4 y - 4x^4 - 2x^3 y - 3xy + 10x - 2y^3, \\ f_{(1)}(x, y) &:= x^4 y - 4x^4 - 2x^3 y - 3xy + 10x - 2y^3 + 1. \end{aligned}$$

Since $f_{(1)}(x, y) = f_{(0)}(x, y) + 1$ *for all* $(x, y) \in \mathbb{R}^2$*, we have* $f(x, y) \neq 0$ *for all* $(x, y) \in \mathbb{R}^2$*. For any* $(x^*, y^*) \in [0, 1.3] \times [0, 1.3]$*, we choose the piecewise* C^1 *path* $\gamma_{(x^*, y^*)}$ *as*

$$\gamma_{(x^*, y^*)}(t) := \begin{cases} (t, 0) & \textit{if } 0 \leq t \leq x^*, \\ (x^*, t - x^*) & \textit{if } x^* \leq t \leq x^* + y^*. \end{cases}$$

Figure 2 depicts the unwrapped phase $\theta_f(x, y)$ *computed by using Theorem 1 repeatedly for two subintervals* $[0, x^*]$ *and* $[x^*, x^* + y^*]$*.*

4. STABILIZATION OF THE ALGEBRAIC PHASE UNWRAPPING ALONG THE REAL AXIS

4.1. Numerical Instabilities of Algorithm 1

To implement Algorithm 1 (Sturm-$\mathcal{R}$) precisely, we need large number of digits to express the rational coefficients of the polynomials $\Psi_k(t)$ (e.g., see Example 1). This phenomenon is exactly same as the *coefficient growth* well-known in the computation of the *polynomial remainder sequence* through the Euclidean algorithm [39]. In computer implementation of $\theta_A(t)$ in Eq. (4) through Algorithm 1, the coefficient growth causes the truncation error in the floating-point expression of the rational coefficients (or memory shortages by increasing number of digits for exact expression of the rational coefficients). In particular, once a serious *information loss* (by the addition or subtraction among numbers of ill-balanced absolute values) or *catastrophic cancellation* (by the subtraction number very close numbers) occurs, the gap between theoretical values and numerical values of $\Psi_k(t^*)$ by digital computer becomes unacceptably large.

4.2. Numerical Stabilization by Subresultant Sequence

For a pair of real polynomials

$$\begin{aligned} \Psi_0(t) &:= a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0, \\ \Psi_1(t) &:= b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0, \end{aligned}$$

s.t. $a_m \neq 0$ and $b_n \neq 0$, the i-th *subresultant* $\mathrm{Sres}_i(\Psi_0, \Psi_1, t)$ $(i = 0, 1, \ldots, \min\{m-1, n-1\})$ of $\Psi_0(t)$ and $\Psi_1(t)$ is defined as the determinant of a $(m+n-2i) \times (m+n-2i)$ matrix:

$$\mathrm{Sres}_i(\Psi_0, \Psi_1, t)$$

$$:= \det \begin{pmatrix} a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & & & \Psi_0(t)t^{n-i-1} \\ & a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & a_0 & & \Psi_0(t)t^{n-i-2} \\ & & \ddots & \ddots & & \ddots & \ddots & & \ddots & \vdots \\ & & & a_m & a_{m-1} & \cdots & a_i & a_{i-1} & \cdots & \Psi_0(t)t^i \\ & & & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & & & & a_m & a_{m-1} & \cdots & a_i & \Psi_0(t)t \\ & & & & & & a_m & \cdots & a_{i+1} & \Psi_0(t) \\ b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & & & \Psi_1(t)t^{m-i-1} \\ & b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & b_0 & & \Psi_1(t)t^{m-i-2} \\ & & \ddots & \ddots & & \ddots & \ddots & & \ddots & \vdots \\ & & & b_n & b_{n-1} & \cdots & b_i & b_{i-1} & \cdots & \Psi_1(t)t^i \\ & & & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & & & & b_n & b_{n-1} & \cdots & b_i & \Psi_1(t)t \\ & & & & & & b_n & \cdots & b_{i+1} & \Psi_1(t) \end{pmatrix}.$$

It is well-known [38, 39, 40] that $\deg(\mathrm{Sres}_i(\Psi_0, \Psi_1, t)) \leq i$.

The next proposition gives a relation between the sign of the Sturm sequence, generated by applying Algorithm 1, and that of the subresultant sequence.

Proposition 1 (Relation between the sign of the Sturm sequence and that of the subresultant sequence) *Let* $\{\Psi_k(t)\}_{k=0}^q$ *be the Sturm sequence obtained by applying Algorithm 1 to* $A_{(0)}(t)$ *and* $A_{(1)}(t)$.

If $\deg(\Psi_0) \geq \deg(\Psi_1)$, $q \geq 2$ *and* $\deg(\mathrm{Sres}_i(\Psi_0, \Psi_1, t)) = i$ *for all* $i \in [0, \deg(\Psi_1) - 1]$, *we have* $q = \deg(\Psi_1) + 1$ *and*

$$\mathrm{sgn}(\Psi_k(t^*)) = \mathrm{sgn}\Big[(-1)^{\frac{(k-1)k}{2} + (k-1)(\deg(\Psi_0) - \deg(\Psi_1) + 1)} \times \big(\mathrm{lc}(\Psi_1)\big)^{\deg(\Psi_0) - \deg(\Psi_1) + 1} \mathrm{Sres}_{\deg(\Psi_1) - k + 1}(\Psi_0, \Psi_1, t^*)\Big]$$
$$(k = 2, 3, \ldots, \deg(\Psi_1) + 1). \quad (5)$$

The relations (5) implies that we can compute each sign of $\{\Psi_k(t^*)\}_{k=2}^q$ by $\{\mathrm{Sres}_i(\Psi_0, \Psi_1, t^*)\}_{i=0}^{n-1}$ without computing the coefficients of $\{\Psi(t)\}_{k=2}^q$. Algorithm 2 below evaluates the signs of $\{\Psi_k(t^*)\}_{k=0}^q$ based on (5). In practice, Algorithm 2 plays an adequate role because $\deg(\mathrm{Sres}_i(\Psi_0, \Psi_1, t)) = i$ $(i \in [0, \deg(\Psi_1) - 1])$ holds almost always. (Note: In [35], we have given an algorithm application to general cases including $\deg(\Psi_0) < \deg(\Psi_1)$ or $\deg(\mathrm{Sres}_i(\Psi_0, \Psi_1, t)) < i$ for some $i \in [0, \deg(\Psi_0) - 1]$). The computational complexity for each $\mathrm{Sres}_i(\Psi_0, \Psi_1, t^*)$ is at most $\mathcal{O}\big((\deg(\Psi_0) + \deg(\Psi_1) - 2i)^{\log_2 7}\big)$.

4.3. Numerical Examples

In this section, we examine the numerical performance of the algebraic phase unwrapping, based on Theorem 1 using Algorithm 2. To make the situation likely to cause numerical instability of the algebraic phase unwrapping over $[0, 1]$, based on Theorem 1 using Algorithm 1, we generate randomly a pair of polynomials:

$$\left.\begin{aligned} A_{(0)}(t) &:= (t-0.1)(t-0.21)(t-0.5)(t-0.75)(t-0.8)\overline{A}_{(0)}(t) \\ A_{(1)}(t) &:= (t-0.15)(t-0.2)(t-0.34)(t-0.35)(t-0.81)\overline{A}_{(1)}(t) \end{aligned}\right\},$$

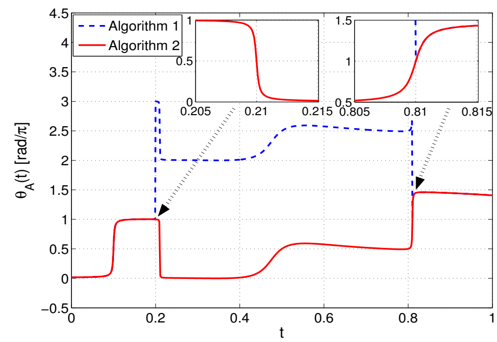

Fig. 3. Estimations of the unwrapped phase with Algorithm 1 and 2

where (i) $\overline{A}_{(0)}(t)$ is a polynomial of degree 35 whose 5 roots are generated by the uniform distribution over $\{(-5, -1) \cup (1, 5)\}$ and 15 complex conjugate pairs of roots are generated by the uniform distribution over $\{(-1, -0.5) \cup (0.5, 1)\} \pm j\{(-1, -0.5) \cup (0.5, 1)\}$, and (ii) $\overline{A}_{(1)}(t)$ is a polynomial of degree 15 whose 5 real roots and 5 complex conjugate pairs of roots are generated as for the above $\overline{A}_{(0)}(t)$. Note that polynomials $A_{(0)}(t)$ and $A_{(1)}(t)$ have close root pairs $(0.21, 0.8) \approx (0.2, 0.81)$, which likely causes the catastrophic cancellation [41, 42] explained in Sec. 4.1. Figure 3 depicts one example where Algorithm 1 fails in phase unwrapping at $t = 0.2$ and $t = 0.81$ while Algorithm 2 succeeds in phase unwrapping over $[0, 1]$. Table 1 summarizes the result for 1000 trials, where we observe that the total number of polynomials in failure by Algorithm 1 is reduced to less than $1/24$ by replacing it with Algorithm 2.

Algorithm 2 Proposed algorithm for computing (4)

Input: $A_{(0)}(t), A_{(1)}(t) \in \mathbb{R}[t]$, $a \in \mathbb{R}$ and $t^* \in (a, b]$

$\Psi_0(t) \leftarrow \dfrac{A_{(0)}(t)}{(t-a)^{e_0}}$, $\Psi_1(t) \leftarrow \dfrac{A_{(1)}(t)}{(t-a)^{e_1}}$ (where e_i denotes the order of $t = a$ as a zero of polynomial $A_{(i)}(t)$ $(i = 0, 1)$)

$m \leftarrow \deg(\Psi_0)$, $n \leftarrow \deg(\Psi_1)$, $\mathrm{lc}_1 \leftarrow \mathrm{lc}(\Psi_1)$

for $k = 2$ **to** $(n+1)$ **do**

  $\mathrm{sgn}(\Psi_k(t^*)) \leftarrow (-1)^{\frac{(k-1)k}{2} + (k-1)(m-n+1)} \times \mathrm{sgn}\left(\mathrm{lc}_1^{m-n+1} \mathrm{Sres}_{n-k+1}(\Psi_0, \Psi_1, t^*)\right)$

end for

Output: $\{\mathrm{sgn}(\Psi_k(t^*))\}_{k=0}^{\deg(\Psi_1)+1}$

Table 1. Performance comparison for pairs of random polynomials

Algorithm	Total number of pairs $(A_{(0)}, A_{(1)})$ in failure
Algorithm 1	249 (among 1000, in 64-bit floating point arithmetic)
Algorithm 2	10 (among 1000, in 64-bit floating point arithmetic)

5. CONCLUSION

In this paper, we have extended and stabilized the algebraic phase unwrapping along the real axis. First, we have removed a assumption premised in the original algebraic phase unwrapping. Second, we have elucidated the path independence of two-dimensional phase unwrapping completely, and extended the algebraic phase unwrapping for a pair of bivariate polynomials. Third, after clarifying the relation between the Sturm sequence and the subresultant sequence, we have shown that the algebraic phase unwrapping along the real axis can be stabilized significantly, by evaluating directly the signs of the Sturm sequence, in the terms of the subresultant sequence.

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