ESTIMATION OF BANDLIMITED SIGNALS FROM THE SIGNS OF NOISY SAMPLES

Animesh Kumar[*]

Electrical Engineering
Indian Institute of Technology Bombay
Mumbai, India
animesh@ee.iitb.ac.in

Vinod M. Prabhakaran[†]

School of Technology and Computer Science
Tata Institute of Fundamental Research, Bombay
Mumbai, India
vinodmp@tifr.res.in

ABSTRACT

The sampling, quantization, and estimation of a bounded dynamic-range bandlimited signal affected by additive independent Gaussian noise is studied in this work. Considering the desirability of cheap, low-precision sensors, the use of single-bit analog to digital convertors (ADCs) is considered. For bandlimited signals, the distortion due to additive independent Gaussian noise can be reduced by oversampling (statistical diversity). The pointwise expected mean-squared error is used as a distortion metric for signal estimate in this work. If N is the oversampling ratio with respect to the Nyquist rate, then we show that a distortion of $O(1/N)$ can be achieved with *single-bit* ADCs that record the signs of the observed noisy signal. This improves the (best known) distortion result by Masry for quantizing bandlimited signals in noise, using signs of noisy signal samples, from $O(1/N^{2/3})$. This improvement comes by exploiting the structure of bandlimited signals in the estimation of original signal from noisy quantized bits.

Index Terms— sampling methods, signal sampling, estimation, quantization

1. INTRODUCTION

Consider a bandlimited signal (or field) quantization problem, where the samples are affected by additive independent and identically distributed (i.i.d.) Gaussian noise. In sensor-networks, low-cost sensing devices are desirable; therefore, single-bit analog to digital converters (ADCs) are desirable for quantization. In a distributed setup, where sampling precedes filtering, noise induced distortion can be reduced by oversampling. The distortion and oversampling tradeoff for bandlimited signals in the presence of noise under single-bit quantization is the main theme of this work.

A single-bit ADC with comparison threshold at zero can record the sign of a sample. The term single-bit quantization will be used instead of sign of a sample from now on. Estimation from the perfect samples of a signal in additive noise is a classical topic [1]. However, estimation of noise-affected bandlimited signals with single-bit quantization has been rarely addressed. Sampling of continuous signals using a random process as a dither has been studied by Masry [2], where consistent estimates for a continuous signal are derived from the single-bit quantized samples. Recovery of finite support signals from single-bit quantized samples has been studied by Wang and Ishwar [3], and Masry and Ishwar [4].

Let N be the oversampling rate above the Nyquist rate. Loosely speaking, a bandlimited signal of duration T and bandwidth π has $2\pi T$ degrees of freedom [5]. With NT samples of a noisy signal in duration T, the optimal distortion with *unquantized* samples is *speculated* to be $O(1/N)$.[1] In this work it is shown that a distortion of $O(1/N)$ is *achievable* while sensing with single-bit quantizers. Masry's estimation technique, when applied to bandlimited signals, results in a (mean-squared error) distortion of $O(1/N^{2/3})$.

Our main contribution is the design of a sampling scheme with single-bit quantizers and oversampling rate N, to achieve a distortion of $O(1/N)$ for bounded dynamic-range bandlimited signals in additive independent Gaussian noise.

Notation: The set of bounded signals and the set of finite energy signals will be denoted by $\mathcal{L}^\infty(\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R})$, respectively. The signal of interest will be denoted by $g(t)$. For a signal $s(t)$ in $\mathcal{L}^2(\mathbb{R})$ the Fourier transform will be denoted by $\tilde{s}(\omega)$, and is defined as $\tilde{s}(\omega) = \int_{\mathbb{R}} s(t)\exp(-j\omega t)\mathrm{d}t$. $\mathbb{1}(x \in A)$ denotes the indicator function of a set A. The set of reals, set of integers, convolution, and expectation operator will be denoted by $\mathbb{R}$, $\mathbb{Z}$, $\star$, and $\mathbb{E}$, respectively.

Organization: The mathematical formulation of our sampling problem is discussed in Sec. 2. Short review of stable interpolation kernels and smoothness properties of associated signals are discussed in Sec. 3. The details of single-bit sampling appear in Sec. 4. Conclusions are presented in Sec. 5.

[*]Animesh Kumar's work has been supported by IRCC, IIT Bombay by Grant P09IRCC039. The travel has been supported by the Bharti Centre for Communication, IIT Bombay.

[†]Vinod M. Prabhakaran's work was supported by a Ramanujan Fellowship from the Dept. of Science and Technology, Government of India.

[1]A single bounded constant in additive independent Gaussian noise with N independent readings can be estimated up to a distortion of $O(1/N)$ [6].

2. PROBLEM SETUP

A bandlimited kernel for stable interpolation is as follows (see Fig. 1). For $\lambda > 1$ and $a = (\lambda - 1)/2$, consider

$$\phi(t) = \frac{1}{\pi a t^2} \sin((\pi + a)t) \sin(at); \;\; \phi(0) = 1 + \frac{a}{\pi}. \quad (1)$$

The kernel decreases as $1/t^2$ and therefore it is absolutely and square integrable. This kernel defines the set of Zakai class of

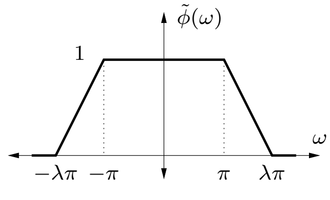

Fig. 1. ***Stable interpolation filter:*** *The kernel $\phi(t) \leftrightarrow \tilde{\phi}(\omega)$ is defined in (1); this kernel will be used to define bandlimited signals.*

bandlimited signals [7]. Bounded bandlimited signals, which form a subset of the Zakai class, are of interest for sampling:

$$BL_{\text{int}} := \{g : |g(t)| \leq 1, \; g(t) \star \phi(t) = g(t) \; \forall t \in \mathbb{R}\}. \quad (2)$$

Any $g(t) \in BL_{\text{int}}$ is continuous everywhere. Any bandlimited signal in $\mathcal{L}^2(\mathbb{R})$ with Fourier spectrum zero outside $[-\pi, \pi]$ also belongs to the set BL_{int}. The set BL_{int} also includes (almost-surely) any sample path of a bounded-dynamic range bandlimited stationary process [8]. The estimation of bandlimited signals in BL_{int}, affected by additive independent Gaussian noise and single-bit quantization, will be studied. *The derived results are applicable to finite energy bounded bandlimited signals as well as (almost-surely) to all sample paths of a bounded stationary bandlimited process.*

The signal affected by additive noise, $g(t) + W(t)$, is available for sampling. It is assumed that $W(t_1), W(t_2), \ldots, W(t_n)$ for distinct $t_1, t_2, \ldots, t_n \in \mathbb{R}$ are i.i.d. with $\mathcal{N}(0, \sigma^2)$ distribution. The sampling rate of $g(t)$ for perfect reconstruction is one sample/second. The reconstruction based on *noisy samples* of $g(t)$ will have distortion that can be reduced by oversampling with a *factor* of N above Nyquist. For any estimate $\widehat{G}_{\text{rec}}(t)$ of the signal $g(t)$, the maximum pointwise mean-squared error D_{rec} is defined as the *distortion*, i.e.,

$$D_{\text{rec}} := \sup_{t \in \mathbb{R}} D_{\text{rec}}(t) = \sup_{t \in \mathbb{R}} \mathbb{E} \left| \widehat{G}_{\text{rec}}(t) - g(t) \right|^2. \quad (3)$$

The dependence of D_{rec} on N is of interest. It will be shown that D_{rec} decreases as $O(1/N)$ with single-bit quantizers.

Samples are quantized using single-bit ADCs as shown in Fig. 2. The role of extra dither noise $W_d(t)$ will be explained later in the following sections. The estimator $\widehat{G}_{\text{1-bit}}(t)$ will be designed and its distortion performance will be analyzed.

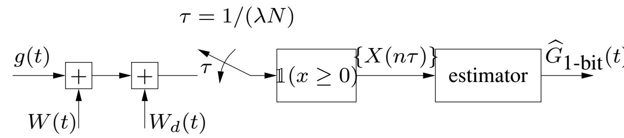

Fig. 2. *The estimator works with poorest precision (one-bit) samples $\{X(n\tau), n \in \mathbb{Z}\}$ where $X(n\tau) = \mathbb{1}(Y(n\tau) \geq 0)$.*

It should be noted that the kernel $\phi(t)$ and its derivative $\phi'(t)$ are absolutely integrable. These properties of $\phi(t)$ are easy to derive and feature in the distortion results:

$$C_\phi := \int_{t \in \mathbb{R}} |\phi(t)| \mathrm{d}t < \infty, \quad (4)$$

$$C'_\phi := \sup_{\{t_k : t_k \in [k/\lambda, (k+1)/\lambda], k \in \mathbb{Z}\}} \sum_{k \in \mathbb{Z}} |\phi'(t_k)| < \infty, \quad (5)$$

$$\text{and } C''_\phi := \sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \phi\left(t - \frac{k}{\lambda}\right) \right|^2 < \infty. \quad (6)$$

3. MATHEMATICAL BACKGROUND

From the interpolation formula for Zakai sense bandlimited signals, the signal of interest $g(t)$ can be perfectly reconstructed from its samples taken on a discrete grid. For $g(t) \in BL_{\text{int}}$, the interpolation formula is given by [9, Lemma 3.1],

$$g(t) = \lambda \sum_{n \in \mathbb{Z}} g\left(\frac{n}{\lambda}\right) \phi\left(t - \frac{n}{\lambda}\right). \quad (7)$$

where $\lambda > 1$ is arbitrary and the equality holds in $\mathcal{L}^\infty(\mathbb{R})$. It is sufficient to sample $g(t)$ at a rate of λ samples/second. The reconstruction in (7) is stable in $\mathcal{L}^\infty(\mathbb{R})$ to bounded perturbations of samples.

The dither $W_d(t)$ in Fig. 2 will now be explained. Due to quantization, minimum risk estimators such as maximum likelihood are non-linear and analytically complex. An analytically tractable reconstruction procedure with $O(1/N)$ is desirable and dithering achieves it. Assume that $W_d(t)$ and $W(t)$ are independent. The block-diagram for sampling with one-bit ADCs is illustrated in Fig. 2. The condition on $\sigma^2 = \text{var}(W(t) + W_d(t))$ is stated using the cumulative distribution function (cdf) of $W + W_d$. Let $F(x)$ and $f(x)$ be the cdf and probability density functions of $W(t) + W_d(t)$. Denote $f(\pm C_\phi) = \delta$ and $f(0) = \Delta$. Observe that $\Delta > \delta$. For our estimation procedure (see Sec. 4), it is required that there is a parameter $\mu > 0$ such that

$$\left(1 - \frac{1}{\sqrt{2} C_\phi^2}\right) \frac{1}{\delta} < \mu < \frac{1}{\Delta}, \quad (8)$$

where C_ϕ is the constant in (4). First fix a $\lambda > 1$. Then, $C_\phi = \int_{t \in \mathbb{R}} |\phi(t)| \mathrm{d}t > \int_{t \in \mathbb{R}} \phi(t) \mathrm{d}t = \tilde{\phi}(0) = 1$. That is,

$C_\phi^2\sqrt{2} > 1$. Therefore, the lower bound on μ in (8) is positive. Next, observe that for a fixed large σ, $\delta = f(C_\phi) \approx f(0) = \Delta$. Then δ and Δ are close enough and the inequality in (8) can be satisfied. That is, for a fixed (λ, C_ϕ), there is a *finite* number σ_0 for which (8) is satisfied for all $\sigma > \sigma_0$. To ensure $\text{var}(W+W_d) \geq \sigma_0^2$, select $\text{var}(W_d) = \max\{0, \sigma_0^2 - \text{var}(W)\}$. If $\text{var}(W(t)) \geq \sigma_0^2$, then the extra dither is not needed.

For our estimation setup, the signal $F(g(t)) - 1/2$ will be used, where $F(x)$ is the cdf of $W + W_d$. The probability density $f(x) = F'(x)$ is finite and non-zero for $x \in [-1, 1]$. Let $l(t) := F(g(t)) - 1/2$. Then $|l(t)| \leq |F(1)| - 1/2$, i.e., $l(t)$ is bounded. Finally $|l'(t)| = |F'(g(t))g'(t)| \leq |F'(0)2\pi^2|$ since $F'(0)$ maximizes $F'(x)$ in $[-1, 1]$ and $|g'(t)| \leq 2\pi^2$ (see [9, Proposition 3.1]).

The definition of BL_{int} involves convolution with a stable kernel and convolution will often appear in the context of error analysis. The following fact will be useful.

Fact 3.1 *Let $p(t)$ be a signal such that $||p||_\infty < \infty$ and $P(t)$ be a random process such that $\sup_{t\in\mathbb{R}} \mathbb{E}(P^2(t)) < \infty$. Then,*

$$||p \star \phi||_\infty \leq C_\phi ||p||_\infty, \tag{9}$$

$$\text{and } \mathbb{E}[(|P(t)| \star |\phi(t)|)^2] \leq C_\phi^2 \sup_{t\in\mathbb{R}} \mathbb{E}(P^2(t)). \tag{10}$$

See [10] for a proof, which uses triangle inequality. The convolutions are well defined since $\phi(t)$ is absolutely integrable. The quantization analysis will now be presented.

4. ESTIMATION FROM SINGLE-BIT SAMPLES

Now a $\widehat{G}_{\text{1-bit}}(t)$ will be obtained such that $D_{\text{1-bit}}$ scales as $O(1/N)$ (see Fig. 2). By dithering or otherwise, it will be assumed that σ is such that (8) is satisfied. Fix $\tau = 1/(N\lambda)$, where $N > 0$ is an integer. The following interpolation is obtained from the single-bit samples $X(n\tau) := \mathbb{1}(g(n\tau) + W(n\tau) \geq 0), n \in \mathbb{Z}$,

$$H_N(t) = \tau \sum_{n\in\mathbb{Z}} (X(n\tau) - 1/2)\phi\,(t - n\tau)\,. \tag{11}$$

Then $H_N(t)$ converges to a mean-square limit.

Proposition 4.1 (Convergence of single-bit interpolation) *Let $l(t) = (F(g(t)) - 1/2)$ and $H_N(t)$ be as in (11). Then*

$$\sup_{t\in\mathbb{R}} \mathbb{E}(H_N(t) - l(t) \star \phi(t))^2 \leq \frac{C_2}{N} + \frac{C_3}{N^2}, \tag{12}$$

where $C_2 > 0$ and $C_3 > 0$ are constants independent of N.

See [10, Sec. VI-B] for a proof. An approximation of $g(t)$ has to be obtained from $H_N(t)$. The signal $l(t) \in \mathcal{L}^\infty(\mathbb{R})$ and the limit $l(t) \star \phi(t)$ is a lowpass version of $l(t)$. The dependence of $l(t) \star \phi(t)$ on $g(t)$ is non-linear due to quantization, which results in the $F(g(t))$ term. The signal $g(t)$ is Zakai sense bandlimited with *one degree of freedom* per unit time. The degree of freedom per unit time of $l(t) \star \phi(t)$ is up to *one*, and $F(x)$ is increasing. We will show that it is possible to invert $l(t) \star \phi(t)$ and find $g(t)$, in spite of non-linearity.

A compandor is a monotonic function $Q(x)$ which has the property that $Q(m(t)) \in \mathcal{L}^2(\mathbb{R})$ if $m(t) \in \mathcal{L}^2(\mathbb{R})$. Landau and Miranker have shown that if $g(t) \in \mathcal{L}^2(\mathbb{R})$ and $\tilde{g}(\omega)$ is zero outside $[-\pi, \pi]$, and if $Q : [-1, 1] \to \mathbb{R}$ is a compandor with non-zero slope, then there is one to one correspondence between $g(t)$ and $Q(g(t)) \star \text{sinc}(t)$ [11]. Further, given any signal $m(t) \in \mathcal{L}^2(\mathbb{R})$ and $\tilde{m}(\omega)$ zero outside $[-\pi, \pi]$, there exists a unique $g_m(t) \in \mathcal{L}^2(\mathbb{R})$ with $\tilde{g_m}(\omega)$ zero outside $[-\pi, \pi]$ and $Q(g_m(t)) \star \text{sinc}(t) = m(t)$.

Our setup is similar where $l(t) = F(g(t)) - 1/2$ is a compandor. But $H_N(t)$ and $l(t)$ need not be in $\mathcal{L}^2(\mathbb{R})$. Thus, the procedure of Landau and Miranker does not extend to our setup, *especially* in the presence of statistical noise. Modifications of their approach will be used to show that $l(t) \star \phi(t)$ can be inverted to obtain $g(t)$, when $g(t) \in BL_{\text{int}}$. This non-linear inversion problem will be cast into a recursive setup, where Banach's fixed-point theorem can be leveraged [12].

A 'clip to one' function Clip$[x]$ is defined first.

$$\text{Clip}[x] = x \;\text{ if } |x| \leq 1, \quad \text{Clip}[x] = \text{sgn}(x) \text{ otherwise.} \tag{13}$$

Since $|g(t)| \leq 1$, it will be unaffected by clipping. Let $\psi(t) = \phi(\lambda t)$. Then $\tilde{\psi}(\omega) = \phi(\omega/\lambda)$. Thus, $\tilde{\psi}(\omega)$ is flat in $[-\lambda\pi, \lambda\pi]$ and in $\pm[\lambda\pi, \lambda^2\pi]$ decreases linearly to zero. Consider the set

$$\mathcal{S}_{\text{BL,bdd}} = \{m : |m(t)| \leq C_\phi,\; m(t) \star \psi(t) = m(t)\}. \tag{14}$$

Then, we have the following Lemma.

Lemma 4.1 ($\mathcal{S}_{\text{BL,bdd}}$ is a complete metric space) *Let $\mathcal{S}_{BL,bdd}$ be as defined in (14). Then $(\mathcal{S}_{BL,bdd}, ||.||_\infty)$ is a complete subset of $(\mathcal{L}^\infty(\mathbb{R}), ||.||_\infty)$.*

Completeness is the non-trivial part of the above lemma. Consider any sequence $m_n(t) \in \mathcal{S}_{\text{BL,bdd}} \subset \mathcal{L}^\infty(\mathbb{R})$ converging to $s(t)$. Such $s(t)$ exists in $\mathcal{L}^\infty(\mathbb{R})$, because latter is complete. By Lemma 3.1, $m_n(t) \star \psi(t)$ will converge to $s(t) \star \psi(t)$. Since $m_n \star \psi \equiv m_n$, therefore, $s(t) = s(t) \star \psi(t)$, or $s(t) \in \mathcal{S}_{\text{BL,bdd}}$. Thus, $\mathcal{S}_{\text{BL,bdd}}$ is complete.

A map $T : \mathcal{S}_{\text{BL,bdd}} \longrightarrow \mathcal{S}_{\text{BL,bdd}}$ will be used to define a recursion for obtaining $g(t)$ from $h(t) := l(t) \star \phi(t)$. Define

$$\begin{aligned} r(t) &:= \mu h(t) + [m(t) - \mu(F(m(t)) - 1/2)] \star \phi(t), \\ T[m(t)] &:= \text{Clip}\,[r(t)] \star \phi(t). \end{aligned} \tag{15}$$

Since $||\text{Clip}\,[r(t)]\,||_\infty \leq 1$, Fact 3.1 ensures that $T[m(t)]$ is bounded by C_ϕ. Also $\phi(t) \star \psi(t) = \phi(t)$, therefore, $T[m(t)] \star \psi(t) = T[m(t)]$. Next, the following lemma is noted.

Lemma 4.2 (T is a contraction) *Let $(\mathcal{S}_{BL,bdd}, ||.||_\infty)$ be the metric space as defined in (14). Let $T : \mathcal{S}_{BL,bdd} \longrightarrow \mathcal{S}_{BL,bdd}$ be a map as defined in (15). If the condition in (8) is satisfied, then there is a choice of μ such that T is a contraction, i.e.,*

$$||T[m_1] - T[m_2]||_\infty \leq \alpha ||m_1 - m_2||_\infty, \tag{16}$$

for some $0 < \alpha < 1$ and any $m_1(t), m_2(t) \in \mathcal{S}_{BL,bdd}$. The parameter α does not depend on the choice of m_1 and m_2.

See [10, Sec. VI-C] for a proof, which uses (8) and calculus. The key point of Lemma 4.2 is that the map T will always carry a signal in $\mathcal{S}_{\text{BL,bdd}}$ towards its fixed point solution. This follows by Banach's fixed-point theorem which is formalized in Proposition 4.2. Now the key recursive equation will be stated. To invert $h(t) = l(t) \star \phi(t)$ and obtain $g(t)$ define

$$r_{k+1}(t) := \mu h(t) + [g_k(t) - \mu(F(g_k(t)) - 1/2)] \star \phi(t),$$
$$g_{k+1}(t) := T[g_k(t)] = \text{Clip}[r_k(t)] \star \phi(t), \qquad (17)$$

where $k \geq 0, k \in \mathbb{Z}$ and $\mu > 0$ is a constant that will be chosen according to Lemma 4.2. Set $g_0(t) \equiv 0$. By substitution the signal $g(t)$ is a fixed point of (17). By using Banach's fixed point theorem [12], the following proposition shows that $g(t)$ is the *only* fixed point of the equation in (17).

Proposition 4.2 (**$g(t)$ is the fixed point of T**) *Let $g(t) \in BL_{int} \subset \mathcal{S}_{BL,bdd}$ be a continuous bounded bandlimited signal. Let $h(t) = l(t) \star \phi(t)$, where $l(t) = F(g(t)) - 1/2$. Consider the recursion $g_k(t) = T[g_{k-1}(t)]$, where T is as defined in (15). Set $g_0(t) \equiv 0$. If μ is selected as in (8), then*

$$\lim_{k\to\infty} ||g_k - g||_\infty = 0. \qquad (18)$$

Proof: Define $d(m_1, m_2) = ||m_1 - m_2||_\infty$ for signals $m_1(t), m_2(t)$. From Lemma 4.1, note that $(\mathcal{S}_{\text{BL,bdd}}, d)$ is a complete metric space. The signal $g(t)$ is in $\mathcal{S}_{\text{BL,bdd}}$ and it is a fixed point for T defined in (15), i.e., $g(t) = T[g(t)]$.

Pick μ as in (8). Then T is a contraction on $(\mathcal{S}_{\text{BL,bdd}}, d)$. Thus, by Banach's fixed point theorem [12, Ch. 5], there is *exactly* one fixed point in $\mathcal{S}_{\text{BL,bdd}}$ for the equation $g(t) = T[g(t)]$. Since $g_k(t)$ converges to a fixed point, it must converge to $g(t)$ in the distance metric d. ♣

The estimation of signal from $H_N(t)$, the statistical approximation of $l(t) \star \phi(t)$, will be discussed now. Let $G_k(t)$ be the sequence of waveforms generated from $H_N(t)$ when the latter is subjected to the recursion in (17). Fix $G_0(t) \equiv 0$ and define

$$R_{k+1}(t) := \mu H_N(t) + [G_k(t) - \mu(F(G_k(t)) - 1/2)] \star \phi(t),$$
$$G_{k+1}(t) := T[G_k(t)] = \text{Clip}[R_k(t)] \star \phi(t). \qquad (19)$$

Let $\widehat{G}_{\text{1-bit}}(t) = \lim_{k\to\infty} G_k(t)$. For the same choice of μ which ensures that T is a contraction on $(\mathcal{S}_{\text{BL,bdd}}, ||.||_\infty)$, the distortion of $|\widehat{G}_{\text{1-bit}}(t) - g(t)|$ has to be established. To this end, the following main result of this work is noted.

Theorem 4.1 *Let $H_N(t)$ be the estimate of $l(t)$ as described in (11) and μ be selected as in (8). With $G_0(t) \equiv 0$, let $G_k(t)$ be the sequence of random waveforms as defined in (19). Define $\lim_{k\to\infty} G_k(t) = \widehat{G}_{1\text{-}bit}(t)$. Then,*

$$D_{1\text{-}bit} := \sup_{t\in\mathbb{R}} \mathbb{E}(\widehat{G}_{1\text{-}bit}(t) - g(t))^2 = O(1/N),$$

i.e., the distortion $D_{1\text{-}bit}$ decreases as $O(1/N)$.

Proof: See [10, Sec. VI-D]. ♣

The key idea of the proof is highlighted. From Lemma 4.2, if $l(t)\star\phi(t)$ is the starting point of recursion in (17), then $g(t)$ is where it ends. However, from Proposition 4.1, we know that $H_N(t) \in \mathcal{S}_{\text{BL,bdd}}$ is available for estimating $g(t)$. Then $H_N(t)$ is subjected to the recursion in (19). Since $H_N(t)$ and $l(t)$ are 'close' (variance $O(1/N)$), successive application of the map T results in signal estimate $\widehat{G}_{\text{1-bit}}$ and $g(t)$ being close (variance $O(1/N)$). The contraction property of T is the key to this result.

Interpretation using degree of freedoms: Assume that a constant $c \in [-1, 1]$ has to be estimated based on N noisy single-bit readings $B_i = \mathbb{1}(c + W_i \geq 0), 1 \leq i \leq N$. The random variables $\{B_i, 1 \leq i \leq N\}$ are i.i.d. Ber(q) where $q = \mathbb{P}(W \geq -c) = \mathbb{P}(W \leq c) = F(c)$. Denote $\widehat{B}_N = (\sum_{i=1}^N B_i)/N$. Define $\widehat{C}_{\text{1-bit}} = F^{-1}(\widehat{B}_N)$ if $\widehat{B}_N \in [F(-1), F(1)]$ and $\widehat{C}_{\text{1-bit}} = \pm 1$ otherwise. Since $F(x)$ is invertible and $\mathrm{d}F^{-1}(x)/\mathrm{d}x$ is bounded for $x \in [F(-1), F(1)]$, therefore, using the delta method, $\widehat{C}_{\text{1-bit}}$ obtained from $\widehat{B}_N$ has a mean-squared error which decreases as $(1/N)$ [6]. Since bandlimited signals have one degree of freedom in every Nyquist interval, an oversampling factor of N means that there are N samples to observe each degree of freedom.

5. CONCLUSIONS AND FUTURE WORK

The sampling, quantization, and estimation of a bounded dynamic-range bandlimited signal affected by additive independent Gaussian noise was studied. The maximum pointwise expected mean-squared error was used as a distortion metric. With single-bit measurements, it was shown that the distortion scales as $O(1/N)$, where N is the oversampling ratio with respect to the Nyquist rate. This improved the (best known) quantization results by Masry for quantizing bandlimited signals in noise from $O(1/N^{2/3})$. This improvement was obtained by exploiting the structure of bandlimited signals in the estimation step. The presented estimation technique hinges upon Banach's fixed point theorem.

This work assumed sufficient dithering by noise because the estimators were linear. It is of interest to look towards estimation techniques which do not require extra dithering.

6. ACKNOWLEDGMENT

The problem of sampling a smooth signal in the presence of noise in a distributed setup was suggested by Prof. Kannan Ramchandran, EECS, University of California, Berkeley, CA. Discussions on this problem with Prof. Kannan Ramchandran and Prof. Martin Wainwright, EECS, University of California, Berkeley, CA, Prof. H. Narayanan, EE, IIT Bombay, and Prof. Prakash Ishwar, ECE, Boston University, Cambridge, MA were insightful.

7. REFERENCES

[1] S. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*. Burlington, MA, USA: Academic Press, 2009.

[2] E. Masry, "The reconstruction of analog signals from the sign of their noisy samples," *IEEE Transactions on Information Theory*, vol. 27, no. 6, pp. 735–745, Nov. 1981.

[3] Y. Wang and P. Ishwar, "Distributed field estimation with randomly deployed, noisy, binary sensors," *IEEE Transactions on Signal Processing*, vol. 57, no. 3, pp. 1177–1189, Mar. 2009.

[4] E. Masry and P. Ishwar, "Field estimation from randomly located binary noisy sensors," *IEEE Transactions on Information Theory*, vol. 55, no. 11, pp. 5197–5210, Nov. 2009.

[5] D. Slepian, "On bandwidth," *Proceedings of the IEEE*, vol. 64, no. 3, pp. 292–300, Mar. 1976.

[6] P. J. Bickel and K. A. Doksum, *Mathematical Statistics Vol I*. Upper Saddle River, NJ, USA: Prentice Hall, 2001.

[7] M. Zakai, "Band-limited functions and the sampling theorem," *Information and Control*, vol. 8, pp. 143–158, 1965.

[8] S. Cambanis and E. Masry, "Zakai's class of bandlimited functions and processes: Its characterization and properties," *SIAM Journal on Applied Mathematics*, vol. 30, no. 1, pp. 10–21, Jan. 1976.

[9] A. Kumar, P. Ishwar, and K. Ramchandran, "High-resolution distributed sampling of bandlimited fields with low-precision sensors," *IEEE Transactions on Information Theory*, vol. 57, no. 1, pp. 476–492, Jan. 2011.

[10] A. Kumar and V. Prabhakaran, "Estimation of bandlimited signals in additive gaussian noise: a "precision indifference" principle," *arXiv, 1211.6598*, 2012. [Online]. Available: http://arxiv.org/abs/1211.6598

[11] H. J. Landau and W. L. Miranker, "The recovery of distorted band-limitd signals," *Journal of Mathematical Analysis and Applications*, vol. 2, no. 1, pp. 97–104, Feb. 1961.

[12] E. Kreyszig, *Introductory Functional Analysis with Applications*, 1st ed. Wiley, 1989.