A SIMPLE OPTIMUM NONLINEAR FILTER FOR STOCHASTIC-RESONANCE-BASED SIGNAL DETECTION

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ABSTRACT

Stochastic resonance (SR) is a physical phenomenon through which system performance is enhanced by noise. Applications of SR in signal processing are expected to realize the detection of a weak signal buried in strong noise. Extraction of the effect of SR requires the design of an effective nonlinear system. Although a number of studies have investigated SR, most have employed conventional nonlinear filters. The present study proposes simple optimum nonlinear characteristics that maximize the performance enhancement, which is measured by the signal-to-noise ratio. The mathematical expression is simple, and the obtained performance is beyond that of linear systems. Surprisingly, the proposed nonlinear method can obtain the Cramér-Rao bounds and is equivalent to the maximum likelihood estimator. In addition, such optimization demonstrates systematically that the applications of SR to signal detection is effective only in non-Gaussian noise environments.

Index Terms— Stochastic resonance, signal detection, nonlinearity, Cramér-Rao Bounds, maximum-likelihood estimation

1. INTRODUCTION

Stochastic resonance (SR) is an interesting phenomenon in that it can enhance system performances in noisy environments. A number of studies have investigated the basic characteristics of SR [1–3] and systems that exhibit SR in various fields, such as superconducting quantum interference devices (SQUIDs) [4, 5], sensory neurons [6, 7].

In the field of signal processing, linear filtering is a traditional approach to detecting weak signals in noisy environments. A novel techniques with SR criterion have been widely discussed [8–12]. Some authors stated that the application of the SR should be effective in Non-Gaussian noise environment [11, 12]. A number of studies have focused on communication systems, and the potential performance improvements have been reported [13, 14]. However, in conventional SR systems, the performance enhancement is limited in the presence of white Gaussian noise, requiring the selection of a linear system [15].

A system exhibiting SR has three components: noise, nonlinearity, and a desired weak signal. In a number of studies on SR, both the characteristics of noise and nonlinearity are fixed in advance, and the effect of SR is measured using the signal-to-noise ratio (SNR), in addition to mutual information and in-out correlation. Of course, there are numerous types of noise and nonlinear systems. A limited number of such investigations cannot determine which type of nonlinearity is optimal for a given noise type, and vice versa.

Signal processing systems usually receive signals that have been corrupted by noise. The characteristics of the noise cannot be designed or adjusted, and the only option is tuning the nonlinearity. This means that the simple optimum nonlinearity, which maximizes SNR for the noise, should be explored. Fundamental nonlinear devices, such as comparators and Schmitt triggers, have also been investigated. Although such devices are easy to implement and evaluate, the resulting performance is poor so that linear systems should be good choice. From an information theory viewpoint, the most reliable scheme involves the use of an optimum detection device, such as a Bayesian detector or a Neyman-Pearson detector. Some studies have evaluated optimum-detector-based SR systems [10, 12, 16], which provide good performance but involve complicated schemes that are not acceptable as simple practical devices.

The present paper describes a theoretical simple optimum nonlinear filter for SR applications. We herein focus on white noise and consider a simple nonlinear system, the length of the impulse response of which is equal to zero (no memory effect). The primary contributions are as follows:

1. Proposal of a simple optimum nonlinearity that maximizes SNR
   We derive a mathematical expression of the optimum nonlinearity in the presence of arbitrary white noise. This simple expression yields the Cramér-Rao bounds and has better performance than that in linear systems.

2. Clarification of the effective situation in which SR-based signal detection is better than in linear systems
   The proposed nonlinearity demonstrates that if the noise follows a non-Gaussian distribution, there exists an SR-based system superior to any linear one. On the

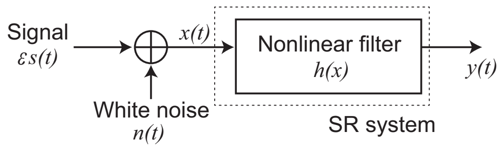


Fig. 1. Schematic diagram of SR-based systems with white noise.

other hand, in the presence of Gaussian noise, no SR-based system can provide a better performance than a linear system.

The remainder of the present paper is organized as follows. In Section 2, we describe the system considered herein. The proposed nonlinearity is introduced and the performance is evaluated in Section 3. The relation to the Cramér-Rao bound is discussed in Section 4. Section 5 concludes the present paper.

2. SYSTEM MODEL

A schematic diagram of the SR-based system is shown in Fig. 1. The SR system receives a desired signal $\epsilon s(t)$ corrupted by a white noise $n(t)$. Since the system consists of a simple nonlinear filter $h(x)$, the output of which is given by

$$y(t) \quad = \quad h\left(\epsilon s(t) + n(t)\right). \tag{1}$$

In this system, we can choose any white noise and the nonlinear function.

We consider the situation in which the signal amplitude ϵ is assumed to be small compared to the noise at the input. In this situation, it is difficult to extract the desired signal $\epsilon s(t)$ from the input. Owing to the SR effect in the nonlinear filter, we can obtain the signal at the output. This effect can be modeled by the SNR, which is widely used in the field of signal detection. The SNR at the output can defined as follows:

$$G = \frac{\frac{1}{T}\int_0^T \langle y(t) - h(n(t))\rangle^2 dt}{\frac{1}{T}\int_0^T \left\{\langle y^2(t)\rangle - \langle y(t)\rangle^2\right\} dt} \tag{2}$$

where $\langle z\rangle = \int zp(n)dn$. This SNR represents the desired signal component at the output, which is divided by the noise variance.If the SNR is larger than that at input, the system can improve signal detection.

3. SIMPLE OPTIMUM NONLINEARITY AND AN EXAMPLE

3.1. Theory of a simple optimum nonlinear filter for SR

The SNR of eq. (2) depends on the filter function $h(x)$. In this section, we determine the optimum $\tilde{h}(x)$, which maximizes the SNR.

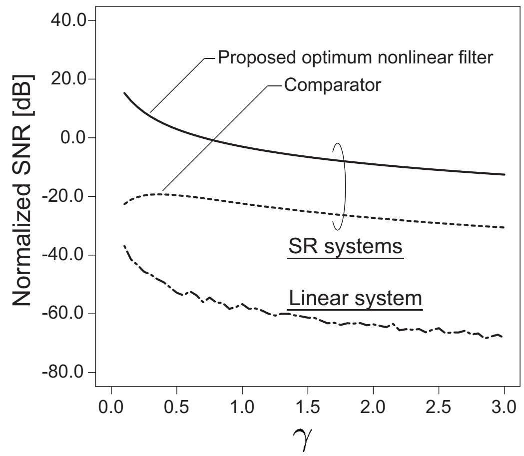


Fig. 2. Normalized SNRs for SR systems with optimal nonlinearity and a simple threshold system, in addition to a linear system. White Cauchy noise is considered. As the increase of γ, the power of the Cauchy noise is also increased.

For a sufficiently small input signal ($\epsilon \approx 0$; low SNR at input), Taylor series of the output around ϵ is,

$$y(t) = h(n(t)) + \epsilon s(t)h'(n(t)) + O(\epsilon^2). \tag{3}$$

Since $\epsilon \approx 0$, $O(\epsilon^2)$ can be negligible, and then eq. (3) is reduced to $y(t) = h(n(t)) + \epsilon s(t)h'(n(t))$. Substituting it into eq. (2), the SNR can be expressed as:

$$G \approx \epsilon^2 P_s g, \qquad g = \frac{\langle h'(n)\rangle^2}{\langle h^2(n)\rangle - \langle h(n)\rangle^2}, \tag{4}$$

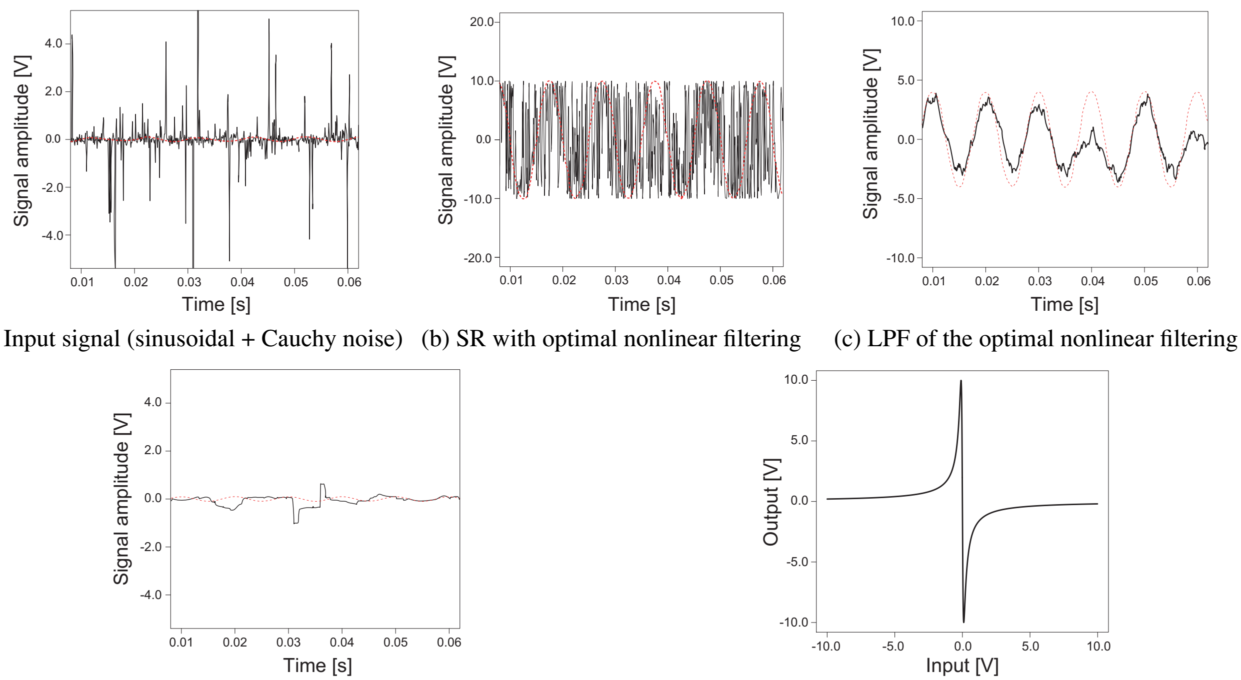
where $P_s = \frac{1}{T}\int_0^T s^2(t)dt$ is the signal power. The coefficient g can be observed that it is normalized by both time and the signal power. In this sense, we call it as the normalized SNR, which is key parameter in this paper.

Since the normalized SNR contains the nonlinear function, the optimal nonlinearity $\tilde{h}(x)$ can be obtained by the maximization. This can be done by the functional derivative of the normalized SNR $\delta g/\delta h(x) = 0$ as follows [17]:

$$\tilde{h}(x) = a - b\frac{\partial \ln p(n)}{\partial n}\bigg|_{n=x} \tag{5}$$

where a and b are constants.

The optimal nonlinearity $\tilde{h}(x)$ given in eq. (5) implies an important clarification of the situation in which SR-based signal detection is better than that in a linear system. If the noise follows a Gaussian distribution, i.e., $p(n) = 1/\sqrt{2\pi\sigma^2}\exp\left\{-\frac{(n-\mu)^2}{2\sigma^2}\right\}$, the optimum nonlinearity can be calculated as $\tilde{h}(x) = cx + d$, where c and d are constants. This optimum filter function appears to be linear,



(a) Input signal (sinusoidal + Cauchy noise) (b) SR with optimal nonlinear filtering (c) LPF of the optimal nonlinear filtering

(d) LPF of the input signal (e) Optimum nonlinearity $\tilde{h}_c(t)$ for Cauchy noise

Fig. 3. A signal example for Cauchy noise ($\gamma = 0.10$). The dotted line indicates the scaled sinusoidal signal. Although the desired signal does not appear in the case of (d) linear filtering case (LPF), (c) LPF of the optimal nonlinear filtering provides the signal fairly well.

which means that *linear filtering is optimum for Gaussian noise*. In the case of non-Gaussian noise, since the optimum function is nonlinear, nonlinear filtering exhibiting SR is beneficial.

3.2. Numerical example and discussion

In order to verify the effect of the proposed nonlinear filter, we evaluate the SNR in the presence of a non-Gaussian noise. Since the SNR depends on the normalized g, the performance is evaluated by it. As an example of non-Gaussian noise, we consider a Cauchy noise with a zero-mean p.d.f., $p(n) = \left\{\pi\gamma\left[1+\frac{n^2}{\gamma^2}\right]\right\}^{-1}$. The SNR for the optimal nonlinearity is plotted as a function of γ in Fig. 2. Here, we consider a sinusoidal waveform as the desired weak signal, and the two constants are set to be $\tilde{a}_c = 0$ and $\tilde{b}_c = 1$. For comparison, the SNRs for a linear system $h(x) = x$ and a simple threshold system $h(x) = \mathrm{sgn}(x - \theta)$, which is a typical nonlinear system in SR research, are shown. The threshold is set to be $\theta = 0.5$. The increase of γ means that the power of the Cauchy noise is also increased. Actually, the SNR at the input side equals to the one in the case of the linear system since the SNR is defined in frequency domain and the system cannot improve it. For reference, Fig. 3 shows signal examples at $\gamma = 0.10$. The optimum nonlinearity is $\tilde{h}_c(x) = \tilde{a}_c + \tilde{b}_c x/(\gamma^2 + x^2)$, as shown in Fig. 3(e).

The nonlinearity function for a weak signal, which is located in the region around zero input voltage in Fig. 3(e), can be viewed as linear function with a large slope. This means that if the noise voltage is weak, the desired weak signal is enhanced significantly. In contrast, the high voltage of the noise is reduced to zero voltage because the nonlinear filter output for the high-voltage input is approximately zero. Fig. 3(a) indicates that this situation often arises, and such a nonlinear effect should improve the SNR, as shown in Fig. 2. The optimum nonlinear function contains the parameter γ, which determines the noise characteristics. The nonlinearity is optimized for every γ, so that optimal nonlinearity always gives a higher SNR than other systems.

The performance of the comparator has a peak at $\gamma \approx 0.30$. This means that the system with the comparator can be enhanced in the certain level of the noise. However, in the case of an optimum nonlinear filter, such peak does not appear and the performance decreases monotonically. If the optimum nonlinearity is fixed and the noise parameter γ is varied, such a peak will appear.

Fig. 2 shows that even if the nonlinearity is not optimum, the SR system is still effective in the case of non-Gaussian noise. A simple comparator can provide better performance than that in the linear case. Such an improvement of the SNR using a nonlinear system cannot be expected in the presence of Gaussian noise, and this effect is due purely to the non-Gaussian property of the input noise.

Although the optimum nonlinear filter is effective in non-Gaussian noise, the SNR remains low. Fig. 3(b) shows an example of the optimum nonlinear filter output. The signal appears to contain the sinusoidal signal, but high-frequency noise corrupts the signal. This means that low-pass filtering (LPF) of the output is effective. The resulting signal obtained after LPF is shown in Fig. 3(c). Note that we use a transversal filter having a cut-off frequency ten times as high as the sinusoidal signal frequency. Then, the desired sinusoidal signal can be obtained after LPF. The signal obtained after only LPF is shown in Fig. 3(d). The output is so distorted that the desired signal does not appear. This signal-based comparison also demonstrates the advantage of the proposed optimum nonlinear filter.

4. RELATION TO CRAMÉR-RAO BOUNDS

In this section, we demonstrate that the proposed optimum nonlinear filter yields the Cramér-Rao bounds. The noise p.d.f. $\rho(n(t))$ can be expressed in terms of the input signal $x(t)$: $\rho(n(t)) = p(x(t) - \epsilon s(t))$. Thus, the current problem can be regarded as an estimation of the unknown input signal $\epsilon s(t)$ by the observed value $x(t)$ with conditional probability $P(x|\epsilon s)$. Since $x(t)$ and the estimated input signal $\hat{s}(t)$ are both random variables, the accuracy of the estimation can be bounded by its variance $\mathrm{Var}(\hat{s}(t))$. The Cramér-Rao inequality [18, 19] gives the lower bound of the variance:

$$\begin{aligned} \mathrm{Var}(\hat{s}) &\geq \mathcal{F}^{-1}(\epsilon s), \\ \mathcal{F}(\epsilon s) &= \int P(x|\epsilon s) \left[\frac{\partial \ln P(x|\epsilon s)}{\partial s} \right]^2 dx. \end{aligned} \tag{6}$$

For a weak input $\epsilon \to 0$, the accuracy is bounded by the Fisher information $\mathcal{F}(0)$. Since $\partial \ln \rho(n(t))/\partial n(t) = -\partial \ln P(x|\epsilon s)/\partial s|_{\epsilon \to 0}$, the noise intensity, which is equal to the variance of the output for the optimal nonlinearity $\tilde{h}(x)$, is given by $\langle \tilde{h}^2(x) \rangle - \langle \tilde{h}(x) \rangle^2 = b^2 \mathcal{F}(0)$. The scale factor of the signal part of the output is $\langle \tilde{h}'(x) \rangle = b\mathcal{F}(0)$. The estimated input signal $\hat{s}$ can be modeled as $\hat{s} = s + \xi$, where ξ is the estimation error, and the input signal is scaled at the output channel by the factor $\langle \tilde{h}'(x) \rangle$. Then, the variance of the estimated signal is obtained as follows:

$$\begin{aligned} \mathrm{Var}(\hat{s}) &= \langle \xi^2 \rangle \\ &= \frac{\langle \tilde{h}^2(x) \rangle - \langle \tilde{h}(x) \rangle^2}{\langle \tilde{h}'(x) \rangle^2} \\ &= \mathcal{F}^{-1}(0). \end{aligned} \tag{7}$$

Eq. (7) indicates that the optimal nonlinearity $\tilde{h}(x)$ yields the Cramér-Rao lower bound of the estimation error, and the maximum SNR can be achieved.

The simple optimum function given by eq. (5) is equivalent to maximum likelihood (ML) estimation in non-Gaussian noise. The ML estimation has been derived from a hypothesis testing problem for a given a priori probability [20]. In this sense, the nonlinear filtering is optimum for the SNR. In general, such estimators are complicated, but the proposed nonlinear filter given by eq. (5) can be realized in a simple form.

5. CONCLUSION

For the application of SR in signal detection, we have proposed an optimal nonlinearity in the present paper. We focus on the white noise, and the nonlinearity is determined only by the p.d.f. of the input noise. The theoretical analysis and the numerical example show that the SNR performance is better than that in linear systems.

The mathematical expression of the optimal nonlinearity eq. (5) is so simple that the calculation and the implementation should not be difficult. In addition, this expression systematically claims that *if the noise p.d.f. is Gaussian, a linear system, rather than an SR system, should be selected. In the non-Gaussian noise case, there exists a nonlinear system that gives a higher SNR than that of a linear system.* This will be helpful for applying SR to signal detection, because one can find easily what situations and/or devices are effective for the SR application, and design a nonlinear system.

The proposed optimum nonlinearity has been discussed from the viewpoint of information theory. The nonlinearity is closely related to Cramér-Rao bounds and maximum likelihood estimation. Such conventional schemes are generally complicated but can provide the minimum error in signal estimation. As discussed in Sections 3 and 4, the proposed framework has the same performance and the design task can be simplified.

The proposed nonlinearity will be expanded to application under colored noise. The current nonlinearity is intended for white noise and can be obtained only from the input noise p.d.f. Colored noise has a time-dependent characteristic (i.e., its auto-correlation function is not delta-shaped). The nonlinearity for colored noise can be found by considering this characteristic. The best performance can be obtained through simple calculation.

In practical situations, the noise p.d.f. will be dynamically changed (e.g., the mean will easily fluctuate). The p.d.f. contains all of the information about the noise, including the mean and variance. Since the proposed nonlinearity is based on the noise p.d.f., the changing of the p.d.f means the redesigning of the nonlinearity. Additional research on estimating the noise p.d.f. and implementation using adaptive filtering theory is necessary.

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